ON POWER ASSOCIATIVE BERNSTEIN ALGEBRAS OF ARBITRARY ORDER

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1 Introduction

A finite-dimensional commutative nonassociative algebra A over a field K is called *baric* if there is a non-trivial homomorphism $\omega : A \to K$. Such an algebra is called a *Bernstein algebra* if

$$x^2x^2 = \omega(x)^2x^2 \text{ for all } x \in A.$$

These algebras were first defined by Holgate [7], motivated by the problem in genetics, posed by S. Bernstein [5], of the conditions required to ensure that a population attains equilibrium after one generation. They have since been studied by a number of authors (see [3], [9], [14], [15], [16], [17], for instance).

A generalisation of Bernstein algebras, describing the situation in which a population is in equilibrium after $k+1$ generations of random mating, was proposed by Abraham [1]. A study of such algebras has been started in [6] and [11]. Here we develop the theory further by focusing on those algebras in this class which are also power associative; these are a little easier to handle and form a natural generalisation of the important class of Jordan Bernstein algebras.

2 Preliminaries

If $x_1, \ldots, x_n$ belong to the algebra A we shall denote by $((x_1, \ldots, x_n))$ the subspace spanned by $x_1, \ldots, x_n$. The symbol $\oplus$ will denote an algebra direct sum, whereas $\dot{+}$ will indicate a direct sum of the vector space structure alone. All algebras will be finite dimensional over a field K.

Let x be an element of the commutative algebra A, and let B be a subspace of A. Then the *principal powers* of x are defined inductively by

$$x^1 = x, x^{r+1} = x^r x \text{ for all } r \in \mathbb{N};$$

the *principal powers* of B are given by $B^1 = B, B^{r+1} = B^r B$ for all $r \in \mathbb{N}$.

Similarly, the *plenary powers* of x and B are defined by

$$x^{[1]} = x, \quad x^{[r+1]} = x^{[r]} x^{[r]} \quad \text{for all } r \in \mathbb{N},$$

$$B^{[1]} = B, \quad B^{[r+1]} = B^{[r]}B^{[r]} \quad \text{for all } r \in \mathbb{N}.$$

Following Abraham [1] and Wörz-Busekros [15] we call the baric algebra A a *k-th order Bernstein algebra* if

$$x^{[k+2]} = \omega(x)^{2^k} x^{[k+1]} \text{ for all } x \in A.$$

Clearly the algebras previously referred to as Bernstein algebras are precisely the first order Bernstein algebras.

It is straightforward to check that the weight homomorphism of a k-th order Bernstein algebra A is unique, and that the set of idempotents of A is precisely the set $A = \{x^{[k+1]} : x \in A \text{ with } \omega(x) = 1\}$. Proofs can be modelled on those for the corresponding results for first order Bernstein algebras.

For the rest of this section A will denote a k-th order Bernstein algebra over a field K of characteristic different from two. We shall also assume that K contains at least 2^{k+1} distinct elements. Put $N =$ Ker ω and let $e \in A$ be an idempotent. The map

$$L_e{}^k : A \to A : x \mapsto e(\ldots(e(ex))\ldots)$$

(where there are k e's in the product) is a linear transformation, and induces the map $L_e{}^k \mid_N : N \to N$. Then

$$A = E \dot{+} U \dot{+} Z$$

where E is spanned by the idempotent $e, U =$ Im $(L_e{}^k \mid_N)$, $Z =$ Ker $(L_e{}^k)$ (see [6]). We shall refer to this as the *Pierce decomposition* of A (relative to e): it clearly depends on the choice of idempotent.

Lemma 2.1 *Let A be a k-th order Bernstein algebra over the field K, and let $A = E \dot{+} U \dot{+} Z$ be its Peirce decomposition relative to the idempotent e. Then* *(i)* $eu = \frac{1}{2}u$ *for all* $u \in U$*, and*

(ii) $U^2 \subseteq Z$*.*

Proof: See the Proposition in [6]. □

Theorem 2.2 *Let* A *be as in Lemma 2.1.*

(i) If ${L_e}^n(z) \neq 0$ *for some* $z \in Z$*, then* Z *has dimension at least* $n+1$*. In particular, if* Z *has dimension* 1 *then* $eZ = 0$*.*

(ii) If $Z = 0$*. Then* A *is a zero order Bernstein algebra, and hence as described in Theorem 9.10 of [15].*

Proof: (i) By elementary linear algebra, the elements $z, L_e(z), ..., {L_e}^n(z)$ are linearly independent.

(ii) We have $U^2 \subseteq Z = 0$. Let $x = \alpha e + u$. Then $x^2 = \alpha^2 e + \alpha u = \alpha x = \omega(x)x$. □

We shall call an algebra B *power solvable* if there is an $n \in \mathbb{N}$ such that $x^{[n]} = 0$ for all $x \in B$; if $x^{[n]} = 0$ for every $x \in B$, but there is an element $y \in B$ for which $y^{[n-1]} \neq 0$ we say that B has *index* n. Finally in this section we have the following result.

Theorem 2.3 *Let* $B = E \dot{+} Z$ *be a baric algebra with weight homomorphism* ω *in which* E *is spanned by the idempotent* e *,* Z *is a subalgebra of* A *contained in the kernel of* ω *and* $eZ = 0$*. Then* B *is a* k*-th order Bernstein algebra if and only if* Z *is power solvable of index* $\leq k+1$*.*

Proof: Let $x = \alpha e + z$, where $\alpha \in K, z \in Z$. Then $x^{[n]} = \alpha^{2^n} e + z^{[n]}$ and $\omega(x) = \alpha$, so that $x^{[k+2]} = \omega(x)^{2^k} x^{[k+1]}$ precisely when $z^{[k+1]} = 0$. □

3 Power associative algebras

The development of an extensive theory for the complete class of k-th order Bernstein algebras appears to be difficult. It has been observed ([7], [14], [15], [16]) that some important classes of first order Bernstein algebras are also Jordan algebras. In the hope of making more rapid progress it therefore seems natural to restrict attention initially to those k-th order Bernstein algebras which are power associative, and so that we now do.

Throughout this section A will denote a power associative k-th order Bernstein algebra over the field K (bearing the same restrictions as in the previous section), and we will write $A = E \dot{+} U \dot{+} Z$, as in section 2, relative to the idempotent e.

Since A is a commutative power associative algebra, it has a Peirce decomposition

$$A = A_0 \dot{+} A_1 \dot{+} A_{\frac{1}{2}}$$

where $A_i = \{a \in A : ea = ia\}$ (see [12], p.131). The following justifies our calling the previous decomposition a Peirce decomposition.

