Symmetric Laman theorems for the groups $\mathcal{C}_2$ and $\mathcal{C}_s$

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Abstract

For a bar and joint framework (G, p) with point group $\mathcal{C}_3$ which describes 3-fold rotational symmetry in the plane, it was recently shown in [19] that the standard Laman conditions, together with the condition derived in [3] that no vertices are fixed by the automorphism corresponding to the 3-fold rotation (geometrically, no vertices are placed on the center of rotation), are both necessary and sufficient for (G, p) to be isostatic, provided that its joints are positioned as generically as possible subject to the given symmetry constraints. In this paper we prove the analogous Laman-type conjectures for the groups $\mathcal{C}_2$ and $\mathcal{C}_s$ which are generated by a half-turn and a reflection in the plane, respectively. In addition, analogously to the results in [19], we also characterize symmetry generic isostatic graphs for the groups $\mathcal{C}_2$ and $\mathcal{C}_s$ in terms of inductive Henneberg-type constructions, as well as Crapo-type 3Tree2 partitions - the full sweep of methods used for the simpler problem without symmetry.

1 Introduction

The major problem in generic rigidity is to find a combinatorial characterization of graphs whose generic realizations as bar-and-joint frameworks in d-space are rigid. While for dimension $d \geq 3$, only partial results for this problem have been found, it is completely solved for dimension 2. In fact, using a number of both algebraic and combinatorial techniques, a series of characterizations of generically rigid graphs in the plane have been established, ranging from Laman's famous counts from 1970 on the number of vertices and edges of a graph [12], and Henneberg's inductive construction sequences from 1911 [11], to Crapo's characterization in terms of proper partitions of the edge set of a graph into three trees (3Tree2 partitions) from 1989 [4].

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Using techniques from representation theory, it was recently shown in [3] that if a 2-dimensional isostatic bar and joint framework possesses non-trivial symmetries, then it must not only satisfy the Laman conditions, but also some very simply stated extra conditions concerning the number of joints and bars that are fixed by various symmetry operations of the framework (see also [15, 17, 16]). In particular, these restrictions imply that a 2-dimensional isostatic framework must belong to one of only six possible point groups. In the Schoenflies notation [2], these groups are denoted by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_s, \mathcal{C}_{2v}$, and $\mathcal{C}_{3v}$.

It was conjectured in [3] that for these groups, the Laman conditions, together with the corresponding additional conditions concerning the number of fixed structural components, are not only necessary, but also sufficient for a symmetric framework to be isostatic, provided that its joints are positioned as generically as possible subject to the given symmetry constraints.

Using the definition of 'generic' for symmetry groups established in [18], this conjecture was confirmed in [19] for the symmetry group $\mathcal{C}_3$ which describes 3-fold rotational symmetry in the plane ($\mathbb{Z}_3$ as an abstract group). In this paper, we verify the analogous conjectures for the symmetry groups $\mathcal{C}_2$ and $\mathcal{C}_s$ which are generated by a half-turn and a reflection in the plane, respectively ($\mathbb{Z}_2$ as abstract groups).

Similarly to the $\mathcal{C}_3$ case, these results are striking in their simplicity: to test a 'generic' framework with $\mathcal{C}_2$ or $\mathcal{C}_s$ symmetry for isostaticity, we just need to check the number of joints and bars that are 'fixed' by the corresponding symmetry operations, as well as the standard conditions for generic rigidity without symmetry.

By defining appropriate symmetrized inductive construction techniques, as well as appropriate symmetrized 3Tree2 partitions of a graph, we also establish symmetric versions of Henneberg's Theorem (see [7, 11]) and Crapo's Theorem ([4, 7, 22]) for the groups $\mathcal{C}_2$ and $\mathcal{C}_s$. These results provide us with some alternate techniques to give a 'certificate' that a graph is 'generically' isostatic modulo the given symmetry, and they also enable us to generate all such graphs by means of an inductive construction sequence.

With each of the main results presented in this paper, we also lay the foundation to design algorithms that decide whether a given graph is generically isostatic modulo the given symmetry.

As we will see in Sections 4.2 and 5.2, it turns out that the proofs for the characterizations of symmetry generically isostatic graphs for the group $\mathcal{C}_2$, and in particular for the group $\mathcal{C}_s$, are considerably more complex than the ones for $\mathcal{C}_3$. An initial indication for this is that Crapo's Theorem uses partitions of the edges of a graph into *three* edge-disjoint trees, so that it is less obvious how to extend this result to the groups $\mathcal{C}_2$ and $\mathcal{C}_s$ of order 2 than to the cyclic group $\mathcal{C}_3$ of order 3. Moreover, due to the nature of the necessary conditions for a graph to be generically isostatic modulo $\mathcal{C}_2$ or $\mathcal{C}_s$ symmetry derived in [3], the simple number-theoretic arguments used in the proof of the symmetric Laman theorem for $\mathcal{C}_3$ (see [19]) cannot be used in the proofs of the corresponding Laman-type theorems for the groups $\mathcal{C}_2$ and $\mathcal{C}_s$.

The Laman-type conjectures for the dihedral groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ still remain open. A discussion on some of the difficulties that arise in proving these conjectures is given in Section 6 (see also [16, 19] for further comments).

2 Preliminaries on frameworks

2.1 Graph theory terminology

All graphs considered in this paper are finite graphs without loops or multiple edges. We denote the *vertex set* of a graph G by $V(G)$ and the *edge set* of G by $E(G)$. Two vertices $u \neq v$ of G are said to be *adjacent* if $\{u, v\} \in E(G)$, and *independent* otherwise. A set S of vertices of G is *independent* if every two vertices of S are independent. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices that are adjacent to v and the elements of $N_G(v)$ are called the *neighbors* of v.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. For $v \in V(G)$ and $e \in E(G)$ we write $G - \{v\}$ for the subgraph of G that has $V(G) \setminus \{v\}$ as its vertex set and whose edges are those of G that are not incident with v. Similarly, we write $G - \{e\}$ for the subgraph of G that has $V(G)$ as its vertex set and $E(G) \setminus \{e\}$ as its edge set. The deletion of a set of vertices or a set of edges from G is defined and denoted analogously.

If u and v are independent vertices of G, then we write $G + \big\{\{u, v\}\big\}$ for the graph that has $V(G)$ as its vertex set and $E(G) \cup \big\{\{u, v\}\big\}$ as its edge set. The addition of a set of edges is again defined and denoted analogously.

For a nonempty subset U of $V(G)$, the subgraph $\langle U \rangle$ of G *induced* by U is the graph having vertex set U and whose edges are those of G that are incident with two elements of U.

The *intersection* $G = G_1 \cap G_2$ of two graphs G_1 and G_2 is the graph with $V(G) = V(G_1) \cap V(G_2)$ and $E(G) = E(G_1) \cap E(G_2)$. Similarly, the *union* $G = G_1 \cup G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

An *automorphism* of a graph G is a permutation α of $V(G)$ such that $\{u, v\} \in E(G)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(G)$. The group of automorphisms of a graph G is denoted by $\mathrm{Aut}(G)$.

Let H be a subgraph of G and $\alpha \in \mathrm{Aut}(G)$. We define $\alpha(H)$ to be the subgraph of G that has $\alpha\big(V(H)\big)$ as its vertex set and $\alpha\big(E(H)\big)$ as its edge set, where $\{u, v\} \in \alpha\big(E(H)\big)$ if and only if $\alpha^{-1}(\{u, v\}) = \{\alpha^{-1}(u), \alpha^{-1}(v)\} \in E(H)$.

We say that H is *invariant under* α if $\alpha\big(V(H)\big) = V(H)$ and $\alpha\big(E(H)\big) = E(H)$, in which case we write $\alpha(H) = H$.

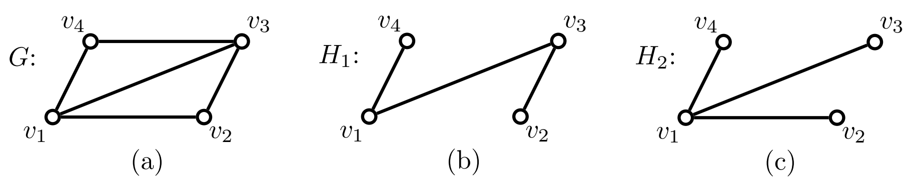


Figure 1: *An invariant* (b) *and a non-invariant subgraph* (c) *of the graph G under $\alpha = (v_1\, v_3)(v_2\, v_4) \in Aut(G)$.*

The graph G in Figure 1 (a), for example, has the automorphism $\alpha = (v_1\, v_3)(v_2\, v_4)$. The subgraph H_1 of G is invariant under α, but the subgraph H_2 of G is not, because $\alpha\big(E(H_2)\big) \neq E(H_2)$.

Let u and v be two (not necessarily distinct) vertices of a graph G. A u-v

path in G is a finite alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of vertices and edges of G in which no vertex is repeated and $e_i = \{u_{i-1}, u_i\}$ for $i = 1, 2, \ldots, k$. A u-v path is called a *cycle* if $k \geq 3$ and $u = v$.

Let a *u-v path* P in G be given by $u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ and let $\alpha \in \mathrm{Aut}(G)$. Then we denote $\alpha(P)$ to be the $\alpha(u)$-$\alpha(v)$ path $\alpha(u) = \alpha(u_0), \alpha(e_1), \alpha(u_1), \alpha(e_2), \ldots, \alpha(u_{k-1}), \alpha(e_k), \alpha(u_k) = \alpha(v)$ in G.

A vertex u is said to be *connected* to a vertex v in G if there exists a $u - v$ path in G. A graph G is *connected* if every two vertices of G are connected.

A graph with no cycles is called a *forest* and a connected forest is called a *tree*.

A connected subgraph H of a graph G is a *component* of G if $H = H'$ whenever H' is a connected subgraph of G containing H.

2.2 Infinitesimal rigidity

A *framework* in $\mathbb{R}^d$ is a pair (G, p), where G is a graph and $p : V(G) \to \mathbb{R}^d$ is a map with the property that $p(u) \neq p(v)$ for all $\{u, v\} \in E(G)$ [6, 7, 28]. We also say that (G, p) is a d-dimensional *realization* of the *underlying graph* G. An ordered pair $\big(v, p(v)\big)$, where $v \in V(G)$, is a *joint* of (G, p), and an unordered pair $\big\{\big(u, p(u)\big), \big(v, p(v)\big)\big\}$ of joints, where $\{u, v\} \in E(G)$, is a *bar* of (G, p). For a framework (G, p) whose underlying graph G has a vertex set that is indexed from 1 to n, say $V(G) = \{v_1, v_2, \ldots, v_n\}$, we will frequently denote $p(v_i)$ by p_i for $i = 1, 2, \ldots, n$. The k^{th} component of a vector x is denoted by $(x)_k$.

Let (G, p) be a framework in $\mathbb{R}^d$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. An *infinitesimal motion* of (G, p) is a function $u : V(G) \to \mathbb{R}^d$ such that

$$\big(p_i - p_j\big) \cdot \big(u_i - u_j\big) = 0 \quad \text{for all } \{v_i, v_j\} \in E(G), \tag{1}$$

where $u_i = u(v_i)$ for each $i = 1, \ldots n$.

An infinitesimal motion u of (G, p) is an *infinitesimal rigid motion* if there exists a skew-symmetric matrix S (a rotation) and a vector t (a translation) such that $u(v) = Sp(v) + t$ for all $v \in V(G)$. Otherwise u is an *infinitesimal flex* of (G, p).

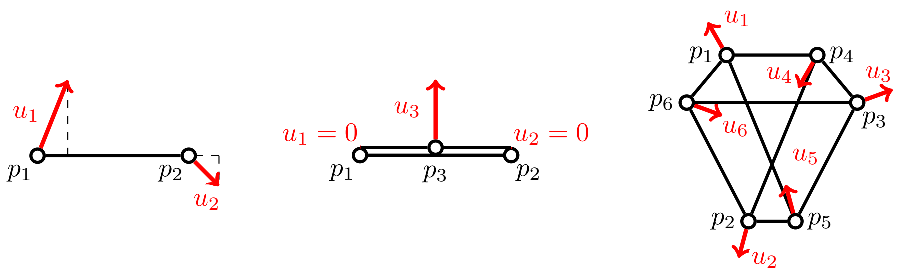
(G, p) is *infinitesimally rigid* if every infinitesimal motion of (G, p) is an infinitesimal rigid motion. Otherwise (G, p) is said to be *infinitesimally flexible* [6, 7, 28].

The *rigidity matrix* $\mathbf{R}(G, p)$ of (G, p) is the $|E(G)| \times dn$ matrix

$$\begin{array}{c} \\ \{v_i, v_j\} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & v_i & & & & v_j & & & \end{array} \\ \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & p_i - p_j & 0 & \ldots & 0 & p_j - p_i & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} \end{array},$$

that is, for each edge $\{v_i, v_j\} \in E(G)$, $\mathbf{R}(G, p)$ has the row with $(p_i - p_j)_1, \ldots, (p_i - p_j)_d$ in the columns $d(i-1)+1, \ldots, di$, $(p_j - p_i)_1, \ldots, (p_j - p_i)_d$ in the columns $d(j-1)+1, \ldots, dj$, and 0 elsewhere [6, 7, 28].

Note that if we identify an infinitesimal motion u of (G, p) with a column vector in $\mathbb{R}^{dn}$ (by using the order on $V(G)$), then the equations in (1) can be



(a) (b) (c)

Figure 2: *The arrows indicate the non-zero displacement vectors of an infinitesimal rigid motion* (a) *and infinitesimal flexes* (b, c) *of frameworks in* $\mathbb{R}^2$.

written as $\mathbf{R}(G,p)u = 0$. So, the kernel of the rigidity matrix $\mathbf{R}(G,p)$ is the space of all infinitesimal motions of (G,p). It is well known that a framework (G,p) in $\mathbb{R}^d$ is infinitesimally rigid if and only if either the rank of its associated rigidity matrix $\mathbf{R}(G,p)$ is precisely $dn - \binom{d+1}{2}$, or G is a complete graph K_n and the points p_i, $i = 1, \ldots, n$, are affinely independent [1].

Remark 2.1 Let $1 \leq m \leq d$ and let (G,p) be a framework in $\mathbb{R}^d$. If (G,p) has at least $m+1$ joints and the points $p(v)$, $v \in V(G)$, span an affine subspace of $\mathbb{R}^d$ of dimension less than m, then (G,p) is infinitesimally flexible (recall Figure 2 (b)). In particular, if (G,p) is infinitesimally rigid and $|V(G)| \geq d$, then the points $p(v)$, $v \in V(G)$, span an affine subspace of $\mathbb{R}^d$ of dimension at least $d-1$.

A framework (G,p) is *independent* if the row vectors of the rigidity matrix $\mathbf{R}(G,p)$ are linearly independent. A framework which is both independent and infinitesimally rigid is called *isostatic* [6, 7, 28].

Theorem 2.1 *[7] For a d-dimensional realization (G,p) of a graph G with $|V(G)| \geq d$, the following are equivalent:*

(i) (G,p) *is isostatic;*

(ii) (G,p) *is infinitesimally rigid and* $|E(G)| = d|V(G)| - \binom{d+1}{2}$*;*

(iii) (G,p) *is independent and* $|E(G)| = d|V(G)| - \binom{d+1}{2}$*;*

(iv) (G,p) *is minimal infinitesimally rigid, i.e.,* (G,p) *is infinitesimally rigid and the removal of any bar results in a framework that is not infinitesimally rigid.*

2.3 Generic rigidity

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and K_n be the complete graph on $V(G)$. A framework (G,p) is called *generic* if the determinant of any submatrix of $\mathbf{R}(K_n,p)$ is zero only if it is (identically) zero in the variables p'_i [7].

Note that it follows immediately from this definition that the set of all generic realizations of a given graph G in $\mathbb{R}^d$ forms a dense open subset of all possible realizations of G in $\mathbb{R}^d$. Moreover, it is known that the framework (G,p) is infinitesimally rigid (independent, isostatic) for some map $p : V(G) \to \mathbb{R}^d$ if and only if every d-dimensional generic realization of G is infinitesimally rigid (independent, isostatic) [7]. Thus, for generic frameworks, infinitesimal rigidity is purely combinatorial, and hence a property of the underlying graph. We say that a graph G is *generically d-rigid (d-independent, d-isostatic)* if d-dimensional generic realizations of G are infinitesimally rigid (independent, isostatic).

In 1970, Laman gave a complete characterization of generically 2-isostatic graphs:

Theorem 2.2 (Laman, 1970) *[12] A graph G with $|V(G)| \geq 2$ is generically 2-isostatic if and only if*

(i) $|E(G)| = 2|V(G)| - 3$;

(ii) $|E(H)| \leq 2|V(H)| - 3$ *for all* $H \subseteq G$ *with* $|V(H)| \geq 2$.

Various proofs of Laman's Theorem can be found in [6], [7], [14], [22], and [27], for example. Throughout this paper, we will refer to the conditions (i) and (ii) in Theorem 2.2 as the *Laman conditions*.

