1 Introduction

This paper is a continuation of the study of power associative higher order Bernstein algebras initiated in [3]. We first introduce the notion of a reduced algebra and show that if $A$ is reduced then it is a Jordan algebra. We then prove some important identities that hold in power associative higher order Bernstein algebras and deduce that such algebras satisfy the identity $x^{2k+1} = \omega(x)x^{2k+1} - 1$ for all $x \in A$.

If $x_1, \ldots, x_n$ belong to the algebra $A$ we shall denote by $((x_1, \ldots, x_n))$ the subspace spanned by $x_1, \ldots, x_n$. The symbol $\oplus$ will denote an algebra direct sum, whereas $\oplus$ will indicate a direct sum of the vector space structure alone. All algebras will be finite dimensional over a field $K$. 
Let \( x \) be an element of the commutative algebra \( A \), and let \( B \) be a subspace of \( A \). Then the \textit{principal powers} of \( x \) are defined inductively by
\[
x^1 = x, \quad x^{r+1} = x^r x \quad \text{for all} \quad r \in \mathbb{N};
\]
the \textit{principal powers} of \( B \) are given by \( B^1 = B, B^{r+1} = B^r B \) for all \( r \in \mathbb{N} \).

Similarly, the \textit{plenary powers} of \( x \) and \( B \) are defined by
\[
x^{[1]} = x, \quad x^{[r+1]} = x^{[r]} x^{[r]} \quad \text{for all} \quad r \in \mathbb{N},
\]
\[
B^{[1]} = B, \quad B^{[r+1]} = B^{[r]} B^{[r]} \quad \text{for all} \quad r \in \mathbb{N}.
\]

Following Abraham [1] and Wörz-Busekros [4] we call the baric algebra \( A \) a \textit{k-th order Bernstein algebra} if
\[
x^{[k+2]} = \omega(x)^{2^k} x^{[k+1]} \quad \text{for all} \quad x \in A.
\]

Clearly the algebras previously referred to as Bernstein algebras are precisely the first order Bernstein algebras.

It is straightforward to check that the weight homomorphism of a \( k \)-th order Bernstein algebra \( A \) is unique, and that the set of idempotents of \( A \) is precisely the set \( A = \{ x^{[k+1]} : x \in A \text{ with } \omega(x) = 1 \} \). Proofs can be modelled on those for the corresponding results for first order Bernstein algebras.

Throughout \( A \) will denote a \( k \)-th order Bernstein algebra over a field \( K \) of characteristic different from two. We shall also assume that \( K \) contains at least \( 2^{k+1} \) distinct elements. Put \( N = \text{Ker } \omega \).
and let \( e \in A \) be an idempotent. The map

\[
L_e^k : A \to A : x \mapsto e(\ldots(e(ex))\ldots)
\]

(where there are \( k \) \( e \)'s in the product) is a linear transformation, and induces the map \( L_e^k |_N : N \to N \). Then

\[
A = E + U + Z
\]

where \( E \) is spanned by the idempotent \( e \), \( U = \text{Im} (L_e^k |_N) \), \( Z = \text{Ker} (L_e^k) \) (see [2]). We shall refer to this as the Pierce decomposition of \( A \) (relative to \( e \)): it clearly depends on the choice of idempotent.

## 2 Reduced algebras

If \( B, C \subseteq A \) the annihilator of \( B \) in \( C \), denoted by \( \text{Ann}_C B \), is defined by

\[
\text{Ann}_C B = \{ c \in C : Bc = 0 \}.
\]

**Lemma 2.1** \( \text{Ann}_Z U = \text{Ann}_Z (U + U^2) \)

**Proof:** Clearly \( \text{Ann}_Z (U + U^2) \subseteq \text{Ann}_Z U \). So suppose that \( zU = 0 \). Then \( z(u_1u_2) = (zu_1)u_2 + (zu_2)u_1 = 0 \) for all \( u_1, u_2 \in U \), and so \( zU^2 = 0 \), giving the reverse inclusion. \( \square \)

**Lemma 2.2** \( \text{Ann}_Z U \) is independent of the choice of idempotent.

**Proof:** Let \( z \in \text{Ann}_Z U \) and let \( f = e + u + u^2 \), where \( u \in U \) be another idempotent. Then

\[
z(x + 2ux - 4u(u^2x)) = 0 \quad \text{for all} \ x \in U,
\]

using Lemma 2.1 and the fact that \( ux, u(u^2x) \in U^2 \).
Thus $z \in \text{Ann}_Z U_f$, by Theorem 5.1 of [3], whence $\text{Ann}_Z U \subseteq \text{Ann}_Z U_f$. The reverse inclusion follows from symmetry. \hfill \Box