Lemma 3.1 *With the above notation $A_0 = Z, A_1 = E$ and $A_{\frac{1}{2}} = U$.*

Proof: It is clear that $E \subseteq A_1$ and $U \subseteq A_{\frac{1}{2}}$. Suppose that $x = \alpha e + u + z \in A_1$ (where $u \in U, z \in Z$). Then $ex = \alpha e + \frac{1}{2}u + ez = x$; in fact,

$$L_e{}^k(x) = \alpha e + (1/2^k)u = x = \alpha e + u + z.$$

Thus $z = ((1/2^k) - 1)u \in U \cap Z = 0$, which implies that $A_1 = E$.

Similarly, if $x = \alpha e + u + z \in A_{\frac{1}{2}}$, then

$$L_e{}^k(x) = \alpha e + (1/2^k)u = (1/2^k)x = (1/2^k)(\alpha e + u + z).$$

Hence $((1/2^k) - 1)\alpha e = (1/2^k)z \in E \cap Z = 0$; that is, $A_{\frac{1}{2}} = U$.

Finally, if $x = \alpha e + u + z \in A_0$, then $0 = L_e{}^k(x) = \alpha e + (1/2^k)u$; hence $x \in Z$. Moreover, if $z = a_0 + a_1 + a_{\frac{1}{2}} \in Z$ (where $a_i \in Ai$) then $0 = L_e{}^k(z) = a_1 + (1/2^k)a_{\frac{1}{2}}$ and $z = a_0 \in A_0$. Then $A_0 = Z$. □

Corollary 3.2 *Let $A = E \dot{+} U \dot{+} Z$ be a power associative k-th order Bernstein algebra. Then*

(i) $U^2 \subseteq Z$;

(ii) $eZ = 0$;

(iii) $Z^2 \subseteq Z$; *and*

(iv) $ZU \subseteq U$.

Proof: (i) This was proved in Lemma 2.1.

(ii),(iii),(iv) These follow quickly from the properties of the Peirce decomposition of a commutative power associative algebra (for which see [12], p. 131, or [2]). □

Next we seek necessary and sufficient conditions for a k-th order Bernstein algebra to be power associative. Such conditions are given in the following result.

Theorem 3.3 *Let $A = E \dot{+} U \dot{+} Z$ be a k-th order Bernstein algebra over a field K of characteristic different from two (and containing at least 2^{k+1} elements). Then for A to be power associative it is necessary that the following identities hold for all $u \in U, z \in Z$. If the characteristic of K is prime to thirty then these identities are also sufficient.*

(3.3.1) properties (i), (ii), (iii), (iv) of Corollary 3.2 hold;

(3.3.2) $u^3 = 0$;

(3.3.3) $2u(uz) = u^2 z$;

(3.3.4) $2z(uz) = uz^2$;

(3.3.5) $u^2(uz) = u(u^2 z) = 0$;

(3.3.6) $2z^2(uz) = 2z(uz^2) = uz^3$;

(3.3.7) $2(uz)^2 + u(uz^2) = z(u^2 z)$;

(3.3.8) $z^4 = z^2 z^2$

Proof: First assume that A is power associative; then (3.3.1) has already been proved. Let $x = e + \lambda u + \mu z$, where $\lambda, \mu \in K, u \in U, z \in Z$. Then

$$
\begin{aligned}
x^2 &= e + \lambda^2 u^2 + \mu^2 z^2 + \lambda u + 2\lambda\mu uz, \\
x^{[3]} &= e + \lambda^4 u^{[3]} + \mu^4 z^{[3]} + \lambda^2 u^2 + 4\lambda^2\mu^2 (uz)^2 + \lambda u \\
&\quad + 2\lambda\mu uz + 2\lambda^2\mu^2 u^2 z^2 + 2\lambda^3 u^3 + 4\lambda^3 \mu u^2 (uz) \\
&\quad + 2\lambda\mu^2 uz^2 + 4\lambda\mu^3 z^2 (uz) + 4\lambda^2 \mu u(uz),
\end{aligned}
$$

$$\begin{aligned} x^3 &= e + \lambda u + \lambda\mu uz + \lambda^3 u^3 + \lambda\mu^2 uz^2 + \lambda^2 u^2 + 2\lambda^2 \mu u(uz) \\ &\quad + \lambda^2 \mu u^2 z + \mu^3 z^3 + \lambda\mu uz + 2\lambda\mu^2 z(uz) \end{aligned}$$

$$\begin{aligned} x^4 &= e + \lambda u + 2\lambda\mu uz + \frac{3}{2}\lambda^3 u^3 + \frac{1}{2}\lambda\mu^2 uz^2 + 3\lambda\mu^2 z(uz) \\ &\quad + \lambda^2 u^2 + 2\lambda^2 \mu u(uz) + \lambda^4 u^4 + \lambda^2\mu^2 u(uz^2) + 2\lambda^3 \mu u(u(uz)) \\ &\quad + \lambda^3 \mu u(u^2 z) + \lambda\mu^3 uz^3 + 2\lambda^2\mu^2 u(z(uz)) + \lambda^3 \mu u^3 z \\ &\quad + \lambda\mu^3 z(uz^2) + \lambda^2 \mu u^2 z + 2\lambda^2\mu^2 z(u(uz)) + \lambda^2\mu^2 z(u^2 z) \\ &\quad + \mu^4 z^4 + 2\lambda\mu^3 z(z(uz)). \end{aligned}$$

Since A is power associative we have $x^{[3]} = x^4$. Equating coefficients of λ^3 gives $2u^3 = \frac{3}{2}u^3$, whence $u^3 = 0$ and we have (3.3.2).

Equating coefficients of $\lambda^2\mu$ and of $\lambda\mu^2$ gives (3.3.3) and (3.3.4) respectively. Equating coefficients of $\lambda^3\mu$ we see that $4u^2(uz) = 2u(u(uz)) + u(u^2 z)$. But $2u(u(uz)) = u(u^2 z)$ from (3.3.3), so that

$$2u^2(uz) = u(u^2 z) \tag{1}$$

Furthermore, replacing z by $u^2 + z$ in (3.3.4) yields

$$2u^2(uz) + 2z(uz) = uz^2 + 2u(u^2 z),$$

whence

$$u^2(uz) = u(u^2 z). \tag{2}$$

Combining (1) and (2) gives (3.3.5).

Next equate coefficients of $\lambda\mu^3$ to see that $4z^2(uz) = uz^3 + z(uz^2) + 2z(z(uz))$. But $2z(z(uz)) = z(uz^2)$ from (3.3.4), so that

$$4z^2(uz) = uz^3 + 2z(uz^2). \tag{3}$$

However, since $uz \in U$ we can replace u by uz in (3.3.4) to deduce that

$$z^2(uz) = 2z(z(uz)) = z(uz^2). \tag{4}$$

Combining (3) and (4) gives (3.3.6).

