ELEMENTARY LIE ALGEBRAS AND LIE A-ALGEBRAS

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1 Introduction

Throughout this paper L will denote a finite-dimensional Lie algebra over a field F. The Frattini ideal of L, $\phi(L)$, is the largest ideal of L contained in all maximal subalgebras of L. The Lie algebra L is called *ϕ-free* if $\phi(L) = 0$, and *elementary* if $\phi(B) = 0$ for every subalgebra B of L. Elementary Lie algebras were introduced by Stitzinger [24] and Towers [27] by analogy to the definition of an elementary group given earlier by Bechtell [3]. An interesting property of an elementary Lie algebra is that it splits over each of its ideals, see [27].

The class of elementary Lie algebras is closely related to the class of Lie algebras all whose nilpotent subalgebras are abelian (called A-algebras) and to the class of Lie algebras L such that $\phi(B) \leq \phi(L)$ for all subalgebras B of L (called E-algebras). A-algebras have been studied by Drensky [7], Sheina [23], Premet and Semenov [21] and Dallmer [6]. Since the Frattini ideal of a nilpotent Lie algebra L is just the derived subalgebra of L, every elementary Lie algebra is an A-algebra. E-algebras were introduced by Stitzinger in [24]. He proved that L is an E-algebra if and only if $L/\phi(L)$ is elementary. A Lie algebra L is called *strongly solvable* if L^2 is nilpotent. Stitizinger also

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proved in [24] that if L is strongly solvable then L is an E-algebra. In this paper it is shown that over a perfect field the converse also holds

For algebraically closed fields of characteristic zero, elementary Lie algebras were determined by Towers in [27]. This classification is shown to remain true for any algebraically closed field of characteristic different from two or three.

Following Jacobson [15], we say that a linear Lie algebra $L \leq \mathrm{gl}(V)$ is *almost algebraic* if L contains the nilpotent and semisimple Jordan components of its elements. Every algebraic Lie algebra is almost algebraic. An abstract Lie algebra L is called almost algebraic if $\mathrm{ad}L \leq \mathrm{gl}(L)$ is almost algebraic. Recently, Zhao and Lu have proved in [29] that every almost-algebraic A-algebra is elementary, whenever the ground field is algebraically closed of characteristic zero. In this paper we prove that every elementary Lie algebra is almost algebraic, provided that $\mathrm{char}(F) = 0$.

The final section of the paper is devoted to classifying the real elementary simple Lie algebras.

We will denote algebra direct sums by $\oplus$, direct sums of the vector space structure alone by $\dot{+}$, and semidirect products by $\rtimes$. The *nilradical* of L will be denoted by $N(L)$, whilst $Asoc(L)$ will denote the sum (necessarily direct) of the minimal abelian ideals of L.

2 The solvable case

Over a field of characteristic zero every solvable Lie algebra is strongly solvable, by Lie's Theorem. This fails in characteristic p for every $p > 0$ (see [22], page 96). However, we have the following result.

Proposition 2.1 *Let L be an elementary solvable Lie algebra over a perfect field F. Then L is strongly solvable.*

Proof. Let L be a minimal counter-example. As the hypotheses are subalgebra closed, every proper subalgebra of L is strongly solvable, and so L has the structure described in Theorem 4 of [5]. Thus, $L = A \rtimes B$, where A is the unique minimal ideal of L, $\dim A \geq 2$, $A^2 = 0$, $B = M \dot{+} Fx$ with $M^2 = 0$, and either M is a minimal ideal of B, or B is the three-dimensional Heisenberg algebra.

Pick any $m \in M$ and put $C = A + Fm$. Then C is ϕ-free, so $A \subseteq N(C) = \mathrm{Asoc}(C)$ by Theorem 7.4 of [26], and A is completely reducible as an Fm-module. Write $A = \oplus_{i=1}^{r} A_i$, where A_i is an irreducible Fm-module for $1 \leq i \leq r$. Then the minimum polynomial of the restriction of ad m to A_i is irreducible for each i, and so $\{(\text{ad}m)|_A : m \in M\}$ is a set of commuting semisimple operators. Let Ω be the algebraic closure of F and put $A_\Omega = A \otimes_F \Omega$, and so on. As F is perfect, $\{(\text{ad}m)|_{A_\Omega} : m \in M\}$ is a set of simultaneously diagonalizable linear maps. So, we can decompose A_Ω into

$$(A_\Omega)_{\alpha_i} = \{a \in A_\Omega : [a, m] = \alpha_i(m)a \ \ \forall m \in M\},$$

where $1 \leq i \leq s$.