A combinatorial characterization of generically isostatic graphs in dimension 3 or higher is not yet known. The so-called 'double banana', for instance, provides a classic counterexample to the existence of a straightforward 3-dimensional analog of Laman's Theorem [6, 7, 23].

In 1911, L. Henneberg showed that generically 2-isostatic graphs can also be characterized using an inductive construction sequence. The two Henneberg construction steps for a graph G are defined as follows (see also Figure 3):

First, let $U \subseteq V(G)$ with $|U| = 2$ and $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = E(G) \cup \big\{\{v,u\} | u \in U\big\}$ is called a *vertex 2-addition (by v) of G* [23, 28].

Secondly, let $U \subseteq V(G)$ with $|U| = 3$ and $\{u_1, u_2\} \in E(G)$ for some $u_1, u_2 \in U$. Further, let $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = \big(E(G) \setminus \big\{\{u_1, u_2\}\big\}\big) \cup \big\{\{v,u\} | u \in U\big\}$ is called an *edge 2-split (on $u_1, u_2; v$) of G*.

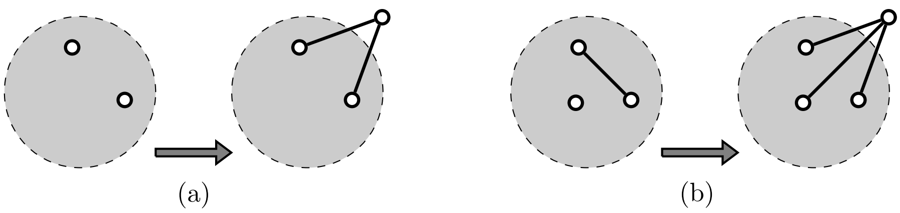


Figure 3: *Illustrations of a vertex 2-addition* (a) *and an edge 2-split* (b).

Theorem 2.3 (Henneberg, 1911) *[11] A graph is generically 2-isostatic if and only if it may be constructed from a single edge by a sequence of vertex 2-additions and edge 2-splits.*

For a proof of Henneberg's Theorem, see [7] or [23], for example.

There exist a few additional inductive construction techniques that are frequently used in rigidity theory. One of these techniques, the 'X-replacement', will play a pivotal role in proving the symmetric version of Laman's Theorem for a symmetry group consisting of the identity and a single reflection.

Let G be a graph, u_1, u_2, u_3, u_4 be four distinct vertices of G with $\{u_1, u_2\}, \{u_3, u_4\} \in E(G)$, and let $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = \big(E(G) \setminus \big\{\{u_1, u_2\}, \{u_3, u_4\}\big\}\big) \cup \big\{\{v, u_i\} | i \in \{1, 2, 3, 4\}\big\}$ is called an *X-replacement (by v) of G* [23, 28] (see also Figure 4).

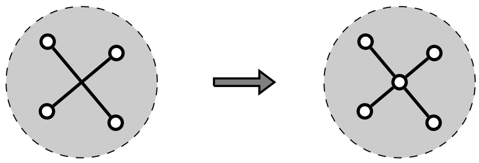


Figure 4: *Illustration of an X-replacement of a graph G.*

Theorem 2.4 (X-Replacement Theorem) *[23, 28] An X-replacement of a generically 2-isostatic graph is generically 2-isostatic.*

The reverse operation of an X-replacement performed on a generically 2-isostatic graph does in general not result in a generically 2-isostatic graph. For more details and some additional inductive construction techniques, we refer the reader to [23].

Another way of characterizing generically 2-isostatic graphs is due to H. Crapo and uses partitions of a graph into edge disjoint trees.

A *3Tree2 partition* of a graph G is a partition of $E(G)$ into the edge sets of three edge disjoint trees T_0, T_1, T_2 such that each vertex of G belongs to exactly two of the trees.

A 3Tree2 partition is called *proper* if no non-trivial subtrees of distinct trees T_i have the same span, i.e., the same vertex sets (see also Figure 5).

(a) (b)

Figure 5: *A proper* (a) *and a non-proper* (b) *3Tree2 partition.*

Remark 2.2 If a graph G has a 3Tree2 partition, then it satisfies $|E(G)| = 2|V(G)| - 3$. This follows from the presence of exactly two trees at each vertex of G and the fact that for every tree T we have $|E(T)| = |V(T)| - 1$. Moreover, note that a 3Tree2 partition of a graph G is proper if and only if every non-trivial subgraph H of G satisfies the count $|E(H)| \leq 2|V(H)| - 3$ [13].

Theorem 2.5 (Crapo, 1989) *[4] A graph G is generically 2-isostatic if and only if G has a proper 3Tree2 partition.*

2.4 Symmetry in frameworks

Throughout this paper, we will only consider 2-dimensional frameworks. A *symmetry operation* of a framework (G,p) in $\mathbb{R}^2$ is an isometry x of $\mathbb{R}^2$ such that for some $\alpha \in \mathrm{Aut}(G)$, we have $x\big(p(v)\big) = p\big(\alpha(v)\big)$ for all $v \in V(G)$ [9, 17, 16, 18, 19].

The set of all symmetry operations of a framework (G,p) forms a group under composition, called the *point group* of (G,p) [2, 9, 16, 18, 19]. Since translating a framework does not change its rigidity properties, we may assume wlog that the point group of any framework in this paper is a *symmetry group*, i.e., a subgroup of the orthogonal group $O(\mathbb{R}^2)$ [16, 17, 18, 19].

We use the Schoenflies notation for the symmetry operations and symmetry groups considered in this paper, as this is one of the standard notations in the literature about symmetric structures (see [2, 3, 5, 8, 9, 16, 17, 18, 19], for example). In particular, we denote the group generated by the half-turn C_2 about the origin in 2D by $\mathcal{C}_2$, and a group generated by a reflection s in 2D by $\mathcal{C}_s$.

Given a symmetry group S and a graph G, we let $\mathscr{R}_{(G,S)}$ denote the set of all 2-dimensional realizations of G whose point group is either equal to S or contains S as a subgroup [16, 17, 18]. In other words, the set $\mathscr{R}_{(G,S)}$ consists of all realizations (G,p) of G for which there exists a map $\Phi : S \to \mathrm{Aut}(G)$ so that

$$x\big(p(v)\big) = p\big(\Phi(x)(v)\big) \text{ for all } v \in V(G) \text{ and all } x \in S. \tag{2}$$

A framework $(G,p) \in \mathscr{R}_{(G,S)}$ satisfying the equations in (2) for the map $\Phi : S \to \mathrm{Aut}(G)$ is said to be *of type* Φ, and the set of all realizations in $\mathscr{R}_{(G,S)}$ which are of type Φ is denoted by $\mathscr{R}_{(G,S,\Phi)}$ (see again [16, 17, 18, 19] as well as Figure 6).

Remark 2.3 Note that a set $\mathscr{R}_{(G,S)}$ can possibly be empty and that for a non-empty set $\mathscr{R}_{(G,S)}$, it is also possible that $\mathscr{R}_{(G,S,\Phi)} = \emptyset$ for some map $\Phi : S \to \mathrm{Aut}(G)$. For examples and further details see [16, 18].

For the set $\mathscr{R}_{(G,S,\Phi)}$, a symmetry-adapted notion of generic was introduced in [18] (see also [16]). Intuitively, an (S,Φ)-generic realization of a graph G is obtained by placing the vertices of a set of representatives for the symmetry orbits $S(v) = \{\Phi(x)(v) | \, x \in S\}$ into 'generic' positions. The positions for the remaining vertices of G are then uniquely determined by the symmetry constraints imposed by S and Φ. It is shown in [18] that the set of (S,Φ)-generic realizations of a graph G forms an open dense subset of the set $\mathscr{R}_{(G,S,\Phi)}$. Moreover, the infinitesimal rigidity properties are the same for all (S,Φ)-generic realizations of G, as the following theorem shows.

Theorem 2.6 *[16, 18] Let G be a graph, S be a symmetry group, and Φ be a map from S to $Aut(G)$ such that $\mathscr{R}_{(G,S,\Phi)} \neq \emptyset$. The following are equivalent.*

(i) There exists a framework $(G,p) \in \mathscr{R}_{(G,S,\Phi)}$ that is infinitesimally rigid (independent, isostatic);

(ii) every (S, Φ)-generic realization of G is infinitesimally rigid (independent, isostatic).

It follows that infinitesimal rigidity (independence, isostaticity) is an (S, Φ)-generic property. So we define a graph G to be (S, Φ)*-generically infinitesimally rigid (independent, isostatic)* if all realizations of G which are (S, Φ)-generic are infinitesimally rigid (independent, isostatic).

Using techniques from group representation theory, it is shown in [3] that if a symmetric isostatic framework (G, p) belongs to a set $\mathscr{R}_{(G,S,\Phi)}$, where S is a non-trivial symmetry group and $\Phi : S \to \mathrm{Aut}(G)$ is a homomorphism, then (G, p) needs to satisfy certain restrictions on the number of joints and bars that are 'fixed' by various symmetry operations of (G, p) (see Theorem 2.7 and [5, 16, 18, 19]). An alternate way of deriving these restrictions is given in [15].

We say that a joint $\big(v, p(v)\big)$ of (G, p) is *fixed* by a symmetry operation $x \in S$ (with respect to Φ) if $\Phi(x)(v) = v$, and a bar $\{(v_i, p_i), (v_j, p_j)\}$ of (G, p) is *fixed* by x (with respect to Φ) if $\Phi(x)\big(\{v_i, v_j\}\big) = \{v_i, v_j\}$.

The number of joints of (G, p) that are fixed by x (with respect to Φ) is denoted by $j_{\Phi(x)}$ and the number of bars of (G, p) that are fixed by x (with respect to Φ) is denoted by $b_{\Phi(x)}$.

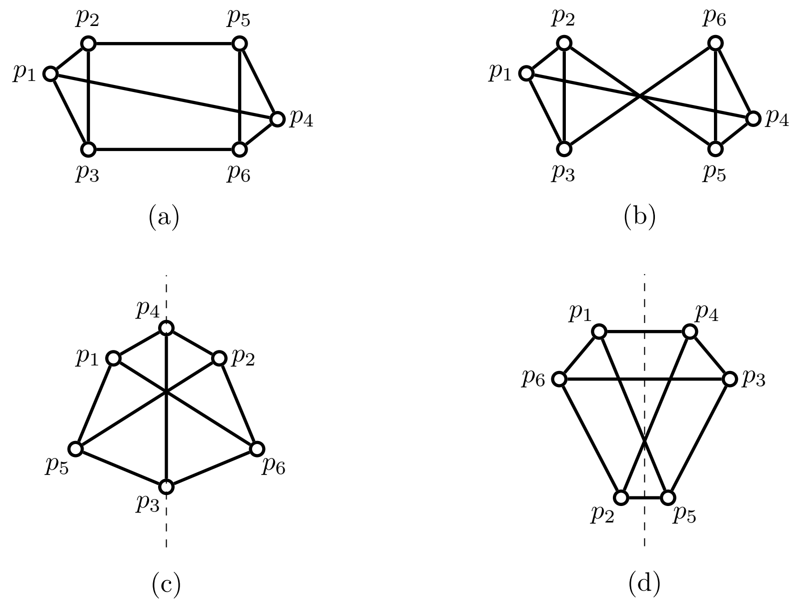


Figure 6: *Examples illustrating Theorem 2.7:* (a,b) *2-dimensional realizations of the graph G_{tp} of the triangular prism in the set $\mathscr{R}_{(G_{tp},\mathcal{C}_2)}$ of different types. While the framework in* (a) *is isostatic, the framework in* (b) *is not, since it has three bars that are fixed by the half-turn in $\mathcal{C}_2$.* (c,d) *2-dimensional realizations of the complete bipartite graph $K_{3,3}$ in the set $\mathscr{R}_{(K_{3,3},\mathcal{C}_s)}$ of different types. While the framework in* (c) *is isostatic, the framework in* (d) *is not, since it has three bars that are fixed by the reflection in $\mathcal{C}_s$.*

Remark 2.4 It follows immediately from these definitions that if a joint of a framework $(G,p) \in \mathscr{R}_{(G,\mathcal{C}_2,\Phi)}$ is fixed by the half-turn C_2, then it must lie at the center of the rotation C_2, i.e., at the origin in $\mathbb{R}^2$. Further, if a bar of (G,p) is fixed by C_2, then it must be centered at the origin.

Similarly, if a joint of a framework $(G,p) \in \mathscr{R}_{(G,\mathcal{C}_s,\Phi)}$ is fixed by the reflection $s \in \mathcal{C}_s$, then it must lie on the mirror line corresponding to s, and if a bar of (G,p) is fixed by s, then it must either lie within the mirror line or perpendicular to and centered at the mirror line corresponding to s [3, 17].

Theorem 2.7 *[3, 16] Let G be a graph, $\Phi : S \to Aut(G)$ be a homomorphism, and (G,p) be an isostatic framework in $\mathscr{R}_{(G,S,\Phi)}$ with the property that the points $p(v)$, $v \in V(G)$, span all of $\mathbb{R}^2$.*

(i) If $S = \mathcal{C}_2$, then $|E(G)| = 2|V(G)| - 3$, $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = 1$;

(ii) if $S = \mathcal{C}_s$, then $|E(G)| = 2|V(G)| - 3$ and $b_{\Phi(s)} = 1$;

In Sections 4.2 and 5.2 we verify the conjectures proposed in [3] that the necessary conditions in Theorem 2.7, together with the Laman conditions, are also *sufficient* for (S,Φ)-generic realizations of G to be isostatic - for both $S = \mathcal{C}_2$ and $S = \mathcal{C}_s$. In addition, we provide Henneberg-type and Crapo-type characterizations of (S,Φ)-generically isostatic graphs for these two groups.

3 Preliminary results and remarks

In our proofs of the symmetric Laman theorems for $\mathcal{C}_2$ and $\mathcal{C}_s$, we will frequently use the following basic lemmas.

Lemma 3.1 *Let G be a graph with $|V(G)| \geq 3$ that satisfies the Laman conditions. Then*

(i) G has a vertex of valence 2 or 3;

(ii) if G has no vertex of valence 2, then G has at least six vertices of valence 3.

Proof. (i) The average valence in G is

$$\frac{2|E(G)|}{|V(G)|} = \frac{2(2|V(G)| - 3)}{|V(G)|} = 4 - \frac{6}{|V(G)|} < 4.$$

Since G satisfies the Laman conditions and $|V(G)| \geq 3$, it is easy to see that G has no vertex of valence 0 or 1.

(ii) Suppose G has no vertex of valence 2 and k vertices of valence 3, where $k < 6$. Then the average valence in G is at least

$$\frac{3k + 4(|V(G)| - k)}{|V(G)|} = 4 - \frac{k}{|V(G)|} > 4 - \frac{6}{|V(G)|}$$

contradicting (i). $\square$

Lemma 3.2 *Let G be a graph that satisfies the Laman conditions and let v be a vertex of G with $N_G(v) = \{v_1, v_2, v_3\}$. Further, let $\alpha \in Aut(G)$ and $\big(v\,\alpha(v)\,\ldots\,\alpha^n(v)\big)$ be the permutation cycle of α containing v. If $\{v, \alpha(v), \ldots, \alpha^n(v)\}$ is an independent set of vertices in G, then*

(i) there exists $\{i, j\} \subseteq \{1, 2, 3\}$ such that for every subgraph H' of $G' = G - \{v, \alpha(v), \ldots, \alpha^n(v)\}$ with $v_i, v_j \in V(H')$, we have $|E(H')| \leq 2|V(H')| - 4$;

(ii) if $\{i, j\} \subseteq \{1, 2, 3\}$ is the only pair for which (i) holds, then $\{\alpha^k(v_i), \alpha^k(v_j)\} \neq \{\alpha^m(v_i), \alpha^m(v_j)\}$ for all $0 \leq k < m \leq n$, and $G' + \big\{\{\alpha^t(v_i), \alpha^t(v_j)\}|\, t = 0, 1, \ldots, n\big\}$ satisfies the Laman conditions.