Now equate coefficients of $\lambda^2\mu^2$ giving

$$4(uz)^2 + 2u^2z^2 = u(uz^2) + 2u(z(uz)) + 2z(u(uz)) + z(u^2z).$$

Using the fact that $2u(z(uz)) = u(uz^2)$ and that $2z(u(uz)) = z(u^2z)$ (deduced from (3.3.4) and (3.3.3) respectively), this reduces to

$$2(uz)^2 + u^2z^2 = u(uz^2) + z(u^2z). \tag{5}$$

But, since $z^2 \in Z$, replacing z by z^2 in (3.3.3) gives

$$2u(uz^2) = u^2z^2. \tag{6}$$

Substituting (6) into (5) produces (3.3.7).

Finally (3.3.8) is clear.

Conversely, suppose that (3.3.1) - (3.3.8) hold. If $x = e + \lambda u + \mu z$ it is easily checked that $x^2x^2 = x^4$: indeed it is clear from the way that these identities were derived that the coefficients of $\lambda^3, \lambda^2\mu, \lambda\mu^2, \lambda^3\mu, \lambda\mu^3$ and $\lambda^2\mu^2$ are the same on each side of the equation; the other coefficients can be compared quickly. It is immediate then that $x^2x^2 = x^4$ when $x = \alpha e + u + z$ and $\alpha \neq 0$. Furthermore, if we were to expand x^2x^2 and x^4 with $x = \lambda u + \mu z$ we would obtain the same terms as before, except that only the terms involving $\lambda^3\mu, \lambda\mu^3$ and $\lambda^2\mu^2$ would be present. It follows that $x^2x^2 = x^4$ all $x \in A$. But, with the extra restrictions on the characteristic, this suffices to ensure that A is power associative ([2], Theorem 1). □

Corollary 3.4 *Let $A = E \dot{+} U \dot{+} Z$ be a power associative k-th order Bernstein algebra over a field K of characteristic different from two (and containing at least 2^{k+1} elements). Then the following identities hold for all $u, u_1, u_2, u_3 \in U, z, z_1, z_2, z_3 \in Z$.*

(3.4.1) $u_1(u_2u_3) + u_2(u_3u_1) + u_3(u_1u_2) = 0;$

(3.4.2) $(u_1u_2)z = u_1(u_2z) + u_2(u_1z);$

(3.4.3) $(z_1z_2)u = z_1(z_2u) + z_2(z_1u);$

(3.4.4) $(u_1u_2)(u_3z) + (u_2u_3)(u_1z) + (u_3u_1)(u_2z) = 0;$

(3.4.5) $u_1((u_2u_3)z) + u_2((u_3u_1)z) + u_3((u_1u_2)z) = 0;$

(3.4.6) $$\begin{aligned}(z_1z_2)(z_3u) + (z_2z_3)(z_1u) + (z_3z_1)(z_2u) &= z_1((z_2z_3)u) + z_2((z_3z_1)u) + z_3((z_1z_2)u) \\ &= \tfrac{1}{2}u(z_1(z_2z_3) + z_2(z_3z_1) + z_3(z_1z_2));\end{aligned}$$

(3.4.7) $$\begin{aligned}&2(u_1z_1)(u_2z_2) + 2(u_1z_2)(u_2z_1) + u_2(u_1(z_1z_2)) + u_1(u_2(z_1z_2)) \\ &= z_1(z_2(u_1u_2)) + z_2(z_1(u_1u_2)).\end{aligned}$$

Proof: (3.4.1): This is the Jacobi identity, and is the well-known linearisation of (3.3.2).

(3.4.2): This is obtained by replacing u by $u_1 + u_2$ in (3.3.3).

(3.4.3): Similarly, replace z by $z_1 + z_2$ in (3.3.4).

(3.4.4), (3.4.5): Putting $u = \lambda u_1 + \mu u_3$ in (3.3.5) and comparing coefficients of $\lambda^2\mu$ gives

$$2(u_1 z)(u_1 u_3) + u_1^2(u_3 z) = 2u_1((u_1 u_3)z) + u_3(u_1^2 z) = 0.$$

Replacing u_1 by $u_1 + u_2$ now gives the identities claimed.

(3.4.6): Putting $z = \lambda z_1 + \mu z_3$ in (3.3.6) and comparing coefficients of $\lambda^2\mu$ gives

$$\begin{aligned} 4(z_1 u)(z_1 z_3) \quad + \quad & 2z_1^2(z_3 u) = 4z_1((z_1 z_3)u) + z_3(z_1^2 u) \\ = \quad & 2u(z_1(z_1 z_3)) + (z_1^2 z_3)u. \end{aligned}$$

Replacing z_1 by $z_1 + z_2$ now gives these identities.

(3.4.7): Putting $u = u_1 + u_2$ in (3.3.7) gives

$$4(u_1 z)(u_2 z) + u_1(u_2 z^2) + u_2(u_1 z^2) = 2z((u_1 u_2)z).$$

The desired identity is now obtained by replacing z by $z_1 + z_2$. □

Corollary 3.5 *Let A be as described in Corollary 2.4. Then A is a Jordan algebra if and only if Z is a Jordan algebra.*

Proof: Let $x = \alpha e + u_1 + z_1$, $y = \beta e + u_2 + z_2$. Then

$$\begin{aligned} x^2 &= \alpha^2 e + u_1^2 + z_1^2 + \alpha u_1 + 2u_1 z_1, \\ xy &= \alpha\beta e + \frac{1}{2}\alpha u_2 + \frac{1}{2}\beta u_1 + u_1 u_2 + u_1 z_2 + z_1 u_2 + z_1 z_2, \\ x^2 y &= \alpha^2\beta e + \frac{1}{2}\alpha^2 u_2 + u_1^2 u_2 + u_1^2 z_2 + z_1^2 u_2 + z_1^2 z_2 + \frac{1}{2}\alpha\beta u_1 \\ & \quad +\alpha u_1 u_2 + \alpha u_1 z_2 + \beta u_1 z_1 + 2u_2(u_1 z_1) + 2(u_1 z_1)z_2, \\ (x^2 y)x &= \alpha^3\beta e + \frac{3}{4}\alpha^2\beta u_1 + \frac{1}{4}\alpha^3 u_2 + \frac{1}{2}\alpha^2 u_1 u_2 + \frac{1}{2}\alpha^2 u_2 z_1 + \frac{1}{2}u_1^2 u_2 \end{aligned}$$