Suppose first that M is a minimal ideal of B. Then M_Ω has a basis $m_1, \ldots, m_t$ of eigenvectors of ad x with corresponding eigenvalues $\beta_1, \ldots, \beta_t$. Let $0 \neq a_i \in (A_\Omega)_{\alpha_i}$. Then

$$[x, a_i] = \sum_{k=1}^{s} a'_k \quad \text{where } a'_k \in (A_\Omega)_{\alpha_k}.$$

But now

$$\begin{aligned} 0 &= [[a_i, m_j], x] + [[m_j, x], a_i] + [[x, a_i], m_j] \\ &= \alpha_i(m_j)[a_i, x] + \beta_j[m_j, a_i] + \textstyle\sum_{k=1}^{s}[a'_k, m_j] \\ &= -\textstyle\sum_{k=1}^{s} \alpha_i(m_j)a'_k - \beta_j\alpha_i(m_j)a_i + \textstyle\sum_{k=1}^{s} \alpha_k(m_j)a'_k \end{aligned}$$

Hence

$$\beta_j\alpha_i(m_j)a_i = \sum_{k=1}^{s}(\alpha_k(m_j) - \alpha_i(m_j))a'_k$$

This yields that $\beta_j\alpha_i(m_j)a_i = 0$ and therefore either $[x, m_j] = 0$ or $[m_j, a_i] = 0$ for all $1 \leq i \leq r$. The former is impossible, since it implies that 0 is a characteristic root of $(\text{ad}x)|_M$, whence B is two-dimensional abelian and L^2 is nilpotent. The latter is also impossible, since then $[M, A] = 0$ and $L^2 \subseteq A \oplus M$, which is abelian.

Hence B is the three-dimensional Heisenberg algebra. But then B is a non-abelian nilpotent subalgebra of L and hence not ϕ-free. This contradiction establishes the result.

A Lie algebra L is called an E-algebra if $\phi(B) \leq \phi(L)$ for all subalgebras B of L. Groups with the analogous property are called E-groups by Bechtell. Stitzinger in [24] proved that a Lie algebra is an E-algebra if and only if $L/\phi(L)$ is elementary. He also proved that every strongly solvable Lie algebra over an arbitrary field is an E-algebra. Next, we prove the converse of this result, provided that the ground field is perfect.

Corollary 2.2 *Let F be perfect. Then every solvable E-algebra is strongly solvable.*

Proof. Let L be a solvable E-algebra. If L is ϕ-free, then L is elementary, by [24], and L^2 is nilpotent, by Proposition 2.1. So suppose that $\phi(L) \neq 0$. Then $L/\phi(L)$ is a solvable elementary Lie algebra, and so $L^2/\phi(L) = (L/\phi(L))^2$ is nilpotent. But then L^2 is nilpotent, by Theorem 5 of [2].

Lemma 2.3 *Let L be a Lie algebra over any field F and let A be a minimal ideal of L with $[L^2, A] = 0$. Then $A \subseteq \mathrm{Asoc}(C)$ for every subalgebra C of L containing A.*

Proof. We have $A^3 = [A^2, A] = 0$. Minimality of A implies that A is abelian. Moreover, since $[L^2, A] = 0$, we have that $(\mathrm{ad}x)|_A$ is C-linear for every $x \in L$. This implies that the sum of the irreducible C-submodules of A is invariant under L, and thus that it coincides with A. The result follows.

If A is a subset of L we denote by $C_L(A)$ the centraliser of A in L. Now we give a construction of elementary solvable Lie algebras.