Proof. (i) It follows from Laman's Theorem (Theorem 2.2) and the Edge 2-Split Theorem (see Proposition 3.3 in [23]) that there exists $\{i_n, j_n\} \subseteq \{1, 2, 3\}$ such that $G_n = G - \{\alpha^n(v)\} + \big\{\{\alpha^n(v_{i_n}), \alpha^n(v_{j_n})\}\big\}$ satisfies the Laman conditions. By the same argument, there exists $\{i_{n-1}, j_{n-1}\} \subseteq \{1, 2, 3\}$ such that $G_{n-1} = G_n - \{\alpha^{n-1}(v)\} + \big\{\{\alpha^{n-1}(v_{i_{n-1}}), \alpha^{n-1}(v_{j_{n-1}})\}\big\}$ satisfies the Laman conditions. Continuing in this fashion, we arrive at a graph G_0 with $V(G_0) = V(G) \setminus \{v, \alpha(v), \ldots, \alpha^n(v)\} = V(G')$ and $E(G_0) = E(G') \cup \big\{\{\alpha^n(v_{i_n}), \alpha^n(v_{j_n})\}, \ldots, \{v_{i_0}, v_{j_0}\}\big\}$ that satisfies the Laman conditions. Therefore, every subgraph H of $G_0 - \big\{\{v_{i_0}, v_{j_0}\}\big\}$ with $v_{i_0}, v_{j_0} \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$. Since $V(G') = V\Big(G_0 - \big\{\{v_{i_0}, v_{j_0}\}\big\}\Big)$ and $E(G') \subseteq E\Big(G_0 - \big\{\{v_{i_0}, v_{j_0}\}\big\}\Big)$, it follows that every subgraph H' of G' with $v_{i_0}, v_{j_0} \in V(H')$ satisfies $|E(H')| \leq 2|V(H')| - 4$.

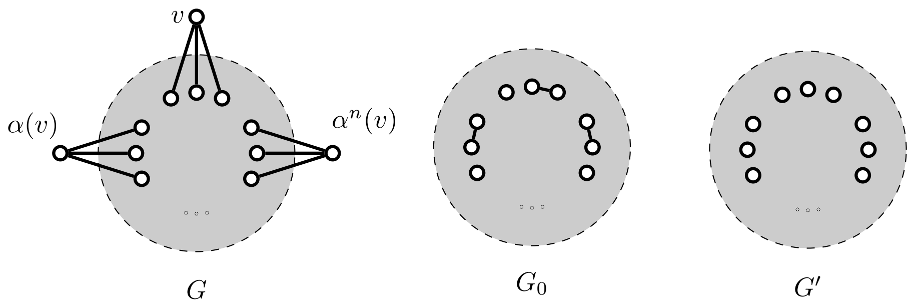


Figure 7: *Illustration of the proof of Lemma 3.2.*

(ii) Wlog we suppose that $\{i, j\} = \{1, 2\}$ is the only pair in $\{1, 2, 3\}$ for which (i) holds. Then there exists a subgraph H_1 of G' with $v_1, v_3 \in V(H_1)$ satisfying $|E(H_1)| = 2|V(H_1)| - 3$ and a subgraph H_2 of G' with $v_2, v_3 \in V(H_2)$ satisfying $|E(H_2)| = 2|V(H_2)| - 3$. Since G' is invariant under α (recall Section 2.1), $\alpha^k(H_1)$ and $\alpha^k(H_2)$ are also subgraphs of G' for all $1 \leq k \leq n$. Moreover, for all $0 \leq k \leq n$, we have

$$\alpha^k(v_1), \alpha^k(v_3) \in V\big(\alpha^k(H_1)\big)$$

$$|E\big(\alpha^k(H_1)\big)| = 2|V\big(\alpha^k(H_1)\big)| - 3$$

and

$$\alpha^k(v_2), \alpha^k(v_3) \in V\big(\alpha^k(H_2)\big)$$

$$|E(\alpha^k(H_2))| = 2|V(\alpha^k(H_2))| - 3.$$

By Laman's Theorem and the Edge 2-Split Theorem (Proposition 3.3 in [23]), there exists $\{i_n, j_n\} \subseteq \{1,2,3\}$ such that $G_n = G - \{\alpha^n(v)\} + \big\{\{\alpha^n(v_{i_n}), \alpha^n(v_{j_n})\}\big\}$ satisfies the Laman conditions. Likewise, for all $0 \leq k \leq n-1$, there exists $\{i_k, j_k\} \subseteq \{1,2,3\}$ such that $G_k = G_{k+1} - \{\alpha^k(v)\} + \big\{\{\alpha^k(v_{i_k}), \alpha^k(v_{j_k})\}\big\}$ satisfies the Laman conditions. Since for all $0 \leq k \leq n$, we have $G' \subseteq G_k$, and hence $\alpha^k(H_1), \alpha^k(H_2) \subseteq G_k$, we must have $\{i_k, j_k\} = \{1, 2\}$ for all k. In particular, $\{\alpha^k(v_1), \alpha^k(v_2)\} \neq \{\alpha^m(v_1), \alpha^m(v_2)\}$ for all $0 \leq k < m \leq n$ and $G_0 = G' + \big\{\{\alpha^t(v_1), \alpha^t(v_2)\} | \, t = 0, 1, \ldots, n\big\}$ satisfies the Laman conditions. $\square$

For both of the groups $\mathcal{C}_2$ and $\mathcal{C}_s$, we will prove a symmetrized version of Crapo's Theorem by using an approach that is in the style of Tay's proof (see [22]) of Crapo's original result. This requires the notion of a 'frame', i.e., a generalized notion of a framework that allows joints to be located at the same point in space, even if their corresponding vertices are adjacent. Formally, for a graph G with $V(G) = \{v_1, \ldots, v_n\}$, a *frame* in $\mathbb{R}^2$ is a triple (G, p, q), where $p : V(G) \to \mathbb{R}^2$ and $q : E(G) \to \mathbb{R}^2 \setminus \{0\}$ are maps with the property that for all $\{v_i, v_j\} \in E(G)$ there exists a scalar $\lambda_{ij} \in \mathbb{R}$ (which is possibly zero) such that $p(v_i) - p(v_j) = \lambda_{ij} q(\{v_i, v_j\})$.

The *generalized rigidity matrix* $\mathbf{R}(G, p, q)$ of a frame (G, p, q) in $\mathbb{R}^2$ is the $|E(G)| \times 2n$ matrix

$$\begin{array}{c} \\ \{v_i, v_j\} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & v_i & & & & v_j & & & \end{array} \\ \left(\begin{array}{ccccccccccc} & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & q(\{v_i, v_j\}) & 0 & \ldots & 0 & -q(\{v_i, v_j\}) & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{array}\right), \end{array}$$

i.e., for each edge $\{v_i, v_j\} \in E(G)$, $\mathbf{R}(G, p, q)$ has the row with $\big(q(\{v_i, v_j\})\big)_1$ and $\big(q(\{v_i, v_j\})\big)_2$ in the columns $2i-1$ and $2i$, $-\big(q(\{v_i, v_j\})\big)_1$ and $-\big(q(\{v_i, v_j\})\big)_2$ in the columns $2(j-1)$ and $2j$, and 0 elsewhere.

We say that (G, p, q) is *independent* if $\mathbf{R}(G, p, q)$ has linearly independent rows.

Remark 3.1 If (G, p, q) is a frame with the property that $p(v_i) \neq p(v_j)$ whenever $\{v_i, v_j\} \in E(G)$, then we obtain the rigidity matrix of the framework (G, p) by multiplying each row of $\mathbf{R}(G, p, q)$ by its corresponding scalar λ_{ij}. Therefore, if (G, p, q) is independent, so is (G, p).

Lemma 3.3 *Let (G, p, q) be an independent frame in $\mathbb{R}^2$ and let $p_t : V(G) \to \mathbb{R}[t] \times \mathbb{R}[t]$ and $q_t : E(G) \to \mathbb{R}[t] \times \mathbb{R}[t]$ be such that (G, p_a, q_a) is a frame in $\mathbb{R}^2$ for every $a \in \mathbb{R}$. If $(G, p_a, q_a) = (G, p, q)$ for $a = 0$, then (G, p_a, q_a) is an independent frame in $\mathbb{R}^2$ for almost all $a \in \mathbb{R}$.*

Proof. Note that the rows of $\mathbf{R}(G, p_t, q_t)$ are linearly dependent (over the quotient field of $\mathbb{R}[t]$) if and only if the determinants of all the $|E(G)| \times |E(G)|$ submatrices of $\mathbf{R}(G, p_t, q_t)$ are identically zero. These determinants are polynomials in t. Thus, the set of all $a \in \mathbb{R}$ with the property that $\mathbf{R}(G, p_a, q_a)$ has a non-trivial row dependency is a variety F whose complement, if non-empty, is a dense open set. Since $a = 0$ is in the complement of F we can conclude that for almost all a, (G, p_a, q_a) is independent. □

Each time Lemma 3.3 is applied in this paper, the polynomials in $\mathbf{R}(G, p_t, q_t)$ are *linear* polynomials in t.

4 Characterizations of $(\mathcal{C}_2, \Phi)$-generically isostatic graphs

4.1 Symmetrized Henneberg moves and 3Tree2 partitions for $\mathcal{C}_2$

We need the following inductive construction techniques to obtain a symmetrized Henneberg's Theorem for $\mathcal{C}_2$.

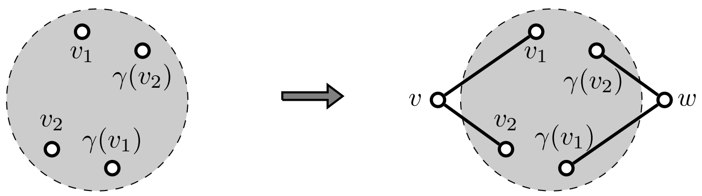


Figure 8: *A* $(\mathcal{C}_2, \Phi)$ *vertex addition of a graph* G*, where* $\Phi(C_2) = \gamma$.

Definition 4.1 Let G be a graph, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(G)$ be a homomorphism. Let v_1, v_2 be two distinct vertices of G and $v, w \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w\}$ and $E(\widehat{G}) = E(G) \cup \big\{\{v, v_1\}, \{v, v_2\}, \{w, \Phi(C_2)(v_1)\}, \{w, \Phi(C_2)(v_2)\}\big\}$ is called a $(\mathcal{C}_2, \Phi)$ *vertex addition (by* (v, w)*) of* G.

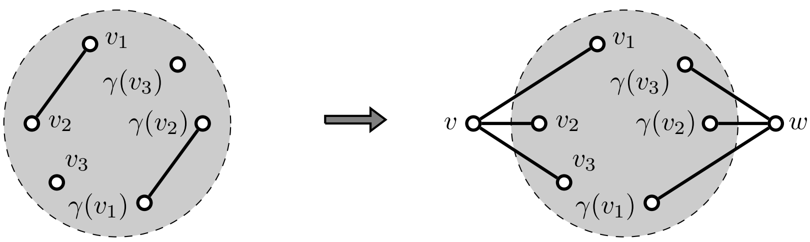


Figure 9: *A* $(\mathcal{C}_2, \Phi)$ *edge split of a graph* G*, where* $\Phi(C_2) = \gamma$.

Definition 4.2 Let G be a graph, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(G)$ be a homomorphism. Let v_1, v_2, v_3 be three distinct vertices of G such that $\{v_1, v_2\} \in E(G)$ and $\{v_1, v_2\}$ is not fixed by $\Phi(C_2)$ and let $v, w \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w\}$ and $E(\widehat{G}) = \big(E(G) \setminus \big\{\{v_1, v_2\}, \{\Phi(C_2)(v_1), \Phi(C_2)(v_2)\}\big\}\big) \cup$

$\big\{\{v, v_i\} | \, i = 1, 2, 3\big\} \cup \big\{\{w, \Phi(C_2)(v_i)\} | \, i = 1, 2, 3\big\}$ is called a $(\mathcal{C}_2, \Phi)$ *edge split (on* $(\{v_1, v_2\}, \{\Phi(C_2)(v_1), \Phi(C_2)(v_2)\}); (v, w))$ *of* G.

Remark 4.1 Each of the constructions in Definitions 4.1 and 4.2 has the property that if the graph G satisfies the Laman conditions, then so does $\widehat{G}$. This follows from Theorems 2.2 and 2.3 and the fact that we can obtain a $(\mathcal{C}_2, \Phi)$ vertex addition of G by a sequence of two vertex 2-additions, and a $(\mathcal{C}_2, \Phi)$ edge split of G by a sequence of two edge 2-splits.

In order to extend Crapo's Theorem to $\mathcal{C}_2$ we need the following symmetrized definition of a 3Tree2 partition.

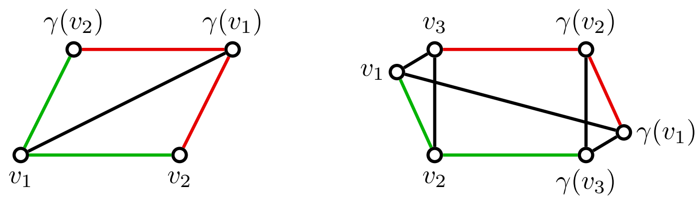


Figure 10: $(\mathcal{C}_2, \Phi)$ *3Tree2 partitions of graphs, where* $\Phi(C_2) = \gamma$*. The edges in black color represent edges of the invariant trees.*

Definition 4.3 Let G be a graph, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(G)$ be a homomorphism. A $(\mathcal{C}_2, \Phi)$ *3Tree2 partition* of G is a 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$ of G such that $\Phi(C_2)(T_1) = T_2$ and $\Phi(C_2)(T_0) = T_0$. The tree T_0 is called the *invariant tree* of $\{E(T_0), E(T_1), E(T_2)\}$.

4.2 The main result for $\mathcal{C}_2$

Theorem 4.1 *Let* G *be a graph with* $|V(G)| \geq 2$, $\mathcal{C}_2 = \{Id, C_2\}$ *be the half-turn symmetry group in dimension* 2*, and* $\Phi : \mathcal{C}_2 \to Aut(G)$ *be a homomorphism. The following are equivalent:*

(i) $\mathscr{R}_{(G,\mathcal{C}_2,\Phi)} \neq \emptyset$ *and* G *is* $(\mathcal{C}_2, \Phi)$*-generically isostatic;*

(ii) $|E(G)| = 2|V(G)| - 3$, $|E(H)| \leq 2|V(H)| - 3$ *for all* $H \subseteq G$ *with* $|V(H)| \geq 2$ *(Laman conditions),* $j_{\Phi(C_2)} = 0$*, and* $b_{\Phi(C_2)} = 1$*;*

(iii) there exists a $(\mathcal{C}_2, \Phi)$ *construction sequence*

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G, \Phi)$$

such that

(a) G_{i+1} *is a* $(\mathcal{C}_2, \Phi_i)$ *vertex addition or a* $(\mathcal{C}_2, \Phi_i)$ *edge split of* G_i *with* $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}, w_{i+1}\}$ *for all* $i = 0, 1, \ldots, k-1$*;*

(b) $\Phi_0 : \mathcal{C}_2 \to Aut(K_2)$ *is a non-trivial homomorphism and for all* $i = 0, 1, \dots, k-1$, $\Phi_{i+1} : \mathcal{C}_2 \to Aut(G_{i+1})$ *is the homomorphism defined by* $\Phi_{i+1}(C_2)|_{V(G_i)} = \Phi_i(C_2)$ *and* $\Phi_{i+1}(C_2)|_{\{v_{i+1}, w_{i+1}\}} = (v_{i+1}\, w_{i+1})$*;*

(iv) *G has a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition whose invariant tree is a spanning tree of G.*

We break the proof of this result up into four Lemmas.

Lemma 4.2 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to Aut(G)$ be a homomorphism. If $\mathscr{R}_{(G,\mathcal{C}_2,\Phi)} \neq \emptyset$ and G is $(\mathcal{C}_2, \Phi)$-generically isostatic, then G satisfies the Laman conditions and we have $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = 1$.*

Proof. The result is trivial if $|V(G)| = 2$, and it follows from Laman's Theorem (Theorem 2.2), Theorem 2.7, and Remark 2.1 if $|V(G)| > 2$. $\square$

Lemma 4.3 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to Aut(G)$ be a homomorphism. If G satisfies the Laman conditions and we also have $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = 1$, then there exists a $(\mathcal{C}_2, \Phi)$ construction sequence for G.*

Proof. We employ induction on $|V(G)|$. Note first that if for a graph G, there exists a homomorphism $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(G)$ such that $j_{\Phi(C_2)} = 0$, then $|V(G)| \equiv 0 \pmod 2$. The only graph with two vertices that satisfies the Laman conditions is the graph K_2 and if $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(K_2)$ is a homomorphism such that $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = 1$, then Φ is clearly a non-trivial homomorphism. This proves the base case.

So we let $n > 2$ and we assume that the result holds for all graphs with n or fewer than n vertices.

Let G be a graph with $|V(G)| = n + 2$ that satisfies the Laman conditions and suppose $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = 1$ for a homomorphism $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(G)$. In the following, we denote $\Phi(C_2)$ by γ. By Lemma 3.1, G has a vertex of valence 2 or 3.

We assume first that G has a vertex v of valence 2, say $N_G(v) = \{v_1, v_2\}$. Then $\gamma(v) \neq v$ since $j_\gamma = 0$. Also, $\gamma(v) \neq v_1, v_2$, for otherwise, say wlog $\gamma(v) = v_1$, the graph $G' = G - \{v, \gamma(v)\}$ satisfies

$$|E(G')| = |E(G)| - 3 = 2|V(G)| - 6 = 2|V(G')| - 2,$$

contradicting the fact that G satisfies the Laman conditions, since $|V(G')| \geq 2$.