$$
\begin{aligned}
& +u_1(u_1^2u_2)+z_1(u_1^2u_2)+u_1(u_1^2z_2)+z_1(u_1^2z_2)+\frac{1}{2}\alpha z_1^2u_2 \\
& +u_1(u_2z_1^2)+z_1(u_2z_1^2)+u_1(z_1^2z_2)+z_1(z_1^2z_2)+\frac{1}{2}\alpha\beta u_1^2 \\
& +\alpha\beta u_1z_1+\alpha u_1(u_1u_2)+\alpha z_1(u_1u_2)+\frac{1}{2}\alpha^2u_1z_2+\alpha u_1(u_1z_2) \\
& +\alpha z_1(u_1z_2)+\beta u_1(u_1z_1)+\beta z_1(u_1z_1)+2u_1(u_2(u_1z_1)) \\
& +2z_1(u_2(u_1z_1))+\alpha(u_1z_1)z_2+2u_1(z_2(u_1z_1))+2z_1(z_2(u_1z_1)), \\
x^2(yx) \quad = \quad & \alpha^3\beta e+\frac{1}{4}\alpha^3u_2+\frac{3}{4}\alpha^2\beta u_1+\frac{1}{2}\alpha^2u_1z_2+\frac{1}{2}\alpha^2z_1u_2+\frac{1}{2}\alpha u_1^2u_2 \\
& +\frac{1}{2}\beta u_1^3+u_1^2(u_1u_2)+u_1^2(u_1z_2)+u_1^2(z_1u_2)+u_1^2(z_1z_2)+\frac{1}{2}\alpha u_2z_1^2 \\
& +\frac{1}{2}\beta u_1z_1^2+z_1^2(u_1u_2)+z_1^2(u_1z_2)+z_1^2(z_1u_2)+z_1^2(z_1z_2) \\
& +\frac{1}{2}\alpha^2u_1u_2+\frac{1}{2}\alpha\beta u_1^2+\alpha u_1(u_1u_2)+\alpha u_1(u_1z_2)+\alpha u_1(z_1u_2) \\
& +\alpha u_1(z_1z_2)+\alpha\beta u_1z_1+\alpha u_2(u_1z_1)+\beta u_1(u_1z_1)+2(u_1z_1)(u_1u_2) \\
& +2(u_1z_1)(u_1z_2)+2(u_1z_1)(z_1u_2)+2(u_1z_1)(z_1z_2).
\end{aligned}
$$

Now

$$
\begin{aligned}
u_1^3 &= 0 \text{ by (3.3.2);} \\
z_1(u_1z_1) &= \frac{1}{2}u_1z_1^2 \text{ by (3.3.4);} \\
u_1(u_1^2u_2) &= u_1^2(u_1u_2) \text{ by (3.4.2);} \\
z_1(u_1z_2) &+ (u_1z_1)z_2 = u_1(z_1z_2) \text{ by (3.4.3);} \\
z_1(u_1u_2) &= u_1(z_1u_2)+u_2(u_1z_1) \text{ by (3.4.2);} \\
u_1(u_1^2z_2) &= u_1^2(u_1z_2) \text{ by (3.4.3);} \\
z_1(u_2z_1^2) &= z_1^2(z_1u_2) \text{ by (3.3.6);} \\
u_1(z_1^2z_2) &+ 2z_1(z_2(u_1z_1)) = (u_1z_1^2)z_2+(u_1z_2)z_1^2+2z_1(z_2(u_1z_1)) \text{ by (3.4.3)} \\
&= 2z_2(z_1(u_1z_1))+(u_1z_2)z_1^2+2z_1(z_2(u_1z_1)) \text{ by (3.3.4)}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1^2(u_1z_2)+2(u_1z_1)(z_1z_2) \text{ by } (3.4.3);\\
u_1(u_2z_1^2) &+ 2z_1(u_2(u_1z_1)) = (u_1u_2)z_1^2 - u_2(u_1z_1^2)+2z_1(u_2(u_1z_1)) \text{ by } (3.4.2)\\
&= (u_1u_2)z_1^2 - 2u_2(z_1(u_1z_1))+2z_1(u_2(u_1z_1)) \text{ by } (3.3.4)\\
&= (u_1u_2)z_1^2 + 2(u_1z_1)(z_1u_2) \text{ by } (3.4.2);\\
z_1(u_1^2z_2) &+ 2u_1(z_2(u_1z_1)) = 2z_1(u_1(u_1z_2)) + 2u_1(z_2(u_1z_1)) \text{ by } (3.3.3)\\
&= 2(z_1u_1)(u_1z_2) + 2u_1(z_1(u_1z_2)) + 2u_1(z_2(u_1z_1)) \text{ by } (3.4.2)\\
&= 2(z_1u_1)(u_1z_2) + 2u_1(u_1(z_1z_2)) \text{ by } (3.4.3)\\
&= 2(z_1u_1)(u_1z_2) + u_1^2(z_1z_2)) \text{ by } (3.3.3);\\
z_1(u_1^2u_2) &+ 2u_1(u_2(u_1z_1)) = -2z_1(u_1(u_1u_2)) + 2u_1(u_2(u_1z_1)) \text{ by } (3.4.1)\\
&= -2u_1(z_1(u_1u_2)) + 2(u_1u_2)(u_1z_1) + 2u_1(u_2(u_1z_1)) \text{ by } (3.4.3)\\
&= -2u_1(u_1(z_1u_2)) + 2(u_1u_2)(u_1z_1) \text{ by } (3.4.2)\\
&= u_1^2(z_1u_2) + 2(u_1u_1)(u_1z_1) \text{ by } (3.4.1).
\end{aligned}
$$

It follows that $(x^2y)x = x^2(yx)$ precisely when $z_1(z_2z_1^2) = (z_1z_2)z_1^2$, which proves the result. □

Corollary 3.6 *Let A be as described in Corollary 3.4. If $z^3 = 0$ for all $z \in Z$ then A is a Jordan algebra.*

Proof: Since $z^3 = 0$ for all $z \in Z$, Z is a Jordan algebra ([4] , Lemma 1). But then A is a Jordan algebra by Corollary 3.5. □

Conditions under which a first order Bernstein algebra is Jordan have been studied elsewhere ([14] and [16]). In [14] it was shown that a first order Bernstein algebra is Jordan if and only if it is power associative. This is not the case for higher order Bernstein algebras, as the following example shows.