**Lemma 2.3** $\text{Ann}_Z U$ is an ideal of $A$.

**Proof:** Let $z \in \text{Ann}_Z U$. Then $ez = 0$, $uz = 0$ for all $u \in U$. Let $z_1 \in Z$. Then

$$u(zz_1) = (uz)z_1 + (uz_1)z = 0$$

since $uz_1 \in U$ and so $(uz_1)z = 0$. Hence $zz_1 \in \text{Ann}_Z U$, and the result follows. \hfill \Box

**Theorem 2.4** $A/\text{Ann}_Z U$ is a Jordan algebra.

**Proof:** Let $z_1, z_2 \in Z$. By Corollary 3.5 of [3] it suffices to show that $z_1^2(z_2z_1) - (z_1^2z_2)z_1 \in \text{Ann}_Z U$; that is, that $u(z_1^2(z_2z_1)) = u((z_1^2z_2)z_1)$. Now

$$u(z_1^2(z_2z_1)) = (uz_1^2)(z_1z_2) + (u(z_2z_1))z_1^2 \quad \text{by (3.4.3)}$$

$$= ((uz_1^2)z_2)z_1 + ((uz_1^2)z_1)z_2 + ((uz_2)z_1)z_1^2 + ((uz_1)z_2)z_1^2 \quad \text{by (3.4.3)}$$

$$= 2(((uz_1)z_1)z_2)z_1 + 2(((uz_1)z_1)z_1)z_2$$

$$+ 2(((uz_2)z_1)z_1)z_1 + 2(((uz_1)z_2)z_1)z_1 \quad \text{by (3.4.3)},$$

and

$$u((z_1^2z_2)z_1) = (u(z_1^2z_2))z_1 + (uz_1)(z_1^2z_2) \quad \text{by (3.4.3)}$$

$$= ((uz_1^2)z_2)z_1 + ((uz_2)z_1^2)z_1 + ((uz_1)z_1^2)z_2 + ((uz_1)z_2)z_1^2 \quad \text{by (3.4.3)}$$

$$= 2(((uz_1)z_1)z_2)z_1 + 2(((uz_2)z_1)z_1)z_1$$

$$+ 2(((uz_1)z_1)z_2)z_2 + 2(((uz_1)z_2)z_1)z_1 \quad \text{by (3.4.3)}.$$
This establishes the result. \[\square\]

We will call \( A \) reduced if \( \text{Ann}_Z U = 0 \).

**Corollary 2.5** If \( A \) is reduced then it is a Jordan algebra.

**Lemma 2.6** \( A/\text{Ann}_Z U \) is reduced.

**Proof:** Suppose that \( z \in Z \) and \( zU \subseteq \text{Ann}_Z U \). Then \( zU \subseteq U \cap Z = 0 \), whence \( z \in \text{Ann}_Z U \). \[\square\]

We shall call \( A \) nuclear if \( A = E + U + U^2 \) (so that \( A = \text{core of } A \)); that is, if \( Z = U^2 \).

**Theorem 2.7** If \( A \) is nuclear, then \( \text{Ann}_Z U = \text{Ann}_A(= \text{Ann}_A A) \).

**Proof:** This is clear from Lemma 2.1 and the fact that \( eZ = 0 \). \[\square\]

**Corollary 2.8** If \( A \) is nuclear then \( A/\text{Ann}_A \) is a Jordan algebra.

**Theorem 2.9** \( A^{[k]} \subseteq E + U + U^2 + Z^k \) for all \( k \in \mathbb{N} \). In particular, if \( Z^k = 0 \) then \( A^{[k]} \) is nuclear.

**Proof:** We proceed by induction on \( k \). First note that

\[
A^2 = E + U + U^2 + Z^2 + UZ \subseteq E + U + U^2 + Z^2 \quad \text{since } UZ \subseteq U.
\]

So suppose the result holds for \( k = n \). Then
\[ A^{[n+1]} = (E + U + U^2 + Z^n)^2 \]
\[ = E + U + U^2 + U^3 + UZ^n + U^{(3)} + U^2Z^n + (Z^n)^2 \]
\[ = E + U + U^2 + Z^{n+1}, \]

since \( U^3, UZ^n \subseteq U, U^{(3)} \subseteq U^2, U^2Z^n + (Z^n)^2 \subseteq Z^{n+1} \). The result follows by induction.

If \( Z^k = 0 \) then \( A^{[k]} \subseteq E + U + U^2 \). But

\[ (E + U + U^2)^2 = E + U + U^2 + U^3 + U^{(3)} = E + U + U^2, \]

so \( E + U + U^2 \subseteq A^{[k]} \) for all \( k \in \mathbb{N} \). \( \square \)

3 Some identities

First we aim to improve (3.3.4) of [3].