Thus, the edges $\{v, v_1\}, \{v, v_2\}, \{\gamma(v), \gamma(v_1)\}, \{\gamma(v), \gamma(v_2)\}$ are pairwise distinct. Therefore,

$$|E(G')| = |E(G)| - 4 = 2|V(G)| - 7 = 2|V(G')| - 3.$$

Also, for $H \subseteq G'$ with $|V(H)| \geq 2$, we have $H \subseteq G$, and hence

$$|E(H)| \leq 2|V(H)| - 3,$$

so that G' satisfies the Laman conditions.

Let $\Phi' : \mathcal{C}_2 \to \mathrm{Aut}(G')$ be the homomorphism with $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in \mathcal{C}_2$. Then we have $j_{\Phi'(C_2)} = 0$ and $b_{\Phi'(C_2)} = 1$, because none of the edges we removed was fixed by γ. Thus, by the induction hypothesis, there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 4.1 (*iii*). Since G is a $(\mathcal{C}_2, \Phi')$ vertex addition of G' with $V(G) = V(G') \cup \{v, \gamma(v)\}$,

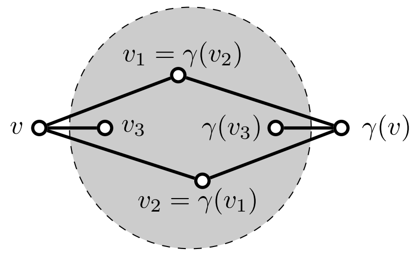
$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G', \Phi'), (G, \Phi)$$

is a sequence with the desired properties.

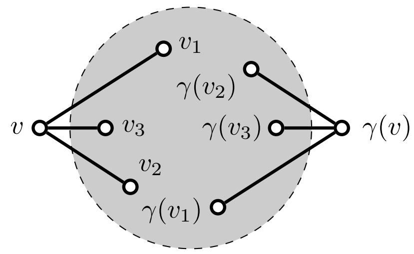
Suppose now that G has a vertex of valence 3 and no vertex of valence 2. Then, by Lemma 3.1, G has at least six vertices of valence 3. Therefore, since $b_\gamma = 1$, there exists a vertex $v \in V(G)$ with $val_G(v) = 3$, say $N_G(v) = \{v_1, v_2, v_3\}$, and $\{v, \gamma(v)\} \notin E(G)$. Since $j_\gamma = 0$, we have $\gamma(v_i) \neq v_i$ for all $i = 1, 2, 3$, and hence we only need to consider the following two cases (see also Figure 11):

Case 1: $v_s = \gamma(v_t)$ for some $\{s, t\} \subseteq \{1, 2, 3\}$. Wlog we assume $v_1 = \gamma(v_2)$. Then we also have $v_2 = \gamma(v_1)$.

Case 2: The six vertices $v_i, \gamma(v_i)$, $i = 1, 2, 3$, are all pairwise distinct.



(Case 1)



(Case 2)

Figure 11: *If a graph G satisfies the conditions in Theorem 4.1 (ii) and has a vertex v of valence 3, then G is a graph of one of the types depicted above.*

Case 1: Since $\gamma(\{v_1, v_2\}) = \{v_1, v_2\}$, it follows from Lemma 3.2 (*i*) and (*ii*) that there exists $\{i, j\} \subseteq \{1, 2, 3\}$ with $\{i, j\} \neq \{1, 2\}$, say wlog $\{i, j\} = \{1, 3\}$, such that for every subgraph H of $G' = G - \{v, \gamma(v)\}$ with $v_i, v_j \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 4$. Since G' is invariant under γ, every subgraph H of G' with $\gamma(v_1), \gamma(v_3) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$.

Note that $\{v_1, v_3\}$ and $\{\gamma(v_1), \gamma(v_3)\}$ are two distinct pairs of vertices (though not edges, by the above counts), for otherwise we have $\gamma(v_1) = v_3$ (since $j_\gamma = 0$), and hence $v_3 = v_2$, a contradiction.

We claim that $\widetilde{G} = G' + \big\{\{v_1, v_3\}, \{\gamma(v_1), \gamma(v_3)\}\big\}$ satisfies the Laman conditions. We clearly have

$$|E(\widetilde{G})| = |E(G')| + 2 = |E(G)| - 4 = 2|V(G)| - 7 = 2|V(\widetilde{G})| - 3.$$

Suppose there exists a subgraph H of G' with $v_1, v_3, \gamma(v_1), \gamma(v_3) \in V(H)$ and $|E(H)| = 2|V(H)| - 4$. Then the subgraph $\widehat{H}$ of G' with $V(\widehat{H}) = V(H) \cup$

$\{v, \gamma(v)\}$ and $E(\widehat{H}) = E(H) \cup \big\{\{v, v_i\} | \, i = 1, 2, 3\big\} \cup \big\{\{\gamma(v), \gamma(v_i)\} | \, i = 1, 2, 3\big\}$ satisfies

$$|E(\widehat{H})| = |E(H)| + 6 = 2|V(H)| + 2 = 2|V(\widehat{H})| - 2,$$

contradicting the fact that G satisfies the Laman conditions.

Therefore, every subgraph H of G' with $v_1, v_3, \gamma(v_1), \gamma(v_3) \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 5$.

Thus, as claimed, the graph $\widetilde{G} = G' + \big\{\{v_1, v_3\}, \{\gamma(v_1), \gamma(v_3)\}\big\}$ satisfies the Laman conditions.

Further, if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_2$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_2$ and $\widetilde{\Phi} : \mathcal{C}_2 \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $j_{\widetilde{\Phi}(C_2)} = 0$ and $b_{\widetilde{\Phi}(C_2)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 4.1 *(iii)*. Since G is a $(\mathcal{C}_2, \widetilde{\Phi})$ edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \gamma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case 2: By Lemma 3.2 *(i)*, there exists $\{i, j\} \subseteq \{1, 2, 3\}$ such that for every subgraph H of $G' = G - \{v, \gamma(v)\}$ with $v_i, v_j \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 4$. Suppose first that wlog $\{i, j\} = \{1, 2\}$ is the only pair in $\{1, 2, 3\}$ with this property. Then, by Lemma 3.2 *(ii)*, $\widetilde{G} = G' + \big\{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}\big\}$ satisfies the Laman conditions.

Further, if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_2$ then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_2$ and $\widetilde{\Phi} : \mathcal{C}_2 \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $j_{\widetilde{\Phi}(C_2)} = 0$ and $b_{\widetilde{\Phi}(C_2)} = 1$ it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 4.1 *(iii)*. Since G is a $(\mathcal{C}_2, \widetilde{\Phi})$ edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \gamma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Suppose now that there exist two distinct pairs in $\{1, 2, 3\}$, say wlog $\{1, 2\}$ and $\{1, 3\}$, such that every subgraph H of G' with $v_1, v_2 \in V(H)$ or $v_1, v_3 \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$. Then every subgraph H of G' with $\gamma(v_1), \gamma(v_2) \in V(H)$ or $\gamma(v_1), \gamma(v_3) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$, because G' is invariant under γ.

Suppose there exists a subgraph H of G' with $v_i, \gamma(v_i) \in V(H)$ for all $i = 1, 2, 3$ and $|E(H)| = 2|V(H)| - 4$. Then the subgraph $\widehat{H}$ of G with $V(\widehat{H}) = V(H) \cup \{v, \gamma(v)\}$ and $E(\widehat{H}) = E(H) \cup \big\{\{v, v_i\} | \, i = 1, 2, 3\big\} \cup \big\{\{\gamma(v), \gamma(v_i)\} | \, i = 1, 2, 3\big\}$ satisfies

$$|E(\widehat{H})| = |E(H)| + 6 = 2|V(H)| + 2 = 2|V(\widehat{H})| - 2,$$

contradicting the fact that G satisfies the Laman conditions.

Thus, every subgraph H of G' with $v_i, \gamma(v_i) \in V(H)$ for all $i = 1, 2, 3$ satisfies the count $|E(H)| \leq 2|V(H)| - 5$.

Now, suppose there exist subgraphs H_1 and H_2 of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H_1)$ and $v_1, v_3, \gamma(v_1), \gamma(v_3) \in V(H_2)$ satisfying $|E(H_i)| = 2|V(H_i)| - 4$ for $i = 1, 2$. Then there also exist $\gamma(H_1) \subseteq G'$ and $\gamma(H_2) \subseteq G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V\big(\gamma(H_1)\big)$ and $v_1, v_3, \gamma(v_1), \gamma(v_3) \in V\big(\gamma(H_2)\big)$ satisfying $|E\big(\gamma(H_i)\big)| = 2|V\big(\gamma(H_i)\big)| - 4$ for $i = 1, 2$. Let $H_i' = H_i \cup \gamma(H_i)$ for $i = 1, 2$. Then

$$\begin{aligned} |E(H_1')| &= |E(H_1)| + |E\big(\gamma(H_1)\big)| - |E\big(H_1 \cap \gamma(H_1)\big)| \\ &\geq 2|V(H_1)| - 4 + 2|V\big(\gamma(H_1)\big)| - 4 - (2|V\big(H_1 \cap \gamma(H_1)\big)| - 4) \\ &= 2|V(H_1')| - 4, \end{aligned}$$

because $H_1 \cap \gamma(H_1)$ is a subgraph of G' with $v_1, v_2 \in V\big(H_1 \cap \gamma(H_1)\big)$. Since H_1' is also a subgraph of G' with $v_1, v_2 \in V(H_1')$, it follows that

$$|E(H_1')| = 2|V(H_1')| - 4.$$

Similarly,

$$|E(H_2')| = 2|V(H_2')| - 4.$$

So, both H_1' and H_2' have an even number of edges. Moreover, both of these graphs are invariant under γ, which says that neither $E(H_1')$ nor $E(H_2')$ contains the edge e of G that is fixed by γ.

Note that $H_1' \cap H_2'$ is a subgraph of G with $v_1, \gamma(v_1) \in V(H_1' \cap H_2')$. Therefore, we have

$$|E(H_1' \cap H_2')| \leq 2|V(H_1' \cap H_2')| - 3,$$

because G satisfies the Laman conditions. Since $H_1' \cap H_2'$ is also invariant under γ and $E(H_1' \cap H_2')$ does not contain the edge e, $|E(H_1' \cap H_2')|$ is an even number. The above upper bound for $|E(H_1' \cap H_2')|$ can therefore be lowered to

$$|E(H_1' \cap H_2')| \leq 2|V(H_1' \cap H_2')| - 4.$$

Thus, $H' = H_1' \cup H_2'$ satisfies

$$\begin{aligned} |E(H')| &= |E(H_1')| + |E(H_2')| - |E(H_1' \cap H_2')| \\ &\geq 2|V(H_1')| - 4 + 2|V(H_2')\big)| - 4 - (2|V(H_1' \cap H_2')| - 4) \\ &= 2|V(H')| - 4. \end{aligned}$$

This is a contradiction, because H' is a subgraph of G' with $v_i, \gamma(v_i) \in V(H')$ for all $i = 1, 2, 3$.

So, for $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{1, 3\}$, say wlog $\{i, j\} = \{1, 2\}$, we have that every subgraph H of G' with $v_i, v_j, \gamma(v_i), \gamma(v_j) \in V(H)$ satisfies $|E(H)| = 2|V(H)| - 5$.

Thus, $\widetilde{G} = G' + \big\{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}\big\}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_2$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_2$ and $\widetilde{\Phi} : \mathcal{C}_2 \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $j_{\widetilde{\Phi}(C_2)} = 0$ and $b_{\widetilde{\Phi}(C_2)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 4.1 (iii). Since G is a $(\mathcal{C}_2, \widetilde{\Phi})$ edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \gamma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties. $\square$

Lemma 4.4 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to Aut(G)$ be a homomorphism. If there exists a $(\mathcal{C}_2, \Phi)$ construction sequence for G, then G has a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition whose invariant tree is a spanning tree of G.*

Proof. We proceed by induction on $|V(G)|$. Let $V(K_2) = \{v_1, v_2\}$ and let $\Phi : \mathcal{C}_2 \to K_2$ be the homomorphism defined by $\Phi(C_2) = (v_1\, v_2)$. Then K_2 has the proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$, where $T_0 = \langle\{v_1, v_2\}\rangle$, $T_1 = \langle\{v_1\}\rangle$, and $T_2 = \langle\{v_2\}\rangle$. Clearly, T_0 is a spanning tree of K_2. This proves the base case.

Assume, then, that the result holds for all graphs with n or fewer than n vertices, where $n > 2$.

Let G be a graph with $|V(G)| = n + 2$ and let $\Phi : \mathcal{C}_2 \to \text{Aut}(G)$ be a homomorphism such that there exists a $(\mathcal{C}_2, \Phi)$ construction sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G, \Phi)$$

satisfying the conditions in Theorem 4.1 (iii). By Remark 4.1, G satisfies the Laman conditions, and hence, by Remark 2.2, any 3Tree2 partition of G must be proper. Therefore, it suffices to show that G has some $(\mathcal{C}_2, \Phi)$ 3Tree2 partition whose invariant tree is a spanning tree of G. In the following, we denote $\Phi(C_2)$ by γ.

By the induction hypothesis, G_{k-1} has a $(\mathcal{C}_2, \Phi_{k-1})$ 3Tree2 partition $\big\{E\big(T_0^{(k-1)}\big), E\big(T_1^{(k-1)}\big), E\big(T_2^{(k-1)}\big)\big\}$ whose invariant tree $T_0^{(k-1)}$ is a spanning tree of G_{k-1}.

Suppose first that G is a $(\mathcal{C}_2, \Phi_{k-1})$ vertex addition by $(v\, w)$ of G_{k-1} with $N_G(v) = \{v_1, v_2\}$. Since $\Phi_{k-1}(C_2) = \gamma|_{V(G_{k-1})}$, we have $N_G(w) = \{\gamma(v_1), \gamma(v_2)\}$. Note that $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V\big(T_0^{(k-1)}\big)$, because $T_0^{(k-1)}$ is a spanning tree of G_{k-1}. Also, v_2 belongs to either $T_1^{(k-1)}$ or $T_2^{(k-1)}$, say wlog $v_2 \in V\big(T_1^{(k-1)}\big)$. Then $\gamma(v_2) \in V\big(T_2^{(k-1)}\big)$. So, if we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_0^{(k)}\big) &= E\big(T_0^{(k-1)}\big) \cup \big\{\{v, v_1\}, \{w, \gamma(v_1)\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_2\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{w\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{w, \gamma(v_2)\}\big\}, \end{aligned}$$

then $\{E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)})\}$ is a $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G whose invariant tree $T_0^{(k)}$ is a spanning tree of G.

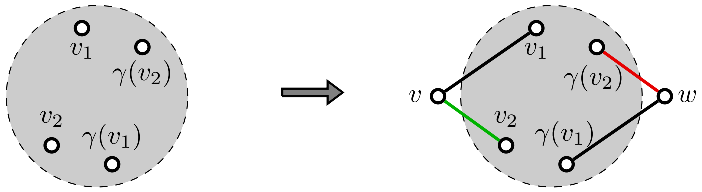


Figure 12: *Construction of a $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_2, \Phi_{k-1})$ vertex addition of G_{k-1}. The edges in black color represent edges of the invariant tree $T_0^{(k)}$.*

Suppose next that G is a $(\mathcal{C}_2, \Phi_{k-1})$ edge split on $(\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}); (v, w)$ of G_{k-1} with $E(G_k) = \big(E(G_{k-1}) \setminus \{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}\}\big) \cup \{\{v, v_i\} | \, i = 1, 2, 3\} \cup \{\{w, \gamma(v_i)\} | \, i = 1, 2, 3\}$. First, we assume that $\{v_1, v_2\} \in E(T_0^{(k-1)})$, and hence $\{\gamma(v_1), \gamma(v_2)\} \in E(T_0^{(k-1)})$. Note that v_3 belongs to either $T_1^{(k-1)}$ or $T_2^{(k-1)}$, say wlog $v_3 \in V(T_1^{(k-1)})$. Then $\gamma(v_3) \in V(T_2^{(k-1)})$. So if we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V(T_0^{(k)}) &= V(T_0^{(k-1)}) \cup \{v, w\} \\ E(T_0^{(k)}) &= \big(E(T_0^{(k-1)}) \setminus \{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}\}\big) \\ &\quad \cup \{\{v, v_1\}, \{v, v_2\}, \{w, \gamma(v_1)\}, \{w, \gamma(v_2)\}\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V(T_1^{(k)}) &= V(T_1^{(k-1)}) \cup \{v\} \\ E(T_1^{(k)}) &= E(T_1^{(k-1)}) \cup \{\{v, v_3\}\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V(T_2^{(k)}) &= V(T_2^{(k-1)}) \cup \{w\} \\ E(T_2^{(k)}) &= E(T_2^{(k-1)}) \cup \{\{w, \gamma(v_3)\}\}, \end{aligned}$$

then $\{E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)})\}$ is a $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G whose invariant tree $T_0^{(k)}$ is a spanning tree of G.