EXAMPLE 3.1 Let A be the six dimensional algebra with basis $e, z_1, z_2, z_3, z_4, z_5$ *and multiplication* $e^2 = e, ez_i = z_i e = 0$ *for* $1 \leq i \leq 5$, $z_1 z_2 = z_2 z_1 = z_3, z_1 z_3 = z_3 z_1 = z_4, z_1 z_5 = z_5 z_1 = -z_3, z_2 z_3 = z_3 z_2 = z_5, z_2 z_4 = z_4 z_2 = z_3$, *all other products being zero. Then* $A = E \oplus Z$ *where* E *is spanned by* e, Z *is spanned by* z_1, z_2, z_3, z_4, z_5, *and* Z *is Suttle's example of a commutative power associative nilalgebra which is not nilpotent (see [13], pp. 50, 51). It is easy to check that* A *is a second order Bernstein algebra which is power associative, but it is not Jordan because* Z *is not nilpotent.*

4 The core

As we noted in section 2 the Peirce decomposition of A generally depends on the choice of idempotent. In this section we seek to clarify the relationship between two such decompositions and to develop the concept of the core of A, first introduced for first order Bernstein algebras by Holgate [7]. As a preliminary we shall give an alternative description of the idempotents, analogous to Lemma 9.8 of [15].

Throughout this section A will denote a power associative k-th order Bernstein algebra over a field K of characteristic different from two (and in which K contains at least 2^{k+1} elements), $e \in A$ will be an idempotent of A and $A = E \dot{+} U \dot{+} Z$ will be its Peirce decomposition relative to e. As before we will put $N = \operatorname{Ker} \omega = U \dot{+} Z$.

Lemma 4.1 *The set of idempotents* I *of* A *is given by*

$$I = \{e + u + u^2 : u \in U\}.$$

Also, $U = \{en : n \in N\}$, $Z = \{n \in N : en = 0\}$

Proof: Let f be an idempotent of A. Then $\omega(f) = 1$, and so $f = e + u + z$, where $u \in U, z \in Z$. Now

$$e + u^2 + z^2 + u + 2uz = f^2 = f = e + u + z,$$

so $uz = 0$ and $u^2 + z^2 = z$ (since $UZ \subseteq U$ and $U^2, Z^2 \subseteq Z$, by Corollary 3.2). Hence

$$\begin{aligned} z^2 = (u^2 + z^2)^2 &= z^4 + 2u^2z^2 \text{ since } u^4 = 0 \text{ by (3.3.2)} \\ &= z^4 + 2(z - z^2)z^2 = 2z^3 - z^4, \end{aligned}$$

and z^2, z^3, z^4 are linearly dependent. But z is nilpotent, so $z^2 = 0$ and $f = e + u + u^2$. Furthermore, $e + u + u^2$ is idempotent for every $u \in U$.

Finally, if $n = u + z$ $(u \in U, z \in Z)$, then $en = \frac{1}{2}u$, by Lemma 2.1 and Corollary 3.2, whence $U = \{en : n \in N\}, Z = \{n \in N : en = 0\}$. □

As in [15], a consequence of the above result is that the Peirce decomposition cannot be further refined. This is implied by the following corollary.

Corollary 4.2 *All idempotents of A are principal and primitive.*

Proof: This is immediate from the fact that the product of any two idempotents of A is non-zero. □

Theorem 4.3 *The sets $C = E \dot{+} U \dot{+} U^2$ and $R = U \dot{+} U^2$ are ideals of A. Moreover, both C and R are independent of the choice of idempotent e.*

Proof: First note that $U^2Z \subseteq U^2$, from (3.4.2). The fact that C and R are ideals follows easily from this and Corollary 3.2.

Next let $A = E_1 \dot{+} U_1 \dot{+} Z_1, A = E_2 \dot{+} U_2 \dot{+} Z_2$ be the Peirce decompositions of A relative to idempotents e_1 and e_2 respectively, and put $C_i = E_i \dot{+} U_i \dot{+} U_i^2$, $R_i = U_i \dot{+} U_i^2$ for $i = 1, 2$. Then $e_2 = e_1 + u_1 + u_1^2$ for some $u_1 \in U_1$, by Lemma 4.1. Thus $e_2 \in C_1$, giving $E_2 \subseteq C_1$. Moreover, $u_2 = 2e_2u_2 \in U_2$ for all $u_2 \in U_2$, since C_1 is an ideal of A. It follows that $U_2 \dot{+} U_2^2 \subseteq C_1$ and hence that $C_2 \subseteq C_1$. The reverse inclusion follows by symmetry, and so $C_1 = C_2$. Finally, $R_1 = C_1 \cap N = C_2 \cap N = R_2$. □

We shall call the ideal $C = E \dot{+} U \dot{+} U^2$ the *core* of A. Our objective next is to investigate some properties of C and R.

Lemma 4.4 *The subspace U satisfies $U^r U^s \subseteq U^{r+s}$ for all $r, s \in \mathbb{N}$. Also, $ZU^r \subseteq U^r$ for all $r \in \mathbb{N}$.*

Proof: The results are clear if $r = 1$, so suppose that they hold for all $r \leq n$ (where $n \geq 1$). Note that, for every $n \in \mathbb{N}$, either $U^n \subseteq Z$ or $U^n \subseteq U$, using Corollary 3.2 (i), (iv). Then

$$\begin{aligned} U^{n+1}U^s &= (UU^n)U^s \\ &\subseteq (U^nU^s)U + (U^sU)U^n, \end{aligned}$$

using (3.4.3) if both U^n, U^s are contained in Z, (3.4.2) if one is inside Z and the other inside U, and (3.4.1) if both are contained in U. But this last set is contained in $U^{s+n}U + U^{s+1}U^n$, which is in U^{s+n+1}, both by the inductive hypothesis. Also

$$\begin{aligned} ZU^{n+1} &= Z(UU^n) \\ &\subseteq (ZU)U^n + (ZU^n)U, \end{aligned}$$

using (3.4.2) if $U^n \subseteq U$ and (3.4.3) if $U^n \subseteq Z$. This last set is contained in $UU^n + U^{n+1}U = U^{n+1}$ using the inductive hypothesis. The results now follow by induction on r. $\square$

Corollary 4.5 *Every product of n elements of U belongs to $U^n (n \in \mathbb{N})$.*

Proof: This is clearly true if $n = 1$. So suppose the result holds for $n \leq t$ $(t \geq 1)$, and let x be a product of $t+1$ elements of U. Then $x = x_1 x_2$ where each of x_1, x_2 is a product of at most t elements of U. By the inductive hypothesis,

$$\begin{aligned} x &\in U^r U^s \text{ where } 1 \leq r, s \leq t, r+s = t+1 \\ &\subseteq U^{r+s} \text{ by Lemma 4.4.} \end{aligned}$$

$\square$

Lemma 4.6 *For each $n \in \mathbb{N}, U^{n+2} \subseteq U^n, R^n = U^n \dot{+} U^{n+1}$ and R^n is an ideal of A.*

Proof: The first two results clearly hold if $n = 1$, so suppose they hold for $n = t (t \geq 1)$. Then

$$U^{t+3} = UU^{t+2} \subseteq UU^t = U^{t+1}, \text{ and}$$

$$\begin{aligned} R^{t+1} = RR^t &= (U \dot{+} U^2)(U^t \dot{+} U^{t+1}) \\ &= U^{t+1} + U^{t+2} + U^2 U^t + U^2 U^{t+1} \\ &= U^{t+1} \dot{+} U^{t+2}, \end{aligned}$$

since $U^2 U^t \subseteq U^{t+2}$, by Lemma 4.4, and

$$U^2 U^{t+1} \subseteq U^{t+3} \subseteq U^{t+1}.$$

These two results thus follow by induction on n.

