# C-SECTIONS OF LIE ALGEBRAS

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**Abstract**

Let $M$ be a maximal subalgebra of a Lie algebra $L$ and $A/B$ a chief factor of $L$ such that $B \subseteq M$ and $A \not\subseteq M$. We call the factor algebra $M \cap A/B$ a $c$-section of $M$. All such $c$-sections are isomorphic, and this concept is related those of $c$-ideals and ideal index previously introduced by the author. Properties of $c$-sections are studied and some new characterizations of solvable Lie algebras are obtained.
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## 1 Preliminary results

Throughout $L$ will denote a finite-dimensional Lie algebra over a field $F$. We denote algebra direct sums by '$\oplus$', whereas vector space direct sums will be denoted by '$\dot{+}$'. If $B$ is a subalgebra of $L$ we define $B_L$, the *core* (with respect to $L$) of $B$ to be the largest ideal of $L$ contained in $B$. In [9] we defined a subalgebra $B$ of $L$ to be a *c-ideal* of $L$ if there is an ideal $C$ of $L$ such that $L = B + C$ and $B \cap C \subseteq B_L$.

Let $M$ be a maximal subalgebra of $L$. We say that a chief factor $C/D$ of $L$ *supplements* $M$ in $L$ if $L = C + M$ and $B \subseteq C \cap M$; if $B = C \cap M$ we say that $C/D$ *complements* $M$ in $L$. In [10] we defined the *ideal index* of a maximal subalgebra $M$ of $L$, denoted by $\eta(L : M)$, to be the well-defined dimension of a chief factor $C/D$ where $C$ is an ideal minimal with respect to supplementing $M$ in $L$. Here we introduce a further concept which is related to the previous two.

Let $M$ be a maximal subalgebra of $L$ and let $C/D$ be a chief factor of $L$ with $D \subseteq M$ and $L = M + C$. Then $(M \cap C)/D$ is called a *c-section* of $M$ in $L$. The analogous concept for groups was introduced by Wang and Shirong in [13] and studied further by Li and Shi in [3].

We say that $L$ is *primitive* if it has a maximal subalgebra $M$ with $M_L = 0$. First we show that all c-sections of $M$ are isomorphic.

**Lemma 1.1** *For every maximal subalgebra $M$ of $L$ there is a unique c-section up to isomorphism.*

**Proof.** Clearly c-sections exist. Let $(M \cap C)/D$ be a c-section of $M$ in $L$, where $C/D$ is a chief factor of $L$, $D \subseteq M$ and $L = M + C$. First we show that this c-section is isomorphic to one in which $D = M_L$. Clearly $D \subseteq M_L \cap C \subseteq C$, so either $M_L \cap C = C$ or $M_L \cap C = D$. If the former holds, then $C \subseteq M_L$, giving $L = M$, a contradiction. In the latter case put $E = C + M_L$. Then $E/M_L \cong C/D$ is a chief factor and $(M \cap E)/M_L$ is a c-section. Moreover,

$$\frac{M \cap E}{M_L} = \frac{M_L + M \cap C}{M_L} \cong \frac{M \cap C}{M_L \cap C} = \frac{M \cap C}{D}.$$

So suppose that $(M \cap C_1)/M_L$ and $(M \cap C_2)/M_L$ are two c-sections, where $C_1/M_L$, $C_2/M_L$ are chief factors and $L = M + C_1 = M + C_2$. Then $L/M_L$ is primitive and so either $C_1 = C_2$ or else $C_1/M_L \cong C_2/M_L$ and $C_1 \cap M = M_L = C_2 \cap M$, by [12, Theorem 1.1]. In the latter case both c-sections are trivial. $\square$

Given a Lie algebra $L$ with a maximal subalgebra $M$ we define $Sec(M)$ to be the Lie algebra which is isomorphic to any c-section of $M$; we call the natural number $\eta^*(L : M) = \dim Sec(M)$ the *c-index* of $M$ in $L$.

The relationship between c-ideals and c-sections, and between ideal index and c-index, for a maximal subalgebra $M$ of $L$ is given by the following lemma.