**Lemma 3.1** \( 2^k u^k z^{k+1} = u z^{k+1} \) for all \( k \in \mathbb{N} \).

**Proof:** The result holds for \( k = 1 \) by (3.3.4) of [3]. Suppose that it is true for \( k = n \). Then

\[ 2^n u z^{n+2} = 2^n (u z^{n+1}) z \]
\[ = (u z^{n+1}) z \quad \text{by the inductive hypothesis} \]
\[ = u z^{n+2} - (u z) z^{n+1} \quad \text{by (3.4.3)} \]
\[ = u z^{n+2} - 2^n (u z) z^{n+1} \quad \text{by the inductive hypothesis again} \]
\[ = u z^{n+2} - 2^n u z^{n+2} \]

Hence \( 2^{n+1} u z^{n+2} = u z^{n+2} \), and the result follows by induction. \( \square \)
Lemma 3.2 \((u \hat{z}^r)(u^2 \hat{z}^s) = 0\) for all \(r, s \geq 0\).

Proof: We use induction on \(s\). If \(s = 0\), then
\[
(u \hat{z}^r)u^2 = \frac{1}{2^{r-1}}(uz^r)u^2 = 0 \quad \text{by (3.3.5)}.
\]
So suppose the result holds for \(s = n\). Then
\[
(u \hat{z}^r)(u^2 \hat{z}^{n+1}) = (u \hat{z}^r)((u^2 \hat{z}^n)z)
\]
\[
= (((u \hat{z}^r)(u^2 \hat{z}^n))z + ((u \hat{z}^r)z)(u^2 \hat{z}^n)) \quad \text{by (3.4.3)}
\]
\[
= (u \hat{z}^{r+1})(u^2 \hat{z}^n) \quad \text{by the inductive hypothesis}
\]
\[
= 0 \quad \text{by the inductive hypothesis again}.
\]
\[\square\]

Lemma 3.3 \((u \hat{z}^r)((u^2 \hat{z}^s) \hat{z}^t) = 0\) for all \(r, s, t \geq 0\).

Proof: We use induction on \(t\). Suppose first that \(t = 0\). Then
\[
(u \hat{z}^r)(u^2 \hat{z}^s) = ((u \hat{z}^r)u^2)z^s + ((u \hat{z}^r)z^s)u^2 \quad \text{by (3.4.3)}
\]
\[
= \frac{1}{2^{r-1}}((u \hat{z}^r)u^2)z^s + 2^{s-1}((u \hat{z}^r) \hat{z}^s)u^2 \quad \text{by Lemma 3.1}
\]
\[
= 2^{s-1}(u \hat{z}^{r+s})u^2
\]
\[
= 2^{s-1} \frac{1}{2^{r+s-1}}(u \hat{z}^{r+s})u^2 \quad \text{by Lemma 3.1 again}
\]
\[
= 0 \quad \text{by (3.3.5)}.
\]
So suppose that the result holds for \(t = n\). Then
\[
(u \hat{z}^r)((u^2 \hat{z}^s) \hat{z}^{n+1}) = (((u \hat{z}^r)((u^2 \hat{z}^s) \hat{z}^n))z + (u \hat{z}^{r+1})(u^2 \hat{z}^s) \hat{z}^n) = 0,
\]
using the inductive hypothesis twice. \[\square\]
Theorem 3.4  \((ae + u + z)^r =\)

\[\alpha^r e + \alpha^{r-2} u^2 + \sum_{i=0}^{r-1} 2i\alpha^{r-i-1} u z^i + \sum_{i=1}^{r-2} 2\alpha^{r-i-2} u^2 z^i + \sum_{j=0}^{r-4} \sum_{i=2}^{r-j-2} \alpha^{r-2-i-j} (u^2 z^i) z^j + z^r\]

for all \(r \in \mathbb{N}\).

Proof: We use induction on \(r\). The result is clear for \(r = 1, 2\), so suppose it holds for \(r\). Then

\[(ae + u + z)^{r+1} = \alpha^{r+1} e + \sum_{i=0}^{r-1} 2i\alpha^{r-i-1} u z^i + \frac{1}{2} \alpha^r u + \sum_{i=0}^{r-1} 2i\alpha^{r-i-1} u^2 z^i + \alpha^r - 2 u^2 z + \sum_{i=2}^{r-4} \sum_{i=2}^{r-j-2} \alpha^{r-2-i-j} (u^2 z^i) z^j + z^{r+1}.