Now

$$\begin{aligned} NR^n = (U+Z)(U^n + U^{n+1}) &\subseteq U^{n+1} + U^{n+2} + U^n + U^{n+1} \text{ using Lemma 4.4} \\ &\subseteq U^n + U^{n+1} \text{ since } U^{n+2} \subseteq U^n. \end{aligned}$$

But one of U^n, U^{n+1} is contained in U and the other in Z, so both are stabilised by e. Hence R^n is an ideal of A. □

An immediate corollary is the following result.

Corollary 4.7 *The ideal $R = U \dot{+} U^2$ is nilpotent if and only if $U^n = 0$ for some $n \in \mathbb{N}$.*

We shall show next that Z always acts nilpotently on U. First we need a lemma.

Lemma 4.8 *For every $u \in U, z \in Z, n \in \mathbb{N}$, we have*

$$z^{2^n} u = 2^{2^n - 1} L_z^{2^n}(u),$$

where $L_z : A \to A : a \mapsto za$.

Proof: We use induction on n. The result holds when $n = 1$ by (3.4.3). So assume that it holds for $n \leq r$, and let $n = r+1$. Then

$$\begin{aligned} z^{2^n} u &= z^{2^{r+1}} u = (z^{2^r})^2 u \text{ since } A \text{ is power associative} \\ &= 2z^{2^r}(z^{2^r} u) \text{ by (3.4.3)} \end{aligned}$$

$$
\begin{aligned}
&= 2.2^{2^r-1} L_z^{2^r}(z^{2^r} u) \text{ by the inductive hypothesis} \\
&= 2^{2^r}.2^{2^r-1} L_z^{2^r}(L_z^{2^r}(u)) \text{ by the inductive hypothesis again} \\
&= 2^{2^{r+1}-1} L_z^{2^{r+1}}(u).
\end{aligned}
$$

□

Now put $\mathcal{L} = \{L_z \mid_U : z \in Z\}$. Then $\mathcal{L}$ is a subalgebra of the Jordan algebra $\text{Hom}(U,U)^+$, since

$$
\begin{aligned}
\frac{1}{2}(L_{z_1}L_{z_2} + L_{z_2}L_{z_1})(u) &= \frac{1}{2}(z_1(z_2u) + z_2(z_1u)) \\
&= \frac{1}{2}(z_1z_2)u \text{ by (3.4.3)} \\
&= L_{\frac{1}{2}z_1z_2}(u).
\end{aligned}
$$

In particular, it is a weakly closed subset of $\text{Hom}(U,U)$ in the sense of Jacobson ([8], p. 31). This enables us to prove the following result.

Theorem 4.9 *The set $\mathcal{L}$ acts nilpotently on U.*

Proof: Let $z \in Z$. Then $z^{[k+1]} = 0$ since A is k-th order Bernstein. It follows that $z^{2^k} = 0$ since A is power associative, and hence that $(L_z \mid_U)^{2^k} = 0$ from Lemma 4.8. Thus $\mathcal{L}$ is a nil weakly closed set of linear transformations acting on U. The result is now a consequence of Theorem 1, p. 33 of [8]. □

If X, Y are subspaces of A we will denote by $X\dot{Y}^r$ the product $(...((XY)Y)...Y)$ where the term Y occurs r times.

Lemma 4.10 *For each $n, r \in \mathbb{N}$ we have*

$$
R^n\dot{N}^r \subseteq R^{n+1} + U^n\dot{Z}^r.
$$

Moreover, if there is an $s \in \mathbb{N}$ *such that* $Z^s = 0$ *then* $R^n \dot{N}^r \subseteq R^{n+1}$ *for some* $r \in \mathbb{N}$.

Proof: We prove the first result by induction on r. If $r = 1$ then

$$\begin{aligned} R^n N = (U^n + U^{n+1})(U + Z) &\subseteq U^{n+1} + U^n Z + U^{n+2} + U^{n+1} Z \\ &\subseteq U^{n+1} + U^{n+2} + U^n Z \text{ since } U^{n+1} Z \subseteq U^{n+1} \text{ by Lemma 4.4} \\ &= R^{n+1} + U^n Z. \end{aligned}$$

Suppose the result holds for $r = m$. Then

$$\begin{aligned} R^n \dot{N}^{m+1} &= ((U^n + U^{n+1})\dot{N}^m)N \\ &\subseteq (U^{n+1} + U^{n+2} + U^n \dot{Z}^m)(U + Z) \text{ by the inductive hypothesis} \\ &\subseteq U^{n+2} + U^{n+3} + (U^n \dot{Z}^m)U + U^{n+1}Z + U^{n+2}Z + U^n \dot{Z}^{m+1} \\ &\subseteq U^{n+1} + U^{n+2} + U^n \dot{Z}^{m+1}, \end{aligned}$$

since $U^{n+3} \subseteq U^{n+1}$ (by Lemma 4.6), $(U^n \dot{Z}^m)U \subseteq U^{n+1}$ (by repeated use of Lemma 4.4), $U^{n+1}Z \subseteq U^{n+1}, U^{n+2}Z \subseteq U^{n+2}$ (by Lemma 4.4). The first result follows by induction on r.

So suppose that $Z^s = 0$. Now $U^n \subseteq Z$ or $U^n \subseteq U$. If the former holds then $R^n \dot{Z}^{s-1} \subseteq R^{n+1} + Z^s = R^{n+1}$; if the latter holds then $R^n \dot{N}^r \subseteq R^{n+1}$ for some $r \in \mathbb{N}$ by Theorem 4.9. □

Recall the definitions of a *genetic algebra* ([15], p. 40) and of a *special train algebra* ([15], p. 55). We close this section by giving a characterisation of those power associative k-th order Bernstein algebras that are genetic. Again we precede the main result by a lemma.