**Lemma 1.2** *Let $M$ be a maximal subalgebra of a Lie algebra $L$. Then* *(i) $M$ is a c-ideal of $L$ if and only if $Sec(M) = 0$; and*

*(ii) $\eta^*(L : M) = \eta(L : M) - \dim(L/M)$.*

**Proof.**

(i) Suppose first that $M$ is a c-ideal of $L$. Then there is an ideal $C$ of $L$ such that $L = M + C$ and $M \cap C \subseteq M_L$. Then $M \cap C = M_L \cap C$ is an ideal of $L$. Let $K$ be an ideal of $L$ with $M \cap C \subset K \subseteq C$. Then $K \not\subseteq M$, so $L = M + K$ and $M \cap C = M \cap K$. This yields that $\dim L = \dim M + \dim K - \dim(M \cap K) = \dim M + \dim C - \dim(M \cap C)$, so $K = C$ and $C/(M \cap C)$ is a chief factor of $L$. It follows that $Sec(M) = 0$.

The converse is clear.

(ii) Let $C/D$ be a chief factor such that $L = M + C$ and $C$ is minimal in the set of ideals supplementing $M$ in $L$. Then $\eta(L : M) = \dim(C/D)$, by the definition of ideal index. Thus,

$$\begin{aligned}\eta(L : M) &= \dim(C/D) = \dim C - \dim D \\ &= \dim C - \dim C \cap M + \dim C \cap M - \dim D \\ &= \dim L - \dim M + \dim(C \cap M/D) \\ &= \dim(L/M) + \eta^*(L : M).\end{aligned}$$

□

**Lemma 1.3** *Let $A/B$ be an abelian chief factor of $L$. Then any maximal subalgebra of $L$ that supplements $A/B$ must complement $A/B$.*

**Proof.** Let $M$ supplement $A/B$, so $L = A + M$ and $B \subseteq M$. Then $[L, M \cap A] = [A + M, M \cap A] \subseteq B + M \cap A = M \cap A$. So $M \cap A$ is an ideal of $L$ and $M \cap A = B$. □

The following lemma will also be useful.

**Lemma 1.4** *Let $B \subseteq M \subseteq L$, where $M$ is maximal in $L$ and $B$ is an ideal of $L$. Then $Sec(M) \cong Sec(M/B)$.*

**Proof.** Clearly $M/B$ is a maximal subalgebra of $L/B$. Let $(C/B)/(D/B)$ be a chief factor of $L/B$ such that $D/B \subseteq M/B$ and $C/B + M/B = L/B$.

Then $C/D$ is a chief factor of $L$ such that $L = C + M$ and $D \subseteq M$. Hence $Sec(M) \cong C \cap M/D \cong Sec(M/B)$. $\square$

In [12] it was shown that a primitive Lie algebra can be one of three types: it is said to be

1. *primitive of type* 1 if it has a unique minimal ideal that is abelian;
2. *primitive of type* 2 if it has a unique minimal ideal that is non-abelian; and
3. *primitive of type* 3 if it has precisely two distinct minimal ideals each of which is non-abelian.

If $M$ is a maximal subalgebra of $L$, then $L/M_L$ is clearly primitive; we say that $M$ is of type $i$ if $L/M_L$ is primitive of type $i$ for $i = 1, 2, 3$. Then we have the following result.

**Lemma 1.5** *Let $L$ be a Lie algebra over a field $F$ and let $M$ be a maximal subalgebra of $L$.*

*(i) If $M$ is of type 1 or 3 then $Sec(M) = 0$.*

*(ii) If $F$ has characteristic zero and $M$ is of type 2 then $Sec(M) \cong M/M_L$.*

**Proof.**

(i) This follows from [12, Theorem 1.1 3(a),(c)].

(ii) Let $A/B$ be a nonabelian chief factor that is supplemented by $M$, so $L = A + M$ and $B = A \cap M_L$. Then $L/M_L$ is simple, by [12, Theorem 1.7 2], which implies that $L = A + M_L$. Hence

$$\frac{M}{M_L} = \frac{M \cap (A + M_L)}{M_L} = \frac{M \cap A + M_L}{M_L} \cong \frac{M \cap A}{M_L \cap A} = \frac{M \cap A}{B} = Sec(M).$$

$\square$

## 2 Main results

First we can state Theorems 3.1, 3.2 and 3.3 of [9] in terms of c-sections as follows.