Proposition 2.4 *Let F be an arbitrary field. Let A be a vector space of finite dimension and let B be an abelian completely reducible subalgebra of* $\mathrm{gl}(A)$. *Then the semidirect product $A \rtimes B$ is an elementary almost-algebraic Lie algebra.*

Proof. Put $L = A \rtimes B$. Then L is strongly solvable and hence an E-algebra. But $A \leq \mathrm{Asoc}(L) \leq C_L(A) = A$, so L is ϕ-free by Theorem 7.3 of [26]. It follows that L is elementary, and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is a minimal ideal of L for $1 \leq i \leq n$.

So suppose now that $A \not\subseteq M$. We may assume that $A_1 \not\subseteq M$. But then $L = A_1 \dot{+} M$, and $M \cong L/A_1 \cong A_2 \oplus \cdots \oplus A_n \dot{+} B$. Since A is a faithful B-module, we have that $A_2 \oplus \cdots \oplus A_n = \mathrm{Asoc}(A_2 \oplus \cdots \oplus A_n \dot{+} B)$, whence $\phi(M) = 0$.

In order to prove that L is almost algebraic, let $x \in L$, $x \notin A$. By Lemma 2.3 we have that $A = \mathrm{Asoc}(A + Fx)$ and so the action of x on A is semisimple. Let $A = E_1 \oplus \cdots \oplus E_k$, where E_i is an irreducible Fx-module. Then $[E_i, x] = 0$ or E_i for each $1 \leq i \leq k$, so we can write $A = C \oplus D$, where

$[C,x]=0$, $[D,x]=D$. Put $x=c+d+b$, where $c\in C, d\in D, b\in B$. Then $[D,b]=[D,x]=D$, so there is a $y\in D$ such that $[y,b]=d$. Consider the automorphism $e^{\mathrm{ad}y}$ of L. We have $e^{\mathrm{ad}y}(b)=(1+\mathrm{ad}y)(b)=b+[y,b]=b+d$, so $b+d\in e^{\mathrm{ad}y}(B)$. Since $e^{\mathrm{ad}y}(B)$ is abelian, it follows that $\mathrm{ad}(b+d)$ is semisimple. Now $(\mathrm{ad}c)^2L=[[L,c],c]\subseteq A^2=0$, whence $\mathrm{ad}c$ is nilpotent. This yields that L is almost algebraic. The proof is complete.

An elementary Lie algebra which can be constructed as in Proposition 2.4 will be called of *type I*. Next, we show that every elementary solvable Lie algebra over a perfect field can be constructed as an algebra direct sum of an abelian Lie algebra and a Lie algebra of type I.

We say that L is *metabelian* if L^2 is abelian. We denote the centre of L by $Z(L)$.

Theorem 2.5 *Let F be perfect. For a solvable Lie algebra L, the following statements are equivalent:*

1. *L is elementary,*
2. *L is ϕ-free and strongly solvable,*
3. *L is ϕ-free and metabelian,*
4. *$L=\mathrm{Asoc}(L)\rtimes B$, where B is an abelian subalgebra of L,*
5. *$L\cong A\oplus E$, where A is an abelian Lie algebra and E is an elementary Lie algebra of type I.*

Proof. (1)⇒(2): This follows from Proposition 2.1.

(2)⇒(3): Let L be ϕ-free and strongly solvable. By Theorems 7.3 and 7.4 of [26], we have that $L=\mathrm{Asoc}(L)\rtimes B$, where B is a subalgebra of L, and that $\mathrm{Asoc}(L)$ is precisely the largest nilpotent ideal of L. As L is strongly solvable, we have $L^2\leq \mathrm{Asoc}(L)$. This yields that both L^2 and B are abelian.

(3)⇒(4): This is clear from Theorems 7.3 and 7.4 of [26] as above.

(4)⇒(5): Decompose $\mathrm{Asoc}(L)=Z(L)\oplus K$, where K is an ideal of L. Put $E=K\dot{+}B$. We have

$$C_E(K)\cap B\leq Z(E)\leq Z(L)\cap E=0,$$

so that $B \lesssim \mathrm{gl}(K)$. Moreover, since $K \leq \mathrm{Asoc}(L)$ and $L = \mathrm{Asoc}(L) \rtimes B$, it follows that K is completely reducible as a B-module. Hence E is of type I.