Lemma 4.11 *For each* $r \in \mathbb{N}$,

$$N^r = (N^r \cap U) \dot{+} (N^r \cap Z).$$

Proof: We use induction on r. The result is apparent for $r = 1$, so suppose that it holds for $r = m$. Then

$$\begin{aligned} N^{m+1} &= (U + Z)((N^m \cap U) \dot{+} (N^m \cap Z)) \text{ by the inductive hypothesis} \\ &= (U(N^m \cap Z) + Z(N^m \cap U)) \dot{+} (U(N^m \cap U) + Z(N^m \cap Z)) \\ &= (N^{m+1} \cap U) \dot{+} (N^{m+1} \cap Z). \end{aligned}$$

□

Corollary 4.12 *All of the principal powers of N are ideals of A.*

Proof: The principal powers of N are clearly ideals of N. Furthermore,

$$eN^r = e(N^r \cap U) = N^r \cap U \subseteq N^r,$$

and so N^r is an ideal of A. □

Theorem 4.13 *The following are equivalent.*

(i) A is genetic;

(ii) A is a special train algebra;

(iii) there are natural numbers r, s such that $U^r = 0$ and $Z^s = 0$.

Proof: (i) $\Rightarrow$ (iii): Since A is genetic, N is nilpotent (Lemma 3.19, [15]). But $U, Z \subseteq N$, whence (iii) holds.
(ii) $\Rightarrow$ (i) : This is Theorem 3.29 of [15].
(iii) $\Rightarrow$ (ii) : We simply need to show that N is nilpotent. First, $N = R + Z$ and R is an ideal of A, by Corollary 4.7. Clearly then $N^k \subseteq R + Z^k$, and so N^k is eventually inside R. But now repeated use of Lemma 4.10 shows that N^k is eventually inside $R^r = 0$. □

Corollary 4.14 *If A is a Jordan algebra then it is a special train algebra.*

Proof: If A is Jordan then N is a Jordan nilalgebra and hence nilpotent. The result now follows from Theorem 4.13. $\square$

5 The type of A

Here A will again denote a power associative k-th order Bernstein algebra over a field K of characteristic different from two (and in which K contains at least 2^{k+1} elements), $e, f \in A$ will be idempotents, and $A = E\dot{+}U_e\dot{+}Z_e = F\dot{+}U_f\dot{+}Z_f$ will be the corresponding Peirce decompositions. Then $N = \text{ Ker } \omega = U_e\dot{+}Z_e = U_f\dot{+}Z_f$.

Theorem 5.1 *Let $f = e + u + u^2$ where $u \in U_e$. Then*

$$\begin{aligned} U_f &= \{x + 2ux - 4u(u^2x) : x \in U_e\}, \text{ and} \\ Z_f &= \{z - 2uz - 4u(u^2z) : z \in Z_e\} \end{aligned}$$

Proof: Let $y = x + z \in U_f$, where $x \in U_e, z \in Z_e$. Then

$$\begin{aligned} y \in U_f &\Leftrightarrow 2fy = y \\ &\Leftrightarrow 2(e + u + u^2)(x + z) = x + z \\ &\Leftrightarrow x + 2ux + 2uz + 2u^2x + 2u^2z = x + z \\ &\Leftrightarrow 2uz + 2u^2x = 0, 2ux + 2u^2z = z, \end{aligned}$$

since $2uz + 2u^2z \in U_e$ and $2ux + 2u^2z \in Z_e$.

Now

$$2u^2z = 4u(uz) \text{ by } (3.3.3)$$

$$= \quad -4u(u^2x) \text{ when } 2uz + 2u^2x = 0,$$

so if $y \in U_f$ then $z = 2ux - 4u(u^2x)$.

Conversely, suppose $y = x + 2ux - 4u(u^2x)$. Then $2uz + 2u^2x = 4u(ux) - 8u(u(u^2x)) + 2u^2x = 0$ since

$$\begin{aligned} 2u(ux) + u^2x &= 0 \text{ from (3.4.1), and} \\ 8u(u(u^2x)) &= -4u^2(u^2x) \text{ from (3.4.1)} \\ &= -2u^{[3]}x \text{ from (3.4.3)} \\ &= 0 \text{ by (3.3.2).} \end{aligned}$$

Moreover, $2ux + 2u^2z = z \Leftrightarrow 2u^2z = -4u(u^2x)$, and this was shown above. Hence $y \in U_f$.

Next let $y_1 = x_1 + z_1 \in Z_f$, where $x_1 \in U_e, z_1 \in Z_e$. Then

$$\begin{aligned} y_1 \in Z_f &\Leftrightarrow 2fy_1 = 0 \\ &\Leftrightarrow 2(e + u + u^2)(x_1 + z_1) = 0 \\ &\Leftrightarrow x_1 + 2ux_1 + 2uz_1 + 2u^2x_1 + 2u^2z_1 = 0 \\ &\Leftrightarrow x_1 + 2uz_1 + 2u^2x_1 = 0, ux_1 + u^2z_1 = 0 \end{aligned}$$

since $x_1 + 2uz_1 + 2u^2x_1 \in U_e$ and $ux_1 + u^2z_1 \in Z_e$.

Now

$$\begin{aligned} 2u^2x_1 &= -4u(ux_1), \text{ putting } u_1 = u_2 = u, u_3 = x_1 \text{ in (3.4.1).} \\ &= 4u(u^2z_1) \text{ when } ux_1 + u^2z_1 = 0, \end{aligned}$$

so $y_1 \in Z_f$ implies that $x_1 = -2uz_1 - 4u(u^2z_1)$.

Conversely, suppose that $x_1 = -2uz_1 - 4u(u^2z_1)$. Then $ux_1 + u^2z_1 = -2u(uz_1) - 4u(u(u^2z_1)) + u^2z_1 = 0$ since

$$u^2z_1 \quad = \quad 2u(uz_1) \text{ from (3.3.3) and}$$

$$
\begin{aligned}
4u(u(u^2z_1)) &= 4u(u^2(uz_1)+z_1u^3) \text{ by (3.4.3)} \\
&= 4u(u^2(uz_1)) \text{ by (3.3.2)} \\
&= 0 \text{ by (3.3.5).}
\end{aligned}
$$

Furthermore $x_1 + 2uz_1 + 2u^2x_1 = 0 \Leftrightarrow 2u^2x_1 = 4u(u^2z_1)$ and this was shown above. Thus $y_1 \in Z_f$. □

Corollary 5.2 *We have* $\dim U_e = \dim U_f$ *and* $\dim Z_e = \dim Z_f$*.*

Proof: The map $\theta : U_e \to U_f : x \mapsto x + 2ux - 4u(u^2x)$ is a surjective linear transformation, and so $\dim U_f \leq \dim U_e$. Since this is true for all idempotents $e, f \in A$ we have $\dim U_e = \dim U_f$, whence $\dim Z_e = \dim Z_f$. □

We call the pair of integers $(\dim U_e + 1, \dim Z_e)$ the *type* of A. In view of Corollary 5.2 the type of A is independent of the choice of idempotent e. Note that if A is of type (r, s) then $r + s = n = \dim A$. Next we consider those power associative k-Bernstein algebras A of type (r, s) where $r \leq 2$ or $s \leq 2$. If A is of type $(r, 0)$ then it is a first order Bernstein algebra and its structure is clear. From now on we write $U = U_e$, $Z = Z_e$.