Assume now that $\{v_1, v_2\} \notin E(T_0^{(k-1)})$. Then wlog $\{v_1, v_2\} \in E(T_1^{(k-1)})$ and $\{\gamma(v_1), \gamma(v_2)\} \in E(T_2^{(k-1)})$. In this case we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V(T_0^{(k)}) &= V(T_0^{(k-1)}) \cup \{v, w\} \\ E(T_0^{(k)}) &= E(T_0^{(k-1)}) \cup \{\{v, v_3\}, \{w, \gamma(v_3)\}\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V(T_1^{(k)}) &= V(T_1^{(k-1)}) \cup \{v\} \\ E(T_1^{(k)}) &= \big(E(T_1^{(k-1)}) \setminus \{\{v_1, v_2\}\}\big) \cup \{\{v, v_1\}, \{v, v_2\}\}, \end{aligned}$$

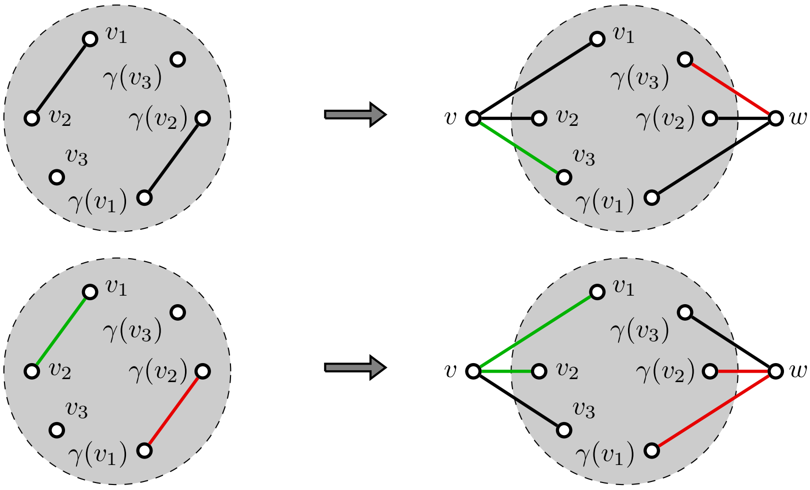


Figure 13: *Construction of a $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_2, \Phi_{k-1})$ edge split of G_{k-1}. The edges in black color represent edges of the invariant trees.*

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{w\} \\ E\big(T_2^{(k)}\big) &= \big(E\big(T_2^{(k-1)}\big) \setminus \{\{\gamma(v_1), \gamma(v_2)\}\}\big) \\ &\quad \cup \big\{\{w, \gamma(v_1)\}, \{w, \gamma(v_2)\}\big\}. \end{aligned}$$

Then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G whose invariant tree $T_0^{(k)}$ is a spanning tree of G. $\square$

Lemma 4.5 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to Aut(G)$ be a homomorphism. If G has a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition whose invariant tree is a spanning tree of G, then $\mathscr{R}_{(G, \mathcal{C}_2, \Phi)} \neq \emptyset$ and G is $(\mathcal{C}_2, \Phi)$-generically isostatic.*

Proof. Suppose G has a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$ whose invariant tree T_0 is a spanning tree of G. By Theorem 2.6, it suffices to find some framework $(G, p) \in \mathscr{R}_{(G, \mathcal{C}_2, \Phi)}$ that is isostatic. Since G has a 3Tree2 partition, G satisfies the count $|E(G)| = 2|V(G)| - 3$, and hence, by Theorem 2.1, it suffices to find a map $p : V(G) \to \mathbb{R}^2$ such that $(G, p) \in \mathscr{R}_{(G, \mathcal{C}_2, \Phi)}$ is independent. In the following, we again denote $\Phi(C_2)$ by γ.

Let V_i be the set of vertices of G that are not in $V(T_i)$ for $i = 0, 1, 2$. Then $V_0 = \emptyset$ since T_0 is a spanning tree of G. Let $e_1 = (0, 0)$ and $e_2 = (0, 1)$ and let (G, p, q) be the frame with $p : V(G) \to \mathbb{R}^2$ and $q : E(G) \to \mathbb{R}^2$ defined by

$$\begin{aligned} p(v) &= e_i \quad \text{if } v \in V_i \\ q(b) &= \begin{cases} (0,1) & \text{if } b \in E(T_0) \\ (-1,0) & \text{if } b \in E(T_1) \\ (1,0) & \text{if } b \in E(T_2) \end{cases} . \end{aligned}$$

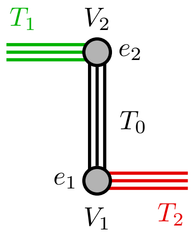


Figure 14: *The frame* (G, p, q).

We claim that the generalized rigidity matrix $\mathbf{R}(G, p, q)$ has linearly independent rows. To see this, we first rearrange the columns of $\mathbf{R}(G, p, q)$ in such a way that we obtain the matrix $\mathbf{R}'(G, p, q)$ which has the $(2i-1)^{st}$ column of $\mathbf{R}(G, p, q)$ in its i^{th} column and the $(2i)^{th}$ column of $\mathbf{R}(G, p, q)$ in its $(|V(G)| + i)^{th}$ column for $i = 1, 2, \ldots, |V(G)|$. Let F_b denote the row vector of $\mathbf{R}'(G, p, q)$ that corresponds to the edge $b \in E(G)$. We then rearrange the rows of $\mathbf{R}'(G, p, q)$ in such a way that we obtain the matrix $\mathbf{R}''(G, p, q)$ which has the vectors F_b with $b \in E(T_0)$ in the rows $1, 2, \ldots, |E(T_0)|$, the vectors F_b with $b \in E(T_1)$ in the following $|E(T_1)|$ rows, and the vectors F_b with $b \in E(T_2)$ in the last $|E(T_2)|$ rows. So $\mathbf{R}''(G, p, q)$ is of the form

$$\left(\begin{array}{ccccc|cccc}
 & & & & & 1 & & -1 & \\
 & & \mathbf{0} & & & & \vdots & & \\
 & & & & & & 1 & & -1 \\
\hline
 & -1 & & 1 & & & & & \\
 & & \vdots & & & & \mathbf{0} & & \\
 & & -1 & 1 & & & & & \\
\hline
1 & -1 & & & & & & & \\
 & & \vdots & & & & \mathbf{0} & & \\
 & 1 & & & -1 & & & &
\end{array}\right).$$

Clearly, $\mathbf{R}(G, p, q)$ has a row dependency if and only if $\mathbf{R}''(G, p, q)$ does. Suppose $\mathbf{R}''(G, p, q)$ has a row dependency of the form

$$\sum_{b \in E(G)} \alpha_b F_b = 0,$$

where $\alpha_b \neq 0$ for some $b \in E(T_0)$. Since T_0 is a tree, it follows that

$$\sum_{b \in E(T_0)} \alpha_b F_b \neq 0.$$

Thus, there exists a vertex $v_r \in V(T_0)$, $r \in \{1, 2, \ldots, |V(G)|\}$, such that

$$\sum_{b \in E(T_0)} \alpha_b (F_b)_{|V(G)|+r} = C \neq 0,$$

and hence

$$\sum_{b \in E(G)} \alpha_b (F_b)_{|V(G)|+r} = C \neq 0,$$

a contradiction.

So, suppose

$$\sum_{b \in E(T_1) \cup E(T_2)} \alpha_b F_b = 0,$$

where $\alpha_b \neq 0$ for some $b \in E(T_1) \cup E(T_2)$, say wlog $b \in E(T_1)$. Since T_1 is a tree, we have

$$\sum_{b \in E(T_1)} \alpha_b F_b \neq 0,$$

and hence there exists a vertex $v_s \in V(T_1)$, $s \in \{1, 2, \dots, |V(G)|\}$, such that

$$\sum_{b \in E(T_1)} \alpha_b (F_b)_s = D \neq 0.$$

Then

$$\sum_{b \in E(T_1) \cup E(T_2)} \alpha_b (F_b)_s = D \neq 0,$$

because the trees T_1 and T_2 have disjoint vertex sets. This is again a contradiction, and hence the frame (G, p, q) is indeed independent.

Now, if (G, p) is not a framework, then we need to symmetrically pull apart those joints of (G, p, q) that have the same location e_i in $\mathbb{R}^2$ and whose vertices are adjacent. So, wlog suppose $|V_1| \geq 2$. Then, since G has the $(\mathcal{C}_2, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$, we have $\gamma(V_1) = V_2$, and hence $|V_1| = |V_2| \geq 2$. Since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, one of $\langle V_1 \rangle \cap T_i$, $i = 0, 2$, is not connected. Note that $T_2 \subseteq \langle V_1 \rangle$, and hence $\langle V_1 \rangle \cap T_2$ is connected. Thus, $\langle V_1 \rangle \cap T_0$ is not connected. Therefore, $\langle V_2 \rangle \cap T_0$ is also not connected. Let A be the set of vertices in one of the components of $\langle V_1 \rangle \cap T_0$ and $\gamma(A)$ be the set of vertices in the corresponding component of $\langle V_2 \rangle \cap T_0$. For $t \in \mathbb{R}$, we define $p_t : V(G) \to \mathbb{R}^2$ and $q_t : E(G) \to \mathbb{R}^2$ by

$$\begin{aligned} p_t(v) &= \begin{cases} (t, 0) & \text{if } v \in A \\ (-t, 1) & \text{if } v \in \gamma(A) \\ p(v) & \text{otherwise} \end{cases} \\ q_t(b) &= \begin{cases} (-t, 1) & \text{if } b \in E_{A, V_2 \setminus \gamma(A)} \\ (-2t, 1) & \text{if } b \in E_{A, \gamma(A)} \\ (-t, 1) & \text{if } b \in E_{\gamma(A), V_1 \setminus A} \\ q(b) & \text{otherwise} \end{cases}, \end{aligned}$$

where for disjoint sets $X, Y \in V(G)$, $E_{X,Y}$ denotes the set of edges of G incident with a vertex in X and a vertex in Y. Then $(G, p_t, q_t) = (G, p, q)$ if $t = 0$. Therefore, by Lemma 3.3, there exists a $t_0 \in \mathbb{R}$, $t_0 \neq 0$, such that the frame (G, p_{t_0}, q_{t_0}) is independent.

If (G, p_{t_0}) is still not a framework, then $V_1 \setminus A$ or A, say wlog $V_1 \setminus A$, contains at least two vertices that are adjacent in G, as does $V_2 \setminus \gamma(A)$. Since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, one of $\langle V_1 \setminus A \rangle \cap T_i$, $i = 0, 2$ is not connected.

If $\langle V_1 \setminus A \rangle \cap T_0$ is not connected, then $\langle V_2 \setminus \gamma(A) \rangle \cap T_0$ is also not connected. Let B and $\gamma(B)$ be the vertex sets of components of $\langle V_1 \setminus A \rangle \cap T_0$ and $\langle V_2 \setminus \gamma(A) \rangle \cap T_0$, respectively. Then we can pull apart the vertices of B from $(V_1 \setminus A) \setminus B$ and the vertices of $\gamma(B)$ from $(V_2 \setminus \gamma(A)) \setminus \gamma(B)$ in an analogous way as before in order to obtain a new independent frame.

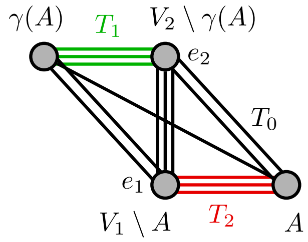


Figure 15: *The frame* (G, p_t, q_t).

If $\langle V_1 \setminus A \rangle \cap T_2$ and $\langle V_2 \setminus \gamma(A) \rangle \cap T_1$ are not connected, then we let B and $\gamma(B)$ be the vertex sets of components of $\langle V_1 \setminus A \rangle \cap T_2$ and $\langle V_2 \setminus \gamma(A) \rangle \cap T_1$, respectively. In this case, we may pull apart the vertices of B from $(V_1 \setminus A) \setminus B$ in direction of the vector $(0, -1)$ and the vertices of $\gamma(B)$ from $(V_2 \setminus \gamma(A)) \setminus \gamma(B)$ in direction of the vector $(0, 1)$ to obtain a new independent frame.

This process can be continued until we obtain an independent frame $(G, \hat{p}, \hat{q})$ with $\hat{p}(u) \neq \hat{p}(v)$ for all $\{u, v\} \in E(G)$. Then, by Remark 3.1, $(G, \hat{p})$ is an independent framework and the right translation of $(G, \hat{p})$ yields an independent framework in the set $\mathscr{R}_{(G, \mathcal{C}_2, \Phi)}$. $\square$

Lemmas 4.2, 4.3, 4.4, and 4.5 provide a complete proof for Theorem 4.1

Remark 4.2 Let G be a graph with $|V(G)| \geq 3$, $\mathcal{C}_2 = \{Id, C_2\}$ be the half-turn symmetry group in dimension 2, and $\Phi : \mathcal{C}_2 \to \mathrm{Aut}(G)$ be a homomorphism. If G is $(\mathcal{C}_2, \Phi)$-generically isostatic, then we can modify the construction in the proof of Lemma 4.4 to obtain proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partitions of G whose invariant trees are not spanning. In particular, it can be shown that if G is $(\mathcal{C}_2, \Phi)$-generically isostatic, then there must exist a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G whose invariant tree is just a single edge of G. However, the existence of a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition of G whose invariant tree is not spanning is not sufficient for G to be $(\mathcal{C}_2, \Phi)$-generically isostatic. This is because a vertex of G that does not belong to the invariant tree of such a $(\mathcal{C}_2, \Phi)$ 3Tree2 partition can possibly be fixed by $\Phi(C_2)$, and hence $j_{\Phi(C_2)}$ may not be zero.

For example, consider the complete graph K_3 with $V(K_3) = \{v_1, v_2, v_3\}$ and let Φ be the homomorphism from the symmetry group $\mathcal{C}_2$ to $\mathrm{Aut}(K_3)$ defined by $\Phi(C_2) = (v_1\, v_2)(v_3)$. Then K_3 has the proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$, where $T_0 = \langle \{v_1, v_2\} \rangle$, $T_1 = \langle \{v_2, v_3\} \rangle$, and $T_2 = \langle \{v_1, v_3\} \rangle$. Since v_3 is fixed by $\Phi(C_2)$, K_3 is not $(\mathcal{C}_2, \Phi)$-generically isostatic. In fact, every realization in the set $\mathscr{R}_{(K_3, \mathcal{C}_2, \Phi)}$ is a degenerate triangle.

If however G has a proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition (whose invariant tree is not necessarily a spanning tree of G) and we also impose the condition that $j_{\Phi(C_2)} = 0$, then it is quite easy to show that we must also have $b_{\Phi(C_2)} = 1$. In other words, the two conditions that G has any proper $(\mathcal{C}_2, \Phi)$ 3Tree2 partition and $j_{\Phi(C_2)} = 0$ are sufficient for G to be $(\mathcal{C}_2, \Phi)$-generically isostatic.

5 Characterizations of $(\mathcal{C}_s, \Phi)$–generically isostatic graphs

5.1 Symmetrized Henneberg moves and 3Tree2 partitions for $\mathcal{C}_s$

We need the following symmetrized inductive construction techniques to obtain a symmetrized Henneberg's Theorem for $\mathcal{C}_s$.