(5)⇒(1): In Towers [27], it is proved that a direct sum of elementary Lie algebras is elementary. So this follows from Proposition 2.4. This completes the proof.

Corollary 2.6 *An elementary solvable Lie algebra over a perfect field is almost algebraic.*

In [13], Gein and Varea showed that solvability was a subalgebra lattice property, provided that L was at least three dimensional and the underlying field was perfect of characteristic different from $2, 3$. We now have that the same is true for strong solvability.

Corollary 2.7 *Let L be a strongly solvable Lie algebra over a perfect field of characteristic different from $2, 3$, and let L^* be a Lie algebra that is lattice isomorphic to L. Then either*

1. *L^* is three-dimensional non-split simple, or*
2. *L^* is strongly solvable and $\dim L = \dim L^*$.*

Proof. Simply combine Theorem 2.5 with Theorem 3.3 of [13].

3 The non-solvable case

If C is a subalgebra of L we denote by $R(C)$ the radical of C, and by $N(C)$ the nilradical of C.

Proposition 3.1 *Let F be perfect. Let L be an elementary Lie algebra which is neither solvable nor semisimple. Then, $L = \mathrm{Asoc}(L) \rtimes (B \dot{+} S)$, where B is abelian, S is a semisimple subalgebra of L and $[B, S] \leq B$. If F has characteristic zero, then $[B, S] = 0$.*

Proof. As L is ϕ-free, $L = \text{Asoc}(L) \rtimes C$ for some subalgebra C of L and $N(L) = \text{Asoc}(L)$, by Theorem 7.3 of [26]. Let $B = R(C)$. Since C is elementary, it splits over B, by [27, Theorem 2.4]. So, $C = B \dot{+} S$, where S is semisimple. It is clear that $R(L) = \text{Asoc}(L) \rtimes B$. By Proposition 2.1, we have that $R(L)$ is strongly solvable. So $B^2 \leq R(L)^2 \cap B \subseteq N(L) \cap B = \text{Asoc}(L) \cap B = 0$. If F has characteristic zero, the final assertion follows from [25, Theorem 4].

A Lie algebra all of whose proper subalgebras are abelian is called *semi-abelian*.

Example Let S be a simple semiabelian Lie algebra over a field of characteristic $p > 0$, let B be a faithful finite-dimensional completely reducible L-module, and put $L = B \dot{+} S$ where $B^2 = 0$ and L acts on B under the given L-module action. Then L is elementary, but S is not an ideal of L.

Notes

- Since elementary Lie algebras are Lie A-algebras it follows from Proposition 2 of [21] that over a field of cohomological dimension ≤ 1 every semisimple elementary Lie algebra is representable as a direct sum of simple ideals, each of which splits over some finite extension of the ground field into a direct sum of ideals isomorphic to $\text{sl}(2, F)$.
- Over a perfect field with non-trivial Brauer group there exists a finite-dimensional simple semiabelian Lie algebra (see Theorem 8.5 of [9]), and this is elementary.
- Let G be the algebra constructed by Gein in Example 2 of [12]. This is a seven-dimensional Lie algebra over a certain perfect field F of characteristic three. Every subalgebra of G of dimension greater than one is simple. So, G is elementary.

We finish this section by proving that the classification of elementary Lie algebras over an algebraically closed field of characteristic zero given by Towers in [27] remains true for any algebraically closed field of characteristic different from 2 or 3.

Theorem 3.2 *Let L be a Lie algebra over an algebraically closed field F of characteristic $\neq 2$ or 3. Then L is elementary if and only if*

1. *L is isomorphic to a direct sum of copies of* $\mathrm{sl}(2, F)$*, or*

2. *there is a basis* $\{a_1, \dots, a_m, b_1 \dots, b_n\}$ *for L such that*

$$[a_i, b_j] = -[b_j, a_i] = \lambda_{ij} a_i \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n$$

   all other products being zero, or

3. *$L \cong A \oplus B$, where A is as in (1) and B is as in (2).*

Proof. If $\mathrm{char}(F) = 0$, then the result is Theorem 3.2 of [27]. Assume $\mathrm{char}(F) = p > 3$. Let L ($\neq 0$) be elementary. If L is solvable, then $L = \mathrm{Asoc}(L) \rtimes B$, by Theorem 2.5. Decompose $\mathrm{Asoc}(L) = A_1 \oplus \cdots \oplus A_m$, where A_i is a minimal ideal of L. As F is algebraically closed, we have $\dim A_i = 1$. Hence L is as in (2).