Theorem 5.3 *Suppose that* $Z^2 = 0$*. Then* A *is a first order Bernstein algebra.*

Proof: Let $x = \alpha e + u + z \in A$, where $u \in U, z \in Z$. Then

$$x^2 = \alpha^2 e + u^2 + \alpha u + 2uz, \text{ and}$$

$$
\begin{aligned}
x^{[3]} &= \alpha^4 e + \alpha^2 u^2 + 4(uz)^2 + \alpha^3 u + 2\alpha^2 uz \\
&\quad +4u^2(uz) + 4\alpha u(uz) \text{ using (3.3.2) and Corollary 3.2 (i).}
\end{aligned}
$$

Now

$$\begin{aligned} 2(uz)^2 &= z(u^2z) - u(uz^2) \text{ by } (3.3.7) \\ &= 0 \text{ since } z^2, u^2z \in Z^2 = 0; \end{aligned}$$

$$\begin{aligned} u^2(uz) &= u(u^2z) \text{ by } (3.3.5) \\ &= 0; \\ 2u(uz) &= u^2z \text{ by } (3.3.3) \\ &= 0. \end{aligned}$$

Thus, $x^{[3]} = \alpha^2 x^2$ and A is a first order Bernstein algebra. □

Corollary 5.4 *If A is of type $(r, 1)$ then A is a first order Bernstein algebra.*

Proof: Let $Z = ((z))$. Then $z^2 = \lambda z$ for some $\lambda \in K$ by Corollary 3.2 (iii). Since $z^{[k+1]} = 0$ we have $\lambda = 0$, whence $Z^2 = 0$. □

Theorem 5.5 *Suppose that $Z^3 = 0$. Then A is a Jordan second order Bernstein algebra.*

Proof: Since $Z^3 = 0$, Z must be a Jordan algebra, and so A is a Jordan algebra, by Corollary 3.5. Let $x = \alpha e + u + z \in A$, where $u \in U, z \in Z$. Then

$$x^2 = \alpha^2 e + u^2 + z^2 + \alpha u + 2uz, \text{ and}$$

$$x^{[3]} = \alpha^4 e + \alpha^2 u^2 + 4(uz)^2 + \alpha^3 u + 2\alpha^2 uz + 4u^2(uz) + 2\alpha uz^2 + 4z^2(uz) + 4\alpha u(uz).$$

Now

$$\begin{aligned} u^2(uz) &= 0 \text{ by } (3.3.5); \\ 2z^2(uz) &= uz^3 \text{ by } (3.3.6) \\ &= 0; \end{aligned}$$

$$\begin{aligned}
4(uz)^2 &= 2z(u^2z) - 2u(uz^2) \text{ by (3.3.7)} \\
&= 2z(u^2z) - u^2z^2 \text{ by (3.3.3)} \\
&\in Z^3 = 0.
\end{aligned}$$

Thus

$$x^{[3]} = \alpha^4 e + \alpha^2 u^2 + \alpha^3 u + 2\alpha^2 (uz) + 2\alpha u z^2 + 4\alpha u(uz).$$

Hence

$$\begin{aligned}
x^{[4]} &= \alpha^8 e + \alpha^6 u^2 + 16\alpha^2 (u(uz))^2 \\
&+ \alpha^7 u + 2\alpha^6 uz + 2\alpha^5 uz^2 + 4\alpha^3 u^2(u(uz)) \\
&+ 4\alpha^5 u(uz) + 4\alpha^4 u(uz^2) + 8\alpha^4 u(u(uz)) \\
&+ 8\alpha^3 (uz)(uz^2) + 16\alpha^3 (uz)(u(uz)) \\
&+ 16\alpha^2 (uz^2)(u(uz))
\end{aligned}$$

(using the fact that $u^2(uz) = u^2(uz^2) = (uz)^2 = (uz^2)^2$ as above).

Now

$$\begin{aligned}
4(u(uz))^2 &= (u^2z)^2 \text{ by (3.3.3)} \\
&\in Z^3 = 0; \\
2u^2(u(uz)) &= u^2(u^2z) \text{ by (3.3.3)} \\
&\in Z^3 = 0; \\
2u(uz^2) &= u^2z^2 \text{ by (3.3.3)} \\
&\in Z^3 = 0; \\
2u(u(uz)) &= u(u^2z) \text{ by (3.3.3)} \\
&= 0 \text{ by (3.3.5);} \\
2(uz)(uz^2) &= 2((uz)u)z^2 - 2u((uz)z^2), \text{putting } u_1 = uz, u_2 = u, z = z^2 \text{ in (3.4.2)} \\
&= 2((uz)u)z^2 - u(uz^3) \text{ by (3.3.6)}
\end{aligned}$$

$$
\begin{aligned}
&= 0 \text{ since } Z^3 = 0; \\
2(uz)(u(uz)) &= -u(uz)^2, \text{ putting } u_1 = u_3 = uz, u_2 = u \text{ in (3.4.1)} \\
&= 0 \text{ as before;} \\
(uz^2)(u(uz)) &= u((u(uz))z^2) - z^2(u(u(uz))) \text{ putting } u_1 = u, z_1 = z^2, z_2 = u(uz) \text{ in (3.4.3)} \\
&= 0 \text{ since } (u(uz))z^2 \in Z^3 = 0 \text{ and } u(u(uz)) = 0 \text{ as above.}
\end{aligned}
$$

It follows that

$$x^{[4]} = \alpha^4 x^{[3]} \text{ and } A \text{ is a second order Bernstein algebra.}$$

□

Corollary 5.6 *If A is of type $(r, 2)$ then A is a Jordan second order Bernstein algebra.*

Proof: Since $\dim Z = 2$ and Z is power associative we have that either

(i) $Z^2 = 0$, or

(ii) $Z = ((z_1, z_2))$ with $z_1^2 = z_2$ and all other products zero. In either case, $Z^3 = 0$.

□

Theorem 5.7 *If A is of type $(1, s)$ then Z is a nilalgebra of index* $\leq \min\{2^k, s+1\}$

Proof: Use Lemma 2.5, the fact that A is power associative, and that if $z^{n+1} = 0, z^n \neq 0$ then the elements $z, z^2, \ldots, z^n$ are linearly independent. □

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