**Theorem 2.1** *Let $L$ be a Lie algebra over a field $F$. Then* *(i) every maximal subalgebra $M$ of $L$ has trivial c-section if and only if $L$ is solvable; and*

*(ii) if $F$ has characteristic zero, or is algebraically closed of characteristic greater than 5, then $L$ has a maximal subalgebra with trivial c-section if and only if $L$ is solvable.*

**Theorem 2.2** *Let $L$ be a Lie algebra over a field $F$ of characteristic zero. Then $Sec(M)$ is solvable for all maximal subalgebras $M$ of $L$ if and only if $L = R\dot{+}S$, where $R$ is the (solvable) radical of $L$ and $S$ is a direct sum of simple algebras which are minimal non-abelian or isomorphic to $sl_2(F)$.*

**Proof.** Suppose first that $Sec(M)$ is solvable for all maximal subalgebras $M$ of $L$, and let $L = R\dot{+}S$ be the Levi decomposition of $L$. Then $Sec(M)$ is solvable for all maximal subalgebras $M$ of $S$, by Lemma 1.4. Let $S = S_1 \oplus \ldots \oplus S_n$, where $S_i$ is simple for each $1 \leq i \leq n$. If $M$ contains all $S_i$ apart from $S_j$, then $Sec(M) \cong M \cap S_j$, so every subalgebra of $S_j$ is solvable. It follows from [8, Theorem 2.2 and the remarks following it] that $S_j$ is minimal non-abelian or isomorphic to $sl_2(F)$ for each $1 \leq j \leq n$.

Suppose conversely that $L$ has the claimed form and let $M$ be a maximal subalgebra of $L$. Every chief factor of $L$ is either abelian or simple, and so every c-section of $M$ is either abelian or isomorphic to a proper subalgebra of one of the simple components of $S$. In either case $Sec(M)$ is solvable. □

**Corollary 2.3** *Let $L$ be a Lie algebra over a field $F$ and suppose that every maximal subalgebra has c-index $k$. Then*

*(i) if $k > 0$, $L$ must be semisimple.*

*Suppose further that $F$ has characteristic zero. Then*

*(ii) every simple ideal of its Levi factor must have all of its maximal subalgebras of dimension $k$;*

*(iii) $k = 0$ if and only if $L$ is solvable;*

*(iv) $k = 1$ if and only if $L$ is a direct sum of non-isomorphic three-dimensional simple ideals and $\sqrt{F} \not\subseteq F$; and*

*(v) $k = 2$ if and only if $L$ is a direct sum of non-isomorphic ideals each of which is a minimal non-abelian simple Lie algebra with all maximal subalgebras of dimension 2.*

**Proof.**

(i) If $L$ has non-trivial radical, it has an abelian chief factor which is supplemented, and hence complemented, by Lemma 1.3, so $k = 0$.

(ii) This is clear.

(iii) This is Theorem 2.1 (i).

(iv) Suppose that $k = 1$. Then $L$ is semisimple and each simple component has all of its maximal subalgebras one dimensional, by (i) and (ii). It follows that they are three-dimensional simple and $\sqrt{F} \not\subseteq F$, by [11, Theorem 3.4]. If there are two that are isomorphic, say $S$ and $\theta(S)$, where $\theta$ is an isomorphism, then the diagonal subalgebra $\{s + \theta(s) : s \in S\}$ is maximal in $S \oplus \theta(S)$. But this together with the simple components other than $S$ and $\theta(S)$ gives a maximal subalgebra $M$ of $L$ with $c$-index 0 in $L$.

Conversely, suppose that $L$ is a direct sum of non-isomorphic three-dimensional simple ideals, $S_1 \oplus \ldots \oplus S_n$, and $\sqrt{F} \not\subseteq F$. Let $M$ be a maximal subalgebra of $L$ with $S_i \not\subseteq M$ and $S_j \not\subseteq M$ for some $1 \leq i, j \leq n$ with $i \neq j$. Then $L = M + S_i = M + S_j$ which yields that $M \cap S_i$ and $M \cap S_j$ are ideals of $L$ and hence are trivial. But then $S_i \cong L/M \cong S_j$, a contradiction. It follows that every maximal subalgebra contains all but one of the simple components and hence that $k = 1$.