Now, let L be semisimple. Since L is an A-algebra, Proposition 2 of Premet and Semenov [21] applies and L is as in (1).

Then suppose that L is neither solvable nor semisimple. By Proposition 3.1 we have that $L = A \rtimes (B \dot{+} S)$, where $0 \neq A = \mathrm{Asoc}(L)$, B is abelian, $0 \neq S$ is a semisimple subalgebra of L and $[B, S] \leq B$. From [21, Proposition 2] again it follows that $S = S_1 \oplus \cdots \oplus S_r$, where S_i is an ideal of S and $S_i \cong \mathrm{sl}(2, F)$ for every $1 \leq i \leq r$. Put $C_i = A \dot{+} S_i$. We have that C_i is a ϕ-free Lie algebra and so $N(C_i) = \mathrm{Asoc}(C_i)$. This yields that $A = \mathrm{Asoc}(C_i)$ and therefore A is a completely reducible S_i-module. Let V be an irreducible S_i-submodule of A.

We claim that $\dim_F V = 1$. By general theory, there exists an element $e \in S_i$ such that V is a cyclic Fe-module on which e acts nilpotently. This can be seen by looking at the representatives of the coadjoint orbits of SL(2) on $\mathrm{sl}(2, F)^*$ (see [11, §2]). Consequently, $M = V \dot{+} Fe$ is a nilpotent subalgebra. This yields that M is abelian, whence $[V, e] = 0$ and $\dim_F V = 1$.

Therefore, we have that $[A, S] = 0$. If $B = 0$, then we have that L is as in (3). Suppose then $B \neq 0$. We have that $B \dot{+} S$ is an elementary Lie algebra, whence $\mathrm{Asoc}(B \dot{+} S) = N(B \dot{+} S) = B$. As above, we obtain that $[B, S] = 0$, from which it follows that $L = (A \dot{+} B) \oplus S$. Since $A \dot{+} B$ is a solvable elementary Lie algebra, it is as in (2), and therefore L is as in (3). This completes the proof in one direction. The converse is easily checked.

The above result does not hold in characteristic 2: over such a field, the three-dimensional simple Lie algebra with basis e_1, e_2, e_3 and products $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ is elementary. The exclusion of characteristic 3 is used in order to invoke the result of Semenov and Premet, which in turn relies on [20, Theorem 3]. This last result fails in characteristic 3, as is shown by the algebra G ([12, Example 2]) referred to earlier. However, we know of no counter-example to the above result in characteristic 3: if we pass to the algebraic closure, then G becomes psl(3), which is no longer elementary.

4 The characteristic zero case

A subalgebra T of L is said to be a *toral* subalgebra of L if T is abelian and $\mathrm{ad}_L t$ is semisimple for every $t \in T$. A Lie algebra L is said to be *ad-semisimple* if $\mathrm{ad}x$ is semisimple for every $x \in L$.

Proposition 4.1 *Let* $\mathrm{char}(F) = 0$. *For a solvable Lie algebra L the following statements are equivalent:*

1. *L is elementary,*
2. *L is ϕ-free and almost algebraic.*

Proof. Assume that L is ϕ-free and almost algebraic. Then $L = N(L)\dot{+}T$, where T is a toral subalgebra of L (see [1] or [26, Theorem 7.5]). Moreover, we have $N(L) = \mathrm{Asoc}(L)$ by Theorem 7.4 of [26]. So, L is elementary by Theorem 2.5. The converse follows from Corollary 2.6.

Proposition 4.2 *Let L be an ad-semisimple Lie algebra over a field of characteristic zero. Then L is elementary*

Proof. Let L be a minimal counter-example. Then it suffices to show that L is ϕ-free. But $L = Z(L) \oplus S$, where S is semisimple, by Levi's Theorem and Theorem 1 of [10]. It follows that $\phi(L) = 0$.