Definition 5.1 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism. Let v_0 be a vertex of G that is not fixed by $\Phi(s)$ and $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$

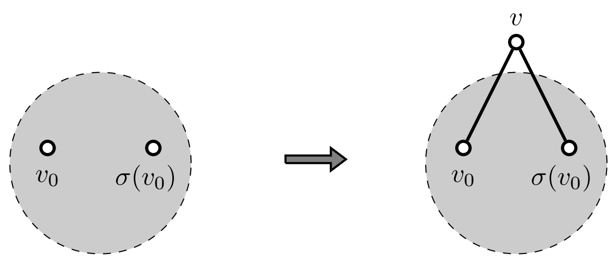


Figure 16: *A $(\mathcal{C}_s, \Phi)$ single vertex addition of a graph G, where $\Phi(s) = \sigma$.*

and $E(\widehat{G}) = E(G) \cup \big\{\{v, v_0\}, \{v, \Phi(s)(v_0)\}\big\}$ is called a $(\mathcal{C}_s, \Phi)$ *single vertex addition (by (v)) of G.*

Definition 5.2 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism. Let v_1, v_2, v_3 be three distinct vertices of G such that $\{v_1, v_2\} \in E(G)$, $\sigma(v_1) = v_2$, and $\sigma(v_3) = v_3$. Further, let $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = \big(E(G) \setminus \big\{\{v_1, v_2\}\big\}\big) \cup \big\{\{v, v_i\} \,|\, i = 1, 2, 3\big\}$ is called a $(\mathcal{C}_s, \Phi)$ *single edge split (on $\{v_1, v_2\}; v$) of G.*

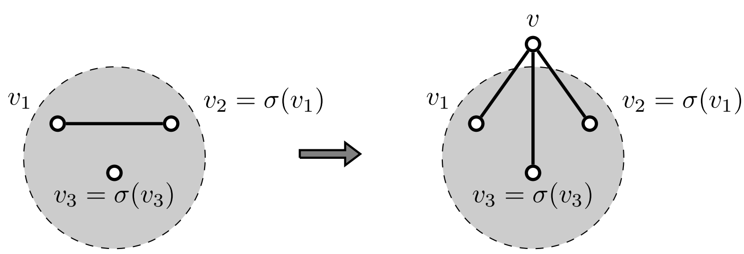


Figure 17: *A $(\mathcal{C}_s, \Phi)$ single edge split of a graph G, where $\Phi(s) = \sigma$.*

Definition 5.3 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism. Let v_1, v_2 be two distinct vertices of G and $v, w \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w\}$ and $E(\widehat{G}) = E(G) \cup \big\{\{v, v_1\}, \{v, v_2\}, \{w, \Phi(s)(v_1)\}, \{w, \Phi(s)(v_2)\}\big\}$ is called a $(\mathcal{C}_s, \Phi)$ *double vertex addition (by (v, w)) of G.*

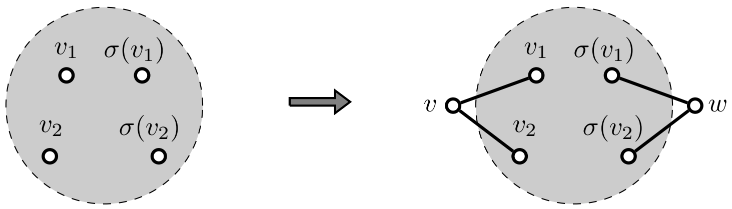


Figure 18: *A $(\mathcal{C}_s, \Phi)$ double vertex addition of a graph G, where $\Phi(s) = \sigma$.*

Definition 5.4 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism. Let v_1, v_2, v_3 be three distinct vertices of G such that $\{v_1, v_2\} \in E(G)$ and $\{v_1, v_2\}$ is not fixed by $\Phi(s)$. Further, let $v, w \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w\}$ and $E(\widehat{G}) = \big(E(G) \setminus \big\{\{v_1, v_2\}, \{\Phi(s)(v_1), \Phi(s)(v_2)\}\big\}\big) \cup \big\{\{v, v_i\} | \, i = 1, 2, 3\big\} \cup \big\{\{w, \Phi(s)(v_i)\} | \, i = 1, 2, 3\big\}$ is called a $(\mathcal{C}_s, \Phi)$ *double edge split (on $(\{v_1, v_2\}, \{\Phi(s)(v_1), \Phi(s)(v_2)\}); (v, w)$) of G.*

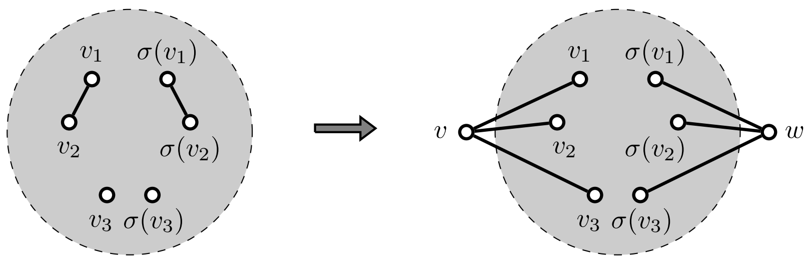


Figure 19: *A $(\mathcal{C}_s, \Phi)$ double edge split of a graph G, where $\Phi(s) = \sigma$.*

Definition 5.5 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism. Let v_1, v_2, v_3, v_4 be four distinct vertices of G with $\{v_1, v_2\}, \{v_3, v_4\} \in E(G)$ and $\Phi(s)(\{v_1, v_2\}) = \{v_3, v_4\}$. Further, let $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = \big(E(G) \setminus \big\{\{v_1, v_2\}, \{v_3, v_4\}\big\}\big) \cup \big\{\{v, v_i\} | \, i \in \{1, 2, 3, 4\}\big\}$ is called a $(\mathcal{C}_s, \Phi)$ *X-replacement (by (v)) of G.*

Remark 5.1 Each of the constructions in Definitions 5.1, 5.2, 5.3, 5.4, and 5.5 has the property that if the graph G satisfies the Laman conditions, then so does $\widehat{G}$. This follows from Theorems 2.2, 2.3, 2.4, and the fact that we can obtain a $(\mathcal{C}_s, \Phi)$ double vertex addition of G by a sequence of two vertex 2-additions and a $(\mathcal{C}_s, \Phi)$ double edge split of G by a sequence of two edge 2-splits.

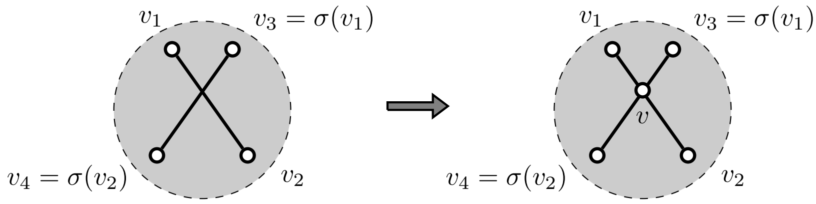


Figure 20: *A $(\mathcal{C}_s, \Phi)$ X-replacement of a graph G, where $\Phi(s) = \sigma$.*

In order to extend Crapo's Theorem to $\mathcal{C}_s$ we need the following symmetrized definitions of a 3Tree2 partition.

Definition 5.6 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism. A $(\mathcal{C}_s, \Phi)$ *3Tree2 $\perp$ partition* of G is a 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$ of G such that $\Phi(s)(T_1) = T_2$ and $\Phi(s)(T_0) = T_0$. The tree T_0 is called the *invariant* tree of $\{E(T_0), E(T_1), E(T_2)\}$.

Remark 5.2 Let $\{E(T_0), E(T_1), E(T_2)\}$ be a $(\mathcal{C}_s, \Phi)$ 3Tree2 $\perp$ partition of a graph G. Then the vertex set of the invariant tree T_0 of $\{E(T_0), E(T_1), E(T_2)\}$ does not contain a vertex $v \in V(G)$ with $\Phi(s)(v) = v$, for otherwise $v \in V(T_1)$ implies $v \in V(T_2)$ and vice versa, contradicting the fact that v only belongs to exactly two of the trees T_i. Therefore, it is easy to see that $E(T_0)$ must contain an edge $e = \{v, w\}$ of G with $\Phi(s)(v) = w$.

Let G be a graph and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism such that $E(G)$ contains an edge $e = \{v, w\}$ with $\Phi(s)(v) = v$ and $\Phi(s)(w) = w$. Then it follows immediately from the previous remark that G cannot have a $(\mathcal{C}_s, \Phi)$ 3Tree2 $\perp$ partition. However, G may have a symmetric 3Tree2 partition of the following kind:

Definition 5.7 Let G be a graph, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism such that there exists an edge $e = \{v, w\} \in E(G)$ with $\Phi(s)(v) = v$ and $\Phi(s)(w) = w$. A $(\mathcal{C}_s, \Phi)$ *3Tree2 $\|$ partition* of G is a 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$ of G such that $e \in E(T_1)$, $\Phi(s)(T_1 - \{v\}) = T_2$ and $\Phi(s)(T_0) = T_0$. The tree T_0 is called the *invariant* tree of $\{E(T_0), E(T_1), E(T_2)\}$.

Remark 5.3 Let $\{E(T_0), E(T_1), E(T_2)\}$ be a $(\mathcal{C}_s, \Phi)$ 3Tree2 $\|$ partition of a graph G. Since T_2 is a tree, so is $\Phi(s)(T_2) = T_1 - \{v\}$, and hence $val_{T_1}(v) = 1$. Also, $v \in V(T_0)$, for otherwise we have $v \in V(T_i)$ for $i = 1, 2$, which contradicts the facts that $\Phi(s)(T_1 - \{v\}) = T_2$ and that $v \notin V\big(\Phi(s)(T_1 - \{v\})\big)$ since v is fixed by $\Phi(s)$. Moreover, there does not exist a vertex $x \in V(G)$ with $x \neq v$, $x \in V(T_0)$, and $\Phi(s)(x) = x$, for otherwise $x \in V(T_1)$ implies $x \in V(T_2)$ and vice versa, contradicting the fact that x only belongs to exactly two of the trees T_i.

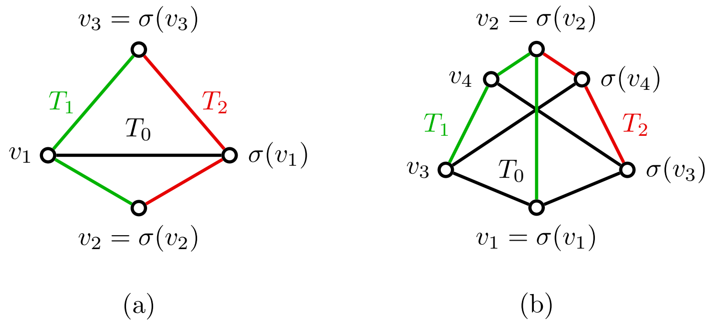


Figure 21: *A* $(\mathcal{C}_s, \Phi)$ *3Tree2* $\perp$ *partition of a graph* (a) *and a* $(\mathcal{C}_s, \Phi)$ *3Tree2* $\|$ *partition of a graph* (b), *where* $\Phi(s) = \sigma$.

Remark 5.4 Let G be a graph and $\Phi : \mathcal{C}_s \to \text{Aut}(G)$ be a homomorphism such that $E(G)$ contains an edge $e = \{v, w\}$ with $\Phi(s)(v) = w$. Then G cannot have a $(\mathcal{C}_s, \Phi)$ 3Tree2 $\|$ partition $\{E(T_0), E(T_1), E(T_2)\}$, for otherwise $e \in E(T_0)$ and, by Remark 5.3, there also exists a vertex in $V(T_0)$ that is fixed by $\Phi(s)$, which implies that there must exist a cycle in T_0.

5.2 The main result for $\mathcal{C}_s$

Theorem 5.1 *Let* G *be a graph with* $|V(G)| \geq 2$, $\mathcal{C}_s = \{Id, s\}$ *be a symmetry group in dimension* 2, *and* $\Phi : \mathcal{C}_s \to Aut(G)$ *be a homomorphism. The following are equivalent:*

(i) $\mathscr{R}_{(G,\mathcal{C}_s,\Phi)} \neq \emptyset$ *and* G *is* $(\mathcal{C}_s, \Phi)$*-generically isostatic;*

(ii) $|E(G)| = 2|V(G)| - 3$, $|E(H)| \leq 2|V(H)| - 3$ *for all* $H \subseteq G$ *with* $|V(H)| \geq 2$ *(Laman conditions), and* $b_{\Phi(s)} = 1$*;*

(iii) there exists a $(\mathcal{C}_s, \Phi)$ *construction sequence*

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G, \Phi)$$

such that

(a) G_{i+1} *is a* $(\mathcal{C}_s, \Phi_i)$ *single or double vertex addition, a* $(\mathcal{C}_s, \Phi_i)$ *single or double edge split, or a* $(\mathcal{C}_s, \Phi_i)$ *X-replacement of* G_i *with* $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}\}$ *or* $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}, w_{i+1}\}$ *for all* $i = 0, 1, \ldots, k-1$*;*

(b) $\Phi_0 : \mathcal{C}_s \to Aut(K_2)$ *is a homomorphism and for all* $i = 0, 1, \ldots, k-1$, $\Phi_{i+1} : \mathcal{C}_s \to Aut(G_{i+1})$ *is the homomorphism defined by* $\Phi_{i+1}(s)|_{V(G_i)} = \Phi_i(s)$ *and* $\Phi_{i+1}(s)(v_{i+1}) = v_{i+1}$ *whenever* $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}\}$ *and* $\Phi_{i+1}(s)|_{\{v_{i+1}, w_{i+1}\}} = (v_{i+1}\, w_{i+1})$ *whenever* $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}, w_{i+1}\}$*;*

(iv) G *has a proper* $(\mathcal{C}_s, \Phi)$ *3Tree2* $\perp$ *partition or a proper* $(\mathcal{C}_s, \Phi)$ *3Tree2* $\|$ *partition.*

We break the proof of this result up into four Lemmas.

Lemma 5.2 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to Aut(G)$ be a homomorphism. If $\mathscr{R}_{(G,\mathcal{C}_s,\Phi)} \neq \emptyset$ and G is $(\mathcal{C}_s, \Phi)$-generically isostatic, then G satisfies the Laman conditions and we have $b_{\Phi(s)} = 1$.*

Proof. The result is trivial if $|V(G)| = 2$, and it follows from Laman's Theorem (Theorem 2.2), Theorem 2.7, and Remark 2.1 if $|V(G)| > 2$. □

Lemma 5.3 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to Aut(G)$ be a homomorphism. If G satisfies the Laman conditions and we also have $b_{\Phi(s)} = 1$, then there exists a $(\mathcal{C}_s, \Phi)$ construction sequence for G.*

Proof. We employ induction on $|V(G)|$. The only graph with two vertices that satisfies the Laman conditions is the graph K_2, and hence the result trivially holds for $|V(G)| = 2$. This proves the base case.

So we let $n > 2$ and we assume that the result holds for all graphs with n or fewer than n vertices.

Let G be a graph with $|V(G)| = n + 1$ that satisfies the Laman conditions and suppose $b_{\Phi(s)} = 1$ for a homomorphism $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$. In the following, we denote $\Phi(s)$ by σ. By Lemma 3.1, G has a vertex of valence 2 or 3.

Case A: G has a vertex v of valence 2, say $N_G(v) = \{v_1, v_2\}$.

Case A.1: Suppose v is fixed by σ. Then $\sigma(v_1) = v_2$, because $b_\sigma = 1$. So, $G' = G - \{v\}$ clearly satisfies the Laman conditions and if we define $\Phi' : \mathcal{C}_s \to \mathrm{Aut}(G')$ to be the homomorphism with $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in \mathcal{C}_s$, then we have $b_{\Phi'(s)} = 1$, and hence, by the induction hypothesis, there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 5.1 (iii). Since G is a $(\mathcal{C}_s, \Phi')$ single vertex addition of G' with $V(G) = V(G') \cup \{v\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G', \Phi'), (G, \Phi)$$

is a sequence with the desired properties.

Case A.2: Suppose $v \neq \sigma(v)$. Then $\sigma(v) \neq v_1, v_2$, for otherwise the graph $G' = G - \{v, \sigma(v)\}$ satisfies

$$|E(G')| = |E(G)| - 3 = 2|V(G)| - 6 = 2|V(G')| - 2,$$

contradicting the fact that G satisfies the Laman conditions.

Thus, the edges $\{v, v_1\}, \{v, v_2\}, \{\sigma(v), \sigma(v_1)\}, \{\sigma(v), \sigma(v_2)\}$ are pairwise distinct. Therefore,

$$|E(G')| = |E(G)| - 4 = 2|V(G)| - 7 = 2|V(G')| - 3$$

and for $H \subseteq G'$ with $|V(H)| \geq 2$, we have $H \subseteq G$, and hence

$$|E(H)| \leq 2|V(H)| - 3,$$

so that G' satisfies the Laman conditions.

Let $\Phi' : \mathcal{C}_s \to \mathrm{Aut}(G')$ be the homomorphism with $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in \mathcal{C}_s$. Then we have $b_{\Phi'(s)} = 1$, and hence, by the induction hypothesis, there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 5.1 (*iii*). Since G is a $(\mathcal{C}_s, \Phi')$ double vertex addition of G' with $V(G) = V(G') \cup \{v, \sigma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G', \Phi'), (G, \Phi)$$

is a sequence with the desired properties.

Case B: G has a vertex v of valence 3, say $N_G(v) = \{v_1, v_2, v_3\}$, and no vertex of valence 2.

Case B.1: Suppose $\sigma(v) = v$. Then wlog $\sigma(v_1) = v_1$ and $\sigma(v_2) = v_3$, because $b_\sigma = 1$. So, the edge $\{v, v_1\}$ of G is fixed by σ and $b_\sigma = 1$ implies that $\{v_2, v_3\} \notin E(G)$.