(v) This is similar to (iv), noting that there are no three-dimensional simple Lie algebras with all maximal subalgebras two dimensional.

□

Note that algebras as described in Corollary 2.3 do exist as the following example shows. This example was constructed by Gejn (see [2, Example 3.5]).

EXAMPLE **2.1** *Let $L$ be the Lie algebra generated by the matrices*

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E \\ 0 & E & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & -A & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -A \\ 0 & A & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 0 & 2E \\ 0 & 0 & 0 \\ -A & 0 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & -2E & 0 \\ A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

*where* $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *with repect to the operation* $[,]$, *over the rational numbers* $\mathbb{Q}$. *Then* $L$ *is simple nonabelian (see [2, Example 3.5]), and the maximal subalgebras are* $\mathbb{Q}f_i + \mathbb{Q}g_i$ *for* $i = 1, 2, 3$.

EXAMPLE **2.2** *Gejn also goes on to construct simple minimal nonabelian Lie algebras over* $\mathbb{Q}$ *of dimension* $3k$ *for* $k \geq 1$ *by putting*

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 2 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

$E$ *as the* $k \times k$ *identity matrix and* $0$ *as the* $k \times k$ *zero matrix (see [2, Example 3.6]). It is straightforward to check that in these every maximal subalgebra has c-index* $k$.

The following corollary is straightforward.

**Corollary 2.4** *Let* $L = R \dot{+} S$ *be a Lie algebra over a field* $F$ *of characteristic zero, where* $R$ *is the radical and* $S$ *is a Levi factor, and suppose that* $L$ *has a maximal subalgebra with c-index* $k$. *Then*

*(i) if* $k > 0$ *then* $S \neq 0$;

*(ii)* $k = 1$ *if and only if* $S$ *has a minimal ideal which is minimal non-abelian or isomorphic to* $sl_2(F)$;

*(iii)* $k > 1$ *if and only if* $S$ *has a minimal ideal with a maximal subalgebra of dimension* $k$.

Let $(L_p, [p], \iota)$ be any finite-dimensional $p$-envelope of $L$. If $S$ is a subalgebra of $L$ we denote by $S_p$ the restricted subalgebra of $L_p$ generated by $\iota(S)$. Then the *(absolute) toral rank* of $S$ in $L$, $TR(S, L)$, is defined by

$$TR(S, L) = \max\{\dim(T) : T \text{ is a torus of } (S_p + Z(L_p))/Z(L_p)\}.$$

This definition is independent of the $p$-envelope chosen (see [7]). We write $TR(L, L) = TR(L)$. A Lie algebra $L$ is *monolithic* if it has a unique minimal ideal (the *monolith* of $L$). The *Frattini ideal*, $\phi(L)$, is the largest ideal contained in every maximal subalgebra of $L$. We put $L^{(0)} = L$, $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$ for $n \in \mathbb{N}$ and $L^{(\infty)} = \cap_{n=0}^{\infty} L^{(n)}$.

**Theorem 2.5** *Let $L$ be a Lie algebra over an algebraically closed field $F$ of characteristic $p > 0$. Then $Sec(M)$ is nilpotent for every maximal subalgebra $M$ of $L$ if and only $L$ is solvable.*

**Proof.** Let $L$ be a minimal non-solvable Lie algebra such that $Sec(M)$ is nilpotent for every maximal subalgebra $M$ of $L$, and let $R$ be the (solvable) radical of $L$. If $L$ is simple then every maximal subalgebra of $L$ is nilpotent, and no such Lie algebra exists over an algebraically closed field. So $L$ has a minimal ideal $A$, and $L/A$ is solvable. If there are two distinct minimal ideals $A_1$ and $A_2$, then $L/A_1$ and $L/A_2$ are solvable, whence $L \cong L/(A_1 \cap A_2)$ is solvable, a contradiction. Hence $L$ is monolithic with monolith $A$. If $A \subseteq R$ then again $L$ would be solvable, so $L$ is semisimple and $\phi(L) = 0$. Thus, there is a maximal subalgebra $M$ of $L$ such that $L = M + A$.