Corollary 4.3 *Over the real field every compact semisimple Lie algebra is elementary.*

A Lie algebra L is said to be *reductive* if its adjoint representation is completely reducible; equivalently, $L = S \oplus Z(L)$, where S is a semisimple ideal of L and $Z(L)$ is the centre of L (see [15]).

Proposition 4.4 *Let* $\mathrm{char}(F) = 0$. *Let* L *be a Lie algebra such that its radical* R *is elementary and* L/R *is ad-semisimple. Then,* L *is elementary and almost algebraic.*

Proof. Let $S(L)$ be the largest semisimple ideal of L. Since $S(L)$ is isomorphic to an ideal of L/R, by Proposition 4.2 it follows that $S(L)$ is elementary. Since $S(L)$ is a direct summand of L and since $S(L)$ is almost algebraic, we may suppose without loss of generality that $S(L) = 0$. By Proposition 4.1 we have that R is almost algebraic. Then L is also almost algebraic by Corollary 3.1 of [1] (see also [18]).

Therefore $L = N(L) \rtimes (B \dot{+} S)$ where B is a toral subalgebra of L, S is a semisimple subalgebra of L and $[B, S] = 0$. We have $N(R) = \mathrm{Asoc}(R)$ since R is ϕ-free. As $\mathrm{char}(F) = 0$, we have that $N(R)$ is a characteristic ideal of R and so $N(R) \leq N(L)$. It follows that $N(L) = \mathrm{Asoc}(R)$ and so $N(L)$ is abelian. Put $A = N(L)$. Since every element of B acts semisimply on A, we have that A is completely reducible as a $(B \oplus S)$-module, see [15]. It follows that $A = \mathrm{Asoc}(L)$. This yields that L is ϕ-free. To prove that L is elementary it suffices to show that every maximal subalgebra M of L is also ϕ-free.

Let us first consider the case when M does not contain R. Then $L = R + M$, $L/R \cong M/M \cap R$ and $M \cap R$ is the radical of M. We have that $M \cap R$ is elementary and $M/M \cap R$ is ad-semisimple. By the above, we obtain that M is ϕ-free.

Now suppose that $R \leq M$. Since $M \cap S$ is ad-semisimple, by [10, Theorem 1] it follows that $M \cap S = Z(M \cap S) \oplus (M \cap S)^2$ and $(M \cap S)^2$ is semisimple. Put $B^* = B \oplus Z(M \cap S)$. We have that $B^* \oplus (M \cap S)^2$ is reductive. Moreover, since every element of $Z(M \cap S)$ acts semisimply on S, it acts also semisimply on A (see [15], page 101). It follows that every element of B^* acts semisimply on A. This yields that A is completely reducible as a $(B^* \oplus (M \cap S)^2)$-module (see [15]) and therefore $A \leq \mathrm{Asoc}(M)$. On the other hand, we have $R(M) = A \dot{+} B^*$ and $A \leq N(M)$. We claim that $A = N(M)$. Let $x \in B^* \cap N(M)$. We have that $(\mathrm{ad}x)|_A$ is nilpotent and semisimple. So, $x \in C_L(A)$. Moreover, we have that $[x, B] \subseteq [S, B] = 0$. This yields that $x \in C_L(R)$. Since $C_L(R) \cap S$ is a semisimple ideal of L, we have $C_L(R) \cap S = 0$, whence $C_L(R) = Z(R)$. Decompose $x = b + z$, $b \in B$, $z \in Z(M \cap S)$. We have $z \in R \cap S = 0$. This yields that $x \in B \cap Z(R) \leq Z(L) = 0$ and therefore $A = N(M)$, as claimed. Hence $A = \mathrm{Asoc}(M)$. Since M splits over A, it follows that M is ϕ-free. This completes the proof.