We claim that the graph $\widetilde{G} = G - \{v\} + \big\{\{v_2, v_3\}\big\}$ satisfies the Laman conditions. Clearly, we have

$$|E(\widetilde{G})| = 2|V(\widetilde{G})| - 3.$$

Let $G' = G - \{v\}$ and suppose there exists a subgraph H of G' with $v_2, v_3 \in V(H)$ and $|E(H)| = 2|V(H)| - 3$. Since G' is invariant under σ, $\sigma(H)$ is also a subgraph of G' and we have $v_2, v_3 \in V\big(\sigma(H)\big)$ and $|E\big(\sigma(H)\big)| = 2|V\big(\sigma(H)\big)| - 3$. Note that $H \cap \sigma(H)$ is a subgraph of G with $v_2, v_3 \in V(H \cap \sigma(H))$, and hence $|E\big(H \cap \sigma(H)\big) \leq 2|V\big(H \cap \sigma(H)\big)| - 3$. Since $H \cap \sigma(H)$ is invariant under σ and $E\big(H \cap \sigma(H)\big)$ does not contain the edge $\{v, v_1\}$, $|E\big(H \cap \sigma(H)\big)|$ is an even number. Thus, we have $|E\big(H \cap \sigma(H)\big)| \leq 2|V\big(H \cap \sigma(H)\big)| - 4$. It follows that the graph $H' = H \cup \sigma(H)$ satisfies

$$\begin{aligned} |E(H')| &= |E(H)| + |E\big(\sigma(H)\big)| - |E\big(H \cap \sigma(H)\big)| \\ &\geq 2|V(H)| - 3 + 2|V\big(\sigma(H)\big)| - 3 - (2|V\big(H \cap \sigma(H)\big)| - 4) \\ &= 2|V(H')| - 2, \end{aligned}$$

contradicting the fact that G satisfies the Laman conditions. So, as claimed, the graph $\widetilde{G}$ satisfies the Laman conditions.

If we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since $\{v_2, v_3\}$ is the only edge that is fixed by $\widetilde{\Phi}$, we also have $b_{\widetilde{\Phi}(s)} = 1$. So, by the induction hypothesis, there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (*iii*). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ single edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case B.2: Suppose v is not fixed by σ. By Lemma 3.1, G has at least six vertices of valence 3. So, since $b_\sigma = 1$, we may assume wlog that $\{v, \sigma(v)\} \notin E(G)$.

Let $G' = G - \{v, \sigma(v)\}$ and suppose there exists a subgraph H of G' with $v_i, \sigma(v_i) \in V(H)$ for all $i = 1, 2, 3$ and $|E(H)| \geq 2|V(H)| - 4$. Then the subgraph $\widehat{H}$ of G with $V(\widehat{H}) = V(H) \cup \{v, \sigma(v)\}$ and $E(\widehat{H}) = E(H) \cup \big\{\{v, v_i\} | \, i = 1, 2, 3\big\} \cup \big\{\{\sigma(v), \sigma(v_i)\} | \, i = 1, 2, 3\big\}$ satisfies

$$|E(\widehat{H})| = |E(H)| + 6 \geq 2|V(H)| + 2 = 2|V(\widehat{H})| - 2,$$

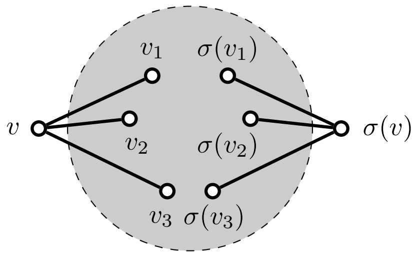
contradicting the fact that G satisfies the Laman conditions.

Thus, every subgraph H of G' with $v_i, \sigma(v_i) \in V(H)$ for all $i = 1, 2, 3$ satisfies the count $|E(H)| \leq 2|V(H)| - 5$. In the following, we will frequently make use of this fact.

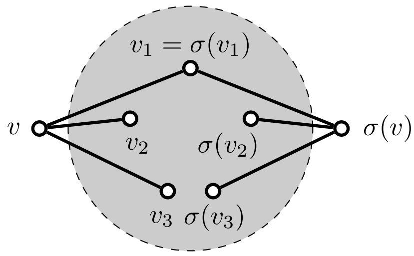
Case B.2.1: Suppose that for every pair $\{i, j\} \subseteq \{1, 2, 3\}$, we have $\sigma(\{v_i, v_j\}) \neq \{v_i, v_j\}$. Then we need to consider the following two subcases (see also Figure 22):

Case B.2.1a: The vertices $v_i, \sigma(v_i)$, $i = 1, 2, 3$, are all pairwise distinct.

Case B.2.1b: One of the v_i, say wlog $v_i = v_1$, is fixed by σ and the vertices $v_2, v_3, \sigma(v_2), \sigma(v_3)$ are pairwise distinct.



(Case B.2.1a)



(Case B.2.1b)

Figure 22: *If a graph G satisfies the conditions in Theorem 5.1 (ii) and has a vertex v with $N_G(v) = \{v_1, v_2, v_3\}$ such that $\sigma(\{v_i, v_j\}) \neq \{v_i, v_j\}$ for all $\{i, j\} \subseteq \{1, 2, 3\}$, then G is a graph of one of the types depicted above.*

Case B.2.1a: By Lemma 3.2 (*i*), there exists $\{i, j\} \subseteq \{1, 2, 3\}$ such that for every subgraph H of $G' = G - \{v, \sigma(v)\}$ with $v_i, v_j \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 4$.

Suppose first that wlog $\{i, j\} = \{1, 2\}$ is the only pair in $\{1, 2, 3\}$ with this property. Then, by Lemma 3.2 (*ii*), $\widetilde{G} = G' + \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $b_{\widetilde{\Phi}(s)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (iii). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ double edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \sigma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

So, suppose there exist two distinct pairs in $\{1,2,3\}$, say wlog $\{1,2\}$ and $\{1,3\}$, such that every subgraph H of G' with $v_1, v_2 \in V(H)$ or $v_1, v_3 \in V(H)$ satisfies $|E(H)| \leq 2|V(H)|-4$. Then every subgraph H of G' with $\sigma(v_1), \sigma(v_2) \in V(H)$ or $\sigma(v_1), \sigma(v_3) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$, because G' is invariant under σ.

Suppose there exist subgraphs H_1 and H_2 of G' with $v_1, v_2, \sigma(v_1), \sigma(v_2) \in V(H_1)$ and $v_1, v_3, \sigma(v_1), \sigma(v_3) \in V(H_2)$ satisfying $|E(H_i)| = 2|V(H_i)| - 4$ for $i = 1, 2$. Then there also exist subgraphs $\sigma(H_1) \subseteq G'$ and $\sigma(H_2) \subseteq G'$ with $v_1, v_2, \sigma(v_1), \sigma(v_2) \in V\big(\sigma(H_1)\big)$ and $v_1, v_3, \sigma(v_1), \sigma(v_3) \in V\big(\sigma(H_2)\big)$ satisfying $|E\big(\sigma(H_i)\big)| = 2|V\big(\sigma(H_i)\big)| - 4$ for $i = 1, 2$. Let $H_i' = H_i \cup \sigma(H_i)$ for $i = 1, 2$. Then

$$\begin{array}{rcl} |E(H_1')| & = & |E(H_1)| + |E\big(\sigma(H_1)\big)| - |E\big(H_1 \cap \sigma(H_1)\big)| \\ & \geq & 2|V(H_1)| - 4 + 2|V\big(\sigma(H_1)\big)| - 4 - (2|V\big(H_1 \cap \sigma(H_1)\big)| - 4) \\ & = & 2|V(H_1')| - 4, \end{array}$$

because $H_1 \cap \sigma(H_1)$ is a subgraph of G' with $v_1, v_2 \in V\big(H_1 \cap \sigma(H_1)\big)$. Since H_1' is also a subgraph of G' with $v_1, v_2 \in V(H_1')$ it follows that

$$|E(H_1')| = 2|V(H_1')| - 4.$$

Similarly, we have

$$|E(H_2')| = 2|V(H_2')| - 4.$$

So, both H_1' and H_2' have an even number of edges. Moreover, both of these graphs are invariant under σ, which says that neither $E(H_1')$ nor $E(H_2')$ contains the edge e of G that is fixed by σ. Note that $H_1' \cap H_2'$ is a subgraph of G with $v_1, \sigma(v_1) \in V(H_1' \cap H_2')$, and hence satisfies the count

$$|E(H_1' \cap H_2')| \leq 2|V(H_1' \cap H_2')| - 3,$$

because G satisfies the Laman conditions. Since $H_1' \cap H_2'$ is also invariant under σ and $E(H_1' \cap H_2')$ does not contain the edge e, $|E(H_1' \cap H_2')|$ is an even number, and hence the above upper bound for $|E(H_1' \cap H_2')|$ can be lowered to

$$|E(H_1' \cap H_2')| \leq 2|V(H_1' \cap H_2')| - 4.$$

Thus, $H' = H_1' \cup H_2'$ satisfies

$$\begin{array}{rcl} |E(H')| & = & |E(H_1')| + |E(H_2')| - |E(H_1' \cap H_2')| \\ & \geq & 2|V(H_1')| - 4 + 2|V(H_2')| - 4 - (2|V(H_1' \cap H_2')| - 4) \\ & = & 2|V(H')| - 4. \end{array}$$

This is a contradiction, because H' is a subgraph of G' with $v_i, \sigma(v_i) \in V(H')$ for all $i = 1, 2, 3$.

So, for $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{1, 3\}$, say wlog $\{i, j\} = \{1, 2\}$, we have that every subgraph H of G' with $v_i, v_j, \sigma(v_i), \sigma(v_j) \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 5$.

Thus, $\widetilde{G} = G' + \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $b_{\widetilde{\Phi}(s)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (iii). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ double edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \sigma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case B.2.1b: By Lemma 3.2 (i), there exists $\{i, j\} \subseteq \{1, 2, 3\}$ such that for every subgraph H of $G' = G - \{v, \sigma(v)\}$ with $v_i, v_j \in V(H)$ we have $|E(H)| \leq 2|V(H)| - 4$. If wlog $\{i, j\} = \{1, 2\}$ is the only pair in $\{1, 2, 3\}$ with this property, then, by Lemma 3.2 (ii), $\widetilde{G} = G' + \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}$ satisfies the Laman conditions and we obtain a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

with the desired properties in the same way as in Case B.2.1a.

So, suppose there exist two distinct pairs $\{i, j\}$ and $\{q, r\}$ in $\{1, 2, 3\}$ such that every subgraph H of G' with $v_i, v_j \in V(H)$ or $v_q, v_r \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$. If one of the pairs $\{i, j\}$ or $\{q, r\}$ is the pair $\{2, 3\}$, then the proof of Case B.2.1a also applies.

So, suppose that every subgraph H of G' with $v_1, v_2 \in V(H)$ or $v_1, v_3 \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$ and that there exists a subgraph A of G' with $v_2, v_3 \in V(A)$ and $|E(A)| = 2|V(A)| - 3$. Then every subgraph H of G' with $v_1, \sigma(v_2) \in V(H)$ or $v_1, \sigma(v_3) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$, because $\sigma(v_1) = v_1$ and G' is invariant under σ.

Suppose there exist subgraphs H_1 and H_2 of G' with $v_1, v_2, \sigma(v_2) \in V(H_1)$ and $v_1, v_3, \sigma(v_3) \in V(H_2)$ satisfying $|E(H_i)| = 2|V(H_i)| - 4$ for $i = 1, 2$. Then there also exist $\sigma(H_1) \subseteq G'$ and $\sigma(H_2) \subseteq G'$ with $v_1, v_2, \sigma(v_2) \in V\big(\sigma(H_1)\big)$ and $v_1, v_3, \sigma(v_3) \in V\big(\sigma(H_2)\big)$ satisfying $|E\big(\sigma(H_i)\big)| = 2|V\big(\sigma(H_i)\big)| - 4$ for $i = 1, 2$. Let $H_i' = H_i \cup \sigma(H_i)$ for $i = 1, 2$. Then

$$\begin{aligned}
|E(H_1')| &= |E(H_1)| + |E\big(\sigma(H_1)\big)| - |E\big(H_1 \cap \sigma(H_1)\big)| \\
&\geq 2|V(H_1)| - 4 + 2|V\big(\sigma(H_1)\big)| - 4 - (2|V\big(H_1 \cap \sigma(H_1)\big)| - 4) \\
&= 2|V(H_1')| - 4,
\end{aligned}$$

because $H_1 \cap \sigma(H_1)$ is a subgraph of G' with $v_1, v_2 \in V\big(H_1 \cap \sigma(H_1)\big)$. Since H_1' is also a subgraph of G' with $v_1, v_2 \in V(H_1')$ it follows that

$$|E(H_1')| = 2|V(H_1')| - 4.$$

Similarly, we have

$$|E(H_2')| = 2|V(H_2')| - 4.$$

Let $H' = H'_1 \cup H'_2$. If $V(H'_1 \cap H'_2)$ contains at least two vertices, then we can derive a contradiction in the same way as in Case B.2.1a. So, suppose $V(H'_1 \cap H'_2) = \{v_1\}$. Then

$$\begin{aligned} |E(H')| &= |E(H'_1)| + |E(H'_2)| - |E(H'_1 \cap H'_2)| \\ &= 2|V(H'_1)| - 4 + 2|V(H'_2)| - 4 - (2|V(H'_1 \cap H'_2)| - 2) \\ &= 2|V(H')| - 6. \end{aligned}$$

Since G' is invariant under σ, $\sigma(A)$ is a subgraph of G' with $\sigma(v_2), \sigma(v_3) \in V\big(\sigma(A)\big)$ and $|E\big(\sigma(A)\big)| = 2|V\big(\sigma(A)\big)| - 3$.

Suppose $|V\big(A \cap \sigma(A)\big)| \geq 2$. Then $|E\big(A \cap \sigma(A)\big)| \leq 2|V\big(A \cap \sigma(A)\big)| - 3$, because G satisfies the Laman conditions, and hence $A' = A \cup \sigma(A)$ is a subgraph of G' with $v_2, v_3, \sigma(v_2), \sigma(v_3) \in V(A')$ satisfying

$$\begin{aligned} |E(A')| &= |E(A)| + |E\big(\sigma(A)\big)| - |E\big(A \cap \sigma(A)\big)| \\ &\geq 2|V(A)| - 3 + 2|V\big(\sigma(A)\big)| - 3 - (2|V\big(A \cap \sigma(A)\big)| - 3) \\ &= 2|V(A')| - 3. \end{aligned}$$

It follows that $|E(A')| = 2|V(A')| - 3$, because $A' \subseteq G$. Since $H_1 \cap A'$ is a subgraph of G with at least two vertices, namely $v_2, \sigma(v_2) \in V(H_1 \cap A')$, we have

$$\begin{aligned} |E(H_1 \cup A')| &= |E(H_1)| + |E(A')| - |E(H_1 \cap A')| \\ &\geq 2|V(H_1)| - 4 + 2|V(A')| - 3 - (2|V(H_1 \cap A')| - 3) \\ &= 2|V(H_1 \cup A')| - 4. \end{aligned}$$

This is a contradiction, because $H_1 \cup A'$ is a subgraph of G' with $v_i, \sigma(v_i) \in V(H_1 \cup A')$ for all $i = 1, 2, 3$

So, suppose every subgraph A of G' with $v_2, v_3 \in V(A)$ and $|E(A)| = 2|V(A)| - 3$ satisfies $|V\big(A \cap \sigma(A)\big)| \leq 1$. Let A_{min} be a subgraph of G' that satisfies $v_2, v_3 \in V(A_{min})$ and $|E(A_{min})| = 2|V(A_{min})| - 3$ and has the smallest number of edges among all such subgraphs of G'. Note that $v_1 \notin V(A_{min})$, for otherwise $v_1, v_2 \in V(A_{min})$, and hence $|E(A_{min})| \leq 2|V(A_{min})| - 4$. Also, A_{min} is connected as we see as follows.

Suppose to the contrary that $A_{min} = A_1 \cup A_2$, where $V(A_1) \cap V(A_2) = \emptyset$. Clearly, one of A_1 or A_2 has at least two vertices. If wlog $|V(A_1)| = 1$ and $|V(A_2)| \geq 2$, then

$$|E(A_{min})| = |E(A_1)| + |E(A_2)| \leq 2|V(A_1)| - 2 + 2|V(A_2)| - 3 = 2|V(A_{min})| - 5$$

and if $|V(A_1)|, |V(A_2)| \geq 2$, then

$$|E(A_{min})| = |E(A_1)| + |E(A_2)| \leq 2|V(A_1)| - 3 + 2|V(A_2)| - 3 = 2|V(A_{min})| - 6.$$

In both cases, we have a contradiction to the fact that $|E(A_{min})| = 2|V(A_{min})| - 3$. So, A_{min} is indeed connected.

Now, consider $H' \cup A_{min}$. We have

$$|E(H' \cup A_{min})| = |E(H')| + |E(A_{min})| - |E(H' \cap A_{min})|.$$

Note that $v_2, v_3 \in V(H' \cap A_{min})$ so that

$$|E(H' \cap A_{min})| \leq 2|V(H' \cap A_{min})| - 3.$$

We claim that $|E(H' \cap A_{min})| < |E(A_{min})|$.

Since A_{min} is connected, there exists a $v_2 - v_3$ path P in A_{min} and P does not contain an edge incident with the vertex v_1, because $v_1 \notin V(A_{min})$. Let $E(P)$ denote the set of edges of P. Since $V(H_1' \cap H_2') = \{v_1\}$, we have $v_2 \in V(H_1')$, $v_2 \notin V(H_2')$, $v_3 \in V(H_2')$, $v_3 \notin V(H_1')$, and every $v_2 - v_3$ path in $H' = H_1' \cup H_2'$ must contain an edge incident with v_1. Thus, $E(P) \nsubseteq E(H')$. So, as claimed, $|E(H' \cap A_{min})| < |E(A_{min})|$.