Now

\[\sum_{i=0}^{r-1} 2i\alpha^{r-i-1} u(u z^i) = \sum_{i=0}^{r-1} \alpha^{r-i-1} u^2 z^i \quad \text{using Lemma 3.1 and (3.3.3)}\]

\[= \alpha^{r-1} u^2 + \alpha^{r-2} u^2 z + \sum_{i=2}^{r-4} \alpha^{r-i-1} u^2 z^i;\]

\[\sum_{i=0}^{r-1} 2i\alpha^{r-i-1} u z^i + \sum_{i=0}^{r-1} 2i\alpha^{r-i-1} u z^i + 1 + \frac{1}{2} \alpha^r u + u z^r\]

\[= \sum_{i=0}^{r} 2i\alpha^{r-i} u z^i + \sum_{i=0}^{r} 2i\alpha^{r-i} u z^i \quad \text{by Lemma 3.1}\]

\[= \sum_{i=0}^{r} 2i\alpha^{r-i} u z^i;\]

\[\sum_{i=1}^{r} \alpha^{r-i-2} u^2 z^i + 2 \alpha^{r-2} u^2 z\]

\[= \sum_{i=2}^{r} 2\alpha^{r-i-1} u^2 z^i + 2 \alpha^{r-2} u^2 z = \sum_{i=1}^{r-1} 2\alpha^{r-i-1} u^2 z^i;\]

\[\sum_{j=0}^{r-4} \sum_{i=2}^{r-j-2} \alpha^{r-2-i-j} (u^2 z^i) z^j + \sum_{i=2}^{r-4} \alpha^{r-i-1} u^2 z^i\]

\[= \sum_{j=1}^{r-4} \sum_{i=2}^{r-j-1} \alpha^{r-1-i-j} (u^2 z^i) z^j + \sum_{i=2}^{r-4} \alpha^{r-i-1} u^2 z^i = \sum_{j=0}^{r-3} \sum_{i=2}^{r-j-1} \alpha^{r-1-i-j} (u^2 z^i) z^j.\]

Combining these with the expression given above for \((ae + u + z)^{r+1}\) confirms the result for \(r + 1\). \(\square\)
Corollary 3.5 \((\alpha e + u + z)^{r+1} = \)

\[
\alpha(\alpha e + u + z)^r + 2^ruz^r + 2u^2z^{r-1} + \sum_{j=0}^{r-3}(u^2z^{r-j-1})z^j + z^{r+1} = \alpha(\alpha e + u + z)^r + (u + z)^{r+1}.
\]

Proof:

\[
(\alpha e + u + z)^{r+1} = \alpha^{r+1}e + \alpha^{r-1}u^2 + \sum_{i=0}^{r-1}2^i\alpha^{r-i}uz^i + 2^ruz^r + \sum_{i=1}^{r-2}2\alpha^{r-i-1}u^2z^i + 2u^2z^{r-1} + \sum_{j=0}^{r-3}(u^2z^{r-j-1})z^j + z^{r+1} = \alpha(\alpha e + u + z)^r + 2^ruz^r + 2u^2z^{r-1} + \sum_{j=0}^{r-3}(u^2z^{r-j-1})z^j + z^{r+1}.
\]

Putting \(\alpha = 0\) shows that the last three terms above are equal to \((u + z)^{r+1}\). \(\square\)

Corollary 3.6 \(x^{r+1} - \omega(x)x^r \in Z^r + UZ^r\) for all \(x \in A\).

Proof: Simply note that \(2^ruz^r = uz^r \in UZ^r\), \(2u^2z^{r-1} \in Z^r\), and \((u^2z^{r-j-1})z^j \in Z^r\). \(\square\)

Corollary 3.7 If \(Z^{2^r} = 0\) then \(A\) is an \(r\)-th order Bernstein algebra.

Proof: Suppose that \(Z^{2^r} = 0\). Then \(x^{2^r+1} = \omega(x)x^{2^r}\), by Corollary 3.5. A straightforward induction argument now shows that \(x^{2^r+k} = \omega(x)^k x^{2^r}\) for all \(k \in \mathbb{N}\); in particular, \(x^{2^r+1} = \omega(x)^{2^r} x^{2^r}\). As \(A\) is power associative this is equivalent to \(x^{[r+2]} = \omega(x)^{2^r} x^{[r+1]}\). \(\square\)

Corollary 3.8 If \(A\) is a power associative \(k\)-th order Bernstein algebra, then \(x^{2^k+1} = \omega(x)x^{2^{k+1} - 1}\) for all \(x \in A\).

Proof: Since \(A\) is power associative and a \(k\)-th order Bernstein algebra, we have

\[
x^{2^k+1} = x^{[k+2]} = \omega(x)^{2^k} x^{[k+1]} = 0
\]
if \( x \in \text{Ker}\omega \). Now put \( r + 1 = 2^{k+1} \) in Corollary 3.5. \( \square \)

References