Corollary 4.5 *Let* $\mathrm{char}(F) = 0$. *Let A be a vector space of finite dimension. Let K be a reductive subalgebra of* $\mathrm{gl}(A)$ *such that K^2 is non-zero and ad-semisimple and every non-zero element of $Z(K)$ is a semisimple transformation of A. Then, the semidirect product $A \rtimes K$ is an elementary almost algebraic Lie algebra.*

Proof. Put $L = A \rtimes K$. We have that $R(L) = A \dot{+} Z(K)$. By Proposition 2.4, it follows that $R(L)$ is an elementary Lie algebra. Also, we have $L/R(L) \cong K^2$. So that $L/R(L)$ is ad-semisimple. The result follows from Proposition 4.4.

A Lie algebra which can be constructed as in the above corollary will be called of *type II*.

Theorem 4.6 *Let* $\mathrm{char}(F) = 0$. *A Lie algebra L is elementary if and only if $L \cong A \oplus B \oplus S$, where A is abelian, B is a Lie algebra of type I or of type II and S is an elementary semisimple Lie algebra.*

Proof. Let L be elementary. Then L splits over $Z(L)$, so $L = Z(L) \oplus \hat{L}$, where $\hat{L}$ is a centerless Lie algebra. Now let $S(\hat{L})$ be the largest semsimple ideal of $\hat{L}$. Then we have that $\hat{L} = K \oplus S(\hat{L})$, where K is a centerless Lie algebra which has no non-zero semisimple ideals. If K is solvable, then we find that K is an elementary Lie algebra of type I.

Then assume that K is not solvable. By Proposition 3.1 it follows that $K = \mathrm{Asoc}(K) \rtimes (B \oplus S)$, where B is abelian, S is semisimple and $[B, S] = 0$. Let $0 \neq s \in S$ such that $\mathrm{ad}_S s$ is nilpotent. Then we have that $\mathrm{Asoc}(K) + Fs$ is a nilpotent subalgebra of L. Hence $[s, \mathrm{Asoc}(L)] = 0$. This yields that $s \in C_K(R(K)) \cap S = 0$, which is a contradiction. Therefore S has no non-zero ad-nilpotent elements. As S is semisimple and $\mathrm{char}(F) = 0$, it follows that S is ad-semisimple. Put $C = C_K(\mathrm{Asoc}(K)) \cap (B \oplus S)$. We have that C is an ideal of the reductive Lie algebra $B \oplus S$, so $C = (C \cap B) \oplus (C \cap S)$. It follows that $C \cap B \leq Z(K) = 0$ and that $C \cap S = 0$ since it is a semisimple ideal of K. This yields that $B \oplus S \lesssim \mathrm{gl}(\mathrm{Asoc}(K))$ and therefore K is of type II. This completes the proof in one direction.

The converse follows from Proposition 2.4 and Corollary 4.5.

Corollary 4.7 *An elementary Lie algebra over a field of characteristic zero is almost algebraic.*

Proof. This follows from Theorem 4.6, Proposition 2.4 and Corollary 4.5.

Corollary 4.8 *Let* $\mathrm{char}(F) = 0$. *For a Lie algebra* L *without non-zero semisimple ideals, the following statements are equivalent:*

1. *L is elementary.*
2. *L is almost algebraic and an A-algebra.*
3. *L is almost algebraic, $N(L)$ is abelian and $L/R(L)$ is ad-semisimple.*
4. *L is almost algebraic, ϕ-free and $L/R(L)$ is ad-semisimple.*

Proof. (1)⇒(2): This follows from Corollary 4.7.

From now on in this proof we assume that L is almost algebraic. Then we have that $L = N(L)\dot{+}B\dot{+}S$, where B is a toral subalgebra of L, S is a semisimple subalgebra of L and $[B, S] = 0$.

(2)⇒(3): Clearly, $N(L)$ is abelian. It remains to prove that S is ad-semisimple. Let $s \in S$ such that $\mathrm{ad}_S s$ is nilpotent. Then $(\mathrm{ad}s)|_{N(L)}$ is nilpotent too. This yields that $N(L) + Fs$ is a nilpotent subalgebra of L and therefore $[N(L), s] = 0$. Thus, $s \in C_L(R(L)) \cap S = 0$ because L has no non-zero semisimple ideals. Hence S has no non-zero ad-nilpotent elements. As $\mathrm{char}(F) = 0$, we have that S is ad-semisimple.