By the minimality of $|E(A_{min})|$, we can conclude that

$$|E(H' \cap A_{min})| \leq 2|V(H' \cap A_{min})| - 4.$$

Thus,

$$\begin{aligned} |E(H' \cup A_{min})| &= |E(H')| + |E(A_{min})| - |E(H' \cap A_{min})| \\ &\geq 2|V(H')| - 6 + 2|V(A_{min})| - 3 - (2|V(H' \cap A_{min})| - 4) \\ &= 2|V(H' \cup A_{min})| - 5. \end{aligned}$$

Now, consider $(H' \cup A_{min}) \cup \sigma(A_{min})$. We have

$$\begin{aligned} |E\big((H' \cup A_{min}) \cup \sigma(A_{min})\big)| &= |E(H' \cup A_{min})| + |E\big(\sigma(A_{min})\big)| \\ &\quad -|E\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)|. \end{aligned}$$

Note that $\sigma(v_2), \sigma(v_3) \in V\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)$ so that

$$|E\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| \leq 2|V\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| - 3.$$

By the definition of A_{min}, $\sigma(A_{min})$ is a subgraph of G' with $\sigma(v_2), \sigma(v_3) \in V\big(\sigma(A_{min})\big)$ and $|E\big(\sigma(A_{min})\big)| = 2|V\big(\sigma(A_{min})\big)| - 3$ and $\sigma(A_{min})$ has the smallest number of edges among all such subgraphs of G'. We claim that $|E\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| < |E\big(\sigma(A_{min})\big)|$.

Since A_{min} is a connected subgraph of G', so is $\sigma(A_{min})$. Therefore, there exists a $\sigma(v_2) - \sigma(v_3)$ path P in $\sigma(A_{min})$ and $E(P)$ does not contain an edge incident with the vertex v_1, because $v_1 \notin V\big(\sigma(A_{min})\big)$. Also, by assumption, $|V\big(A_{min} \cap \sigma(A_{min})\big)| \leq 1$, which says that $E(P)$ does not contain an edge of A_{min}. Since $V(H_1' \cap H_2') = \{v_1\}$, we have $\sigma(v_2) \in V(H_1')$, $\sigma(v_2) \notin V(H_2')$, $\sigma(v_3) \in V(H_2')$, $\sigma(v_3) \notin V(H_1')$, and every $\sigma(v_2) - \sigma(v_3)$ path in $H' \cup A_{min} = (H_1' \cup H_2') \cup A_{min}$ must contain an edge incident with v_1 or an edge of A_{min}. Thus, $E(P) \nsubseteq E(H' \cup A_{min})$. So, as claimed, $|E\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| < |E\big(\sigma(A_{min})\big)|$.

By the minimality of $|E\big(\sigma(A_{min})\big)|$, we can conclude that

$$|E\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| \leq 2|V\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| - 4.$$

Thus,

$$\begin{aligned} |E\big((H' \cup A_{min}) \cup \sigma(A_{min})\big)| &= |E(H' \cup A_{min})| + |E\big(\sigma(A_{min})\big)| \\ &\quad -|E\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| \\ &\geq 2|V(H' \cup A_{min})| - 5 + 2|V\big(\sigma(A_{min})\big)| - 3 \\ &\quad -(2|V\big((H' \cup A_{min}) \cap \sigma(A_{min})\big)| - 4) \\ &= 2|V\big((H' \cup A_{min}) \cup \sigma(A_{min})\big)| - 4. \end{aligned}$$

This is a contradiction, since $(H' \cup A_{min}) \cup \sigma(A_{min})$ is a subgraph of G' with $v_i, \sigma(v_i) \in V\big((H' \cup A_{min}) \cup \sigma(A_{min})\big)$ for all $i = 1, 2, 3$.

So, for $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{1, 3\}$, say wlog $\{i, j\} = \{1, 2\}$, we have that every subgraph H of G' with $v_i, v_j, \sigma(v_i), \sigma(v_j) \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 5$. Thus, $\widetilde{G} = G' + \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $b_{\widetilde{\Phi}(s)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (iii). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ double edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \sigma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case B.2.2: Suppose there exists exactly one pair $\{i, j\}$ in $\{1, 2, 3\}$ such that $\sigma(\{v_i, v_j\}) = \{v_i, v_j\}$. Then we need to consider the following two subcases (see also Figure 23):

Case B.2.2a: Exactly two of the vertices v_i, $i = 1, 2, 3$, say wlog v_1 and v_2, are fixed by σ.

Case B.2.2b: There exists a pair $\{i, j\} \subseteq \{1, 2, 3\}$ such that $\sigma(v_i) = v_j$. Wlog we assume $\sigma(v_1) = v_2$.

Case B.2.2a: Since $\sigma(\{v_1, v_2\}) = \{v_1, v_2\}$, it follows from Lemma 3.2 (i) and (ii) that there exists $\{i, j\} \subseteq \{1, 2, 3\}$ with $\{i, j\} \neq \{1, 2\}$ such that for every subgraph H of $G' = G - \{v, \sigma(v)\}$ with $v_i, v_j \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 4$.

If for every subgraph H of G' with $v_1, v_3 \in V(H)$ or $v_2, v_3 \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 4$, then the proof of Case B.2.1a applies.

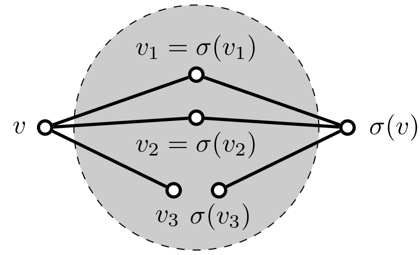
So, suppose wlog that every subgraph H of G' with $v_1, v_3 \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$ and that there exists a subgraph A of G' with $v_2, v_3 \in V(A)$ and $|E(A)| = 2|V(A)| - 3$. Since G' is invariant under σ and $v_i = \sigma(v_i)$ for $i = 1, 2$, every subgraph H of G' with $v_1, \sigma(v_3) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$ and $\sigma(A)$ is a subgraph of G' with $v_2, \sigma(v_3) \in V\big(\sigma(A)\big)$ and $|E\big(\sigma(A)\big)| = 2|V\big(\sigma(A)\big)| - 3$.

We claim that the graph $\widetilde{G} = G' + \big\{\{v_1, v_3\}, \{v_1, \sigma(v_3)\}\big\}$ satisfies the Laman conditions. Clearly,

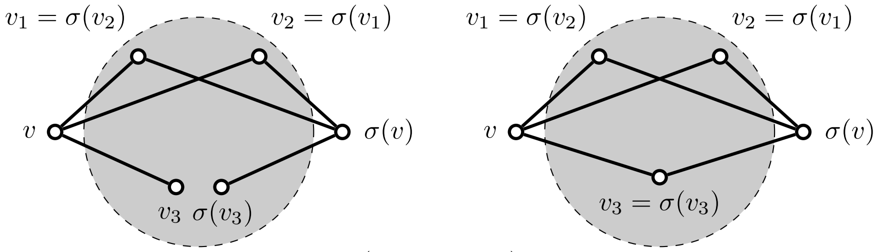
$$|E(\widetilde{G})| = 2|V(\widetilde{G})| - 3.$$

Suppose there exists a subgraph H of G' with $v_1, v_3, \sigma(v_3) \in V(H)$ that satisfies $|E(H)| = 2|V(H)| - 4$. Then $\sigma(H)$ is also a subgraph of G' with $v_1, v_3, \sigma(v_3) \in V\big(\sigma(H)\big)$ that satisfies $|E\big(\sigma(H)\big)| = 2|V\big(\sigma(H)\big)| - 4$. Let $H' = H \cup \sigma(H)$. Then

$$\begin{aligned} |E(H')| &= |E(H)| + |E\big(\sigma(H)\big)| - |E\big(H \cap \sigma(H)\big)| \\ &\geq 2|V(H)| - 4 + 2|V\big(\sigma(H)\big)| - 4 - (2|V\big(H \cap \sigma(H)\big)| - 4) \\ &= 2|V(H')| - 4, \end{aligned}$$



(Case B.2.2a)



(Case B.2.2b)

Figure 23: *If a graph G satisfies the conditions in Theorem 5.1 (ii) and has a vertex v with $N_G(v) = \{v_1, v_2, v_3\}$ such that $\sigma(\{v_i, v_j\}) = \{v_i, v_j\}$ for exactly one pair $\{i, j\} \subseteq \{1, 2, 3\}$, then G is a graph of one of the types depicted above.*

because $H \cap \sigma(H)$ is a subgraph of G' with $v_1, v_3 \in V\big(H \cap \sigma(H)\big)$. Since H' is also a subgraph of G' with $v_1, v_3 \in V(H')$ it follows that

$$|E(H')| = 2|V(H')| - 4.$$

So, since H' has an even number of edges and is invariant under σ, $E(H')$ does not contain the edge e that is fixed by σ.

Let $A' = A \cup \sigma(A)$. Then $H' \cap A'$ is a subgraph of G with $v_3, \sigma(v_3) \in V(H' \cap A')$, and hence satisfies the count

$$|E(H' \cap A')| \leq 2|V(H' \cap A')| - 3.$$

Since $H' \cap A'$ is invariant under σ and $E(H' \cap A')$ does not contain the edge e, $|E(H' \cap A')|$ is an even number. Therefore, the above upper bound for $|E(H' \cap A')|$ can be lowered to

$$|E(H' \cap A')| \leq 2|V(H' \cap A')| - 4.$$

Note that if $V\big(A \cap \sigma(A)\big) = \{v_2\}$, then $|E\big(A \cap \sigma(A)\big)| = 2|V\big(A \cap \sigma(A)\big)| - 2$ and if $|V(A) \cap \sigma(A)| \geq 2$, then $|E\big(A \cap \sigma(A)\big)| \leq 2|V\big(A \cap \sigma(A)\big)| - 3$, because $A \cap \sigma(A)$ is a subgraph of G. Therefore,

$$|E\big(A \cap \sigma(A)\big)| \leq 2|V\big(A \cap \sigma(A)\big)| - 2,$$

and hence

$$\begin{aligned}|E(A')| &= |E(A)| + |E\big(\sigma(A)\big)| - |E\big(A \cap \sigma(A)\big)| \\ &\geq 2|V(A)| - 3 + 2|V\big(\sigma(A)\big)| - 3 - (2|V\big(A \cap \sigma(A)\big)| - 2) \\ &= 2|V(A')| - 4,\end{aligned}$$

Thus,

$$\begin{aligned}|E(H' \cup A')| &= |E(H')| + |E(A')| - |E(H' \cap A')| \\ &\geq 2|V(H')| - 4 + 2|V(A')| - 4 - (2|V(H' \cap A')| - 4) \\ &= 2|V(H' \cup A')| - 4.\end{aligned}$$

This is a contradiction, since $H' \cup A'$ is a subgraph of G' with $v_i, \sigma(v_i) \in V(H' \cup A')$ for all $i = 1, 2, 3$.

Thus, $\widetilde{G} = G' + \big\{\{v_1, v_3\}, \{\sigma(v_1), \sigma(v_3)\}\big\}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $b_{\widetilde{\Phi}(s)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (*iii*). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ double edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \sigma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case B.2.2b: By Lemma 3.2, for $\{i, j\} = \{1, 3\}$ or $\{i, j\} = \{2, 3\}$, say wlog $\{i, j\} = \{1, 3\}$, we have that every subgraph H of $G' = G - \{v, \sigma(v)\}$ with $v_i, v_j \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$. Since G' is invariant under σ, every subgraph H of G' with $\sigma(v_1), \sigma(v_3) \in V(H)$ also satisfies $|E(H)| = 2|V(H)| - 4$. Moreover, if there exists a subgraph H of G' with $v_1, v_3, \sigma(v_1), \sigma(v_3) \in V(H)$, then $v_i, \sigma(v_i) \in V(H)$ for all $i = 1, 2, 3$, and hence $|E(H)| \leq 2|V(H)| - 5$.

Therefore, $\widetilde{G} = G' + \big\{\{v_1, v_3\}, \{\sigma(v_1), \sigma(v_3)\}\big\}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $b_{\widetilde{\Phi}(s)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (*iii*). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ double edge split of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v, \sigma(v)\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case B.2.3: Finally, suppose G has no vertex of valence two, no vertex of valence three that is fixed by σ, and every 3-valent vertex v of G (except possibly the two vertices that are incident with the edge $e \in E(G)$ that is fixed by σ) has the property that $\sigma(u) = u$ for all $u \in N_G(v)$.

Let T denote the set of 3-valent vertices of G. Then $|T| = 2k$, $k \in \mathbb{N}$, and, by Lemma 3.1, $k \geq 3$. Also, let $e = \{y, z\}$ be the edge of G that is fixed by σ.

We claim that G has a vertex v with $N_G(v) = \{v_1, v_2, v_3\}$, $\sigma(v_i) = v_i$ for all $i = 1, 2, 3$, and $val_G(v_i) = 4$ for some $i \in \{1, 2, 3\}$. Suppose to the contrary that there does not exist such a vertex v in $V(G)$.

We suppose first that $y, z \notin T$. Then for every $v \in T$, we have $\sigma(u) = u$ for all $u \in N_G(v)$. Let $N = \bigcup_{v \in T} N_G(v)$ and suppose $|N| = m \leq k + 1$. The subgraph $\langle T \cup N \rangle$ of G induced by $T \cup N$ satisfies $|V(\langle T \cup N \rangle)| = 2k + m$ and $|E(\langle T \cup N \rangle)| \geq 6k$. Thus,

$$2|V(\langle T \cup N \rangle)| - 3 = 2(2k + m) - 3 \leq 6k - 1 < |E(\langle T \cup N \rangle)|,$$

which is a contradiction to the fact that G satisfies the Laman conditions. Therefore, $|N| \geq k + 2$. By assumption, every vertex in N has valence at least 5 in G. Since at most two vertices in N can possibly be incident with e, it follows that at least k vertices in N must have valence at least 6 in G. Therefore, the average valence in G is at least

$$\frac{2k \cdot 3 + 2 \cdot 5 + k \cdot 6 + \big(|V(G)| - (2k + k + 2)\big) \cdot 4}{|V(G)|} = 4 + \frac{2}{|V(G)|},$$

which contradicts the fact that the average valence in G is $4 - \frac{6}{|V(G)|}$ (see Lemma 3.1).

Suppose now that y or z is a vertex in T. Then $\sigma(y) = z$ and both y and z are in T. Let $T' = T \setminus \{y, z\}$ and let $N' = \bigcup_{v \in T'} N_G(v)$. Suppose $|N'| = m \leq k$. The subgraph $\langle T' \cup N' \rangle$ of G induced by $T' \cup N'$ satisfies $|V(\langle T' \cup N' \rangle)| = 2k - 2 + m$ and $|E(\langle T' \cup N' \rangle)| = (2k - 2) \cdot 3 = 6k - 6$. Thus,

$$2|V(\langle T' \cup N' \rangle)| - 3 = 2(2k - 2 + m) - 3 \leq 6k - 7 < |E(\langle T' \cup N' \rangle)|,$$

which is a contradiction to the fact that G satisfies the Laman conditions. Therefore, $|N'| \geq k + 1$. By assumption, every vertex in N' has valence at least 5 in G, and since $y, z \notin N'$, every vertex in N' must even have valence at least 6 in G. Therefore, the average valence in G is at least

$$\frac{2k \cdot 3 + (k + 1) \cdot 6 + \big(|V(G)| - (2k + k + 1)\big) \cdot 4}{|V(G)|} = 4 + \frac{2}{|V(G)|} \ ,$$

which again contradicts the fact that the average valence in G is $4 - \frac{6}{|V(G)|}$ (see Lemma 3.1).

So, as claimed, there exists a vertex $v \in V(G)$ with $N_G(v) = \{v_1, v_2, v_3\}$, $\sigma(v_i) = v_i$ for all $i = 1, 2, 3$, and $val_G(v_i) = 4$ for some $i \in \{1, 2, 3\}$, say wlog $val_G(v_1) = 4$ with $N_G(v_1) = \{v, \sigma(v), w, \sigma(w)\}$.

Let $G' = G - \{v_1\}$. We claim that $\widetilde{G} = G' + \big\{\{v, w\}, \{\sigma(v), \sigma(w)\}\big\}$ satisfies the Laman conditions. We have

$$|E(\widetilde{G})| = |E(G)| - 2 = 2|V(G)| - 5 = 2|V(\widetilde{G})| - 3.$$

Suppose there exists a subgraph H of G' with $v, w, \sigma(v), \sigma(w) \in V(G')$ that satisfies $|E(H)| \geq 2|V(H)| - 4$. Then the subgraph $\widehat{H}$ of G with $V(\widehat{H}) = V(H) \cup$