Put $C = M \cap A$ which is an ideal of $M$. If $\mathrm{ad}\, a$ is nilpotent for all $a \in A$ then $L$ is solvable, a contradiction. Hence there exists $a \in A$ such that $\mathrm{ad}\, a$ is not nilpotent. Let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of $L$ relative to $\mathrm{ad}\, a$. Then $L_0 \neq L$ and $L_1 \subseteq A$, so that if $P$ is a maximal subalgebra containing $L_0$, we have $L = A + P$ and $a \in A \cap P$. We can, therefore, assume that $C \neq 0$.

Then $C$ is nilpotent and $L/A \cong M/C$ is solvable, whence $M$ is solvable. Now $[M, N_A(C)] \subseteq N_A(C)$, so $M + N_A(C)$ is a subalgebra of $L$. But $L = M + N_A(C)$ implies that $C$ is an ideal of $L$, from which $C = A$ and $L$ is solvable, a contradiction. It follows that $M = M + N_A(C)$, and so $N_A(C) = M \cap A = C$, and $C$ is a Cartan subalgebra of $A$. Now $C_p$ is a Cartan subalgebra of $A_p$, by [14, Lemma], and so there is a maximal torus $T \subseteq A_p$ such that $C_p = C_{L_p}(T)$ (see [5]).

Let $A_0(T) + \sum_{i \in \mathbb{Z}_p} A_{i\alpha}$ be a 1-section with respect to $T$. Then every element of $C$ acts nilpotently on $L_0$, the Fitting null-component relative to $T$, and thus so does every element of $C_p$. It follows that $L = L_0 + \sum_{i \in \mathbb{Z}_p} A_{i\alpha}$ so $L^{(\infty)} = A$ is simple with $TR(A) = 1$. We therefore have that

$$p \neq 2, \quad A \in \{sl_2(F), W(1:\underline{1}), H(2:\underline{1})^{(1)}\} \text{ if } p > 3$$

$$\text{and} \quad A \in \{sl_2(F), psl_3(F)\} \text{ if } p = 3,$$

by [4] and [6]. But now, $\dim A_\alpha = 1$ (by [1, Corollary 3.8] for all but $psl_3(F)$, and this is straightforward to check) and $M = L_0 \subset L_0 + A_\alpha \subset L$, a contradiction. It follows that $L$ is solvable.

The converse is clear. □

A subalgebra $U$ of $L$ is *nil* if $\mathrm{ad}\, u$ acts nilpotently on $L$ for all $u \in U$. Notice that we cannot replace 'nilpotent' in Theorem 2.5 by 'solvable' or 'supersolvable' and draw the same conclusion, as $sl_2(F)$ is a counterexample. However, we can prove the same result with 'nilpotent' replaced by the stronger condition 'nil' without any restrictions on the field $F$.

**Theorem 2.6** *Let $L$ be a Lie algebra over any field $F$. Then $Sec(M)$ is nil for every maximal subalgebra $M$ of $L$ if and only if $L$ is solvable.*

**Proof.** Let $L$ be a minimal non-solvable Lie algebra such that $Sec(M)$ is nil for every maximal subalgebra $M$ of $L$. If $L$ is simple then every maximal subalgebra of $L$ is nil. It follows that every element of $L$ is nil and $L$ is nilpotent, by Engel's Theorem. Hence no such Lie algebra exists. So, arguing as in paragraphs 1 and 2 of Theorem 2.5 above, $L$ is monolithic with monolith $A$, $L/A$ is solvable, and there is a maximal subalgebra $M$ of $L$ such that $L = M + A$ with an element $a \in M \cap A$ such that $\text{ad}\, a$ is not nilpotent. But this is a contradiction, since $A \cap M = Sec(M)$ is nil.

Once again, the converse is clear. □

Let $(L, [p])$ be a restricted Lie algebra. Recall that an element $x \in L$ is called *$p$-nilpotent* if there exists an $n \in \mathbb{N}$ such that $x^{[p]^n} = 0$. Then we have the following immediate corollary.

**Corollary 2.7** *Let $L$ be a restricted Lie algebra over a field $F$ of characteristic $p > 0$. Then $Sec(M)$ is $p$-nilpotent for every maximal subalgebra $M$ of $L$ if and only if $L$ is solvable.*

**Proof.** Simply note that that a $p$-nilpotent subalgebra is nil. □

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