(3)⇒(4): Since $N(L)$ is a completely reducible $(B\oplus S)$-module and since $N(L)$ is abelian, it follows that $N(L) = \mathrm{Asoc}(L)$. Since L splits on $N(L)$, we have that L is ϕ-free.

(4)⇒(1): By Theorem 7.4 of [26] we have $N(L) = \mathrm{Asoc}(L)$. It then follows from Theorem 2.5 that $N(L)\dot{+}B$ is elementary. Since $R(L) = N(L)\dot{+}B$, Proposition 4.4 now gives that L is elementary.

5 The real field case

A subalgebra P of L is called *parabolic* if $P \otimes_F \Omega$ contains a Borel subalgebra (that is, a maximal solvable subalgebra) of $L \otimes_F \Omega$, where Ω is the algebraic closure of F. Over a field of characteristic zero, all maximal subalgebras of a reductive Lie algebra are reductive or parabolic (see [4], [16], [19]); it follows that a reductive Lie algebra is elementary if and only if its parabolic subalgebras are ϕ-free.

For results concerning Lie algebras over the real field we refer the reader to the books by Helgason ([14]) and Knapp ([17]).

Theorem 5.1 *Let F be the real field. For a simple Lie algebra L, the following statements are equivalent:*

1. *L is elementary,*
2. *L is an A-algebra,*
3. *L is compact or isomorphic to one of the following Lie algebras:* $\mathrm{sl}(2,\mathbb{R})$, $\mathrm{sl}(2,\mathbb{C})^{\mathbb{R}}$, $\mathrm{so}(n,1)$ $(n > 3)$.

Proof. (1)⇒(2): This is clear.

(2)⇒(3): Let L be a non-compact A-algebra. Suppose first that $L = \bar{L}^{\mathbb{R}}$, the realisation of the complex simple Lie algebra $\bar{L}$. If $\bar{N}$ is a nilpotent subalgebra of $\bar{L}$ then $\bar{N}^{\mathbb{R}}$ is a nilpotent subalgebra of L, and hence abelian. It follows that $\bar{N}$ is abelian and thus that $\bar{L}$ is an A-algebra. The proof of Theorem 3.2 of [27] then shows that $\bar{L} \cong \mathrm{sl}(2,\mathbb{C})$.

So assume now that L is a non-compact real form of a complex simple Lie algebra. The only such algebras for which the nilpotent subalgebra N in the Iwasawa decomposition of L is abelian are $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(n,1)$ $(n > 3)$.

(3)⇒(1): Suppose now that L is one of the algebras described in (3). If L is compact it is elementary by Corollary 4.3, and $\mathrm{sl}(2,\mathbb{R})$ is clearly elementary.

Next suppose that $L \cong \mathrm{sl}(2,\mathbb{C})^{\mathbb{R}}$, and let S be a subalgebra of L. Then $S \otimes \mathbb{C}$ is a subalgebra of $L \otimes \mathbb{C}$, which is elementary. This yields $0 = \phi(S \otimes \mathbb{C}) = \phi(S) \otimes \mathbb{C}$, by [8], whence $\phi(S) = 0$ and L is elementary.

Finally, let $L = \mathrm{so}(n,1)$ $(n > 3)$. We identify L with

$$\left\{ \begin{pmatrix} B & u \\ u^T & 0 \end{pmatrix} : u \in \mathbb{R}^n, B \in M_{n\times n}(\mathbb{R}), B^T = -B \right\}$$

From the remarks at the beginning of this section it suffices to show that the parabolic subalgebras of L are ϕ-free. Now any such subalgebra is conjugate to a standard parabolic subalgebra P with Langlands decomposition $P = (M \oplus A)\dot{+}N$, where $M \oplus A$ is reductive and N is an ideal of P contained in

$$\left\{ \begin{pmatrix} 0 & u & u \\ -u^T & 0 & 0 \\ u^T & 0 & 0 \end{pmatrix} : u \in \mathbb{R}^{n-1} \right\}.$$

Clearly N is abelian and every element of A acts semisimply on N, so P is ϕ-free.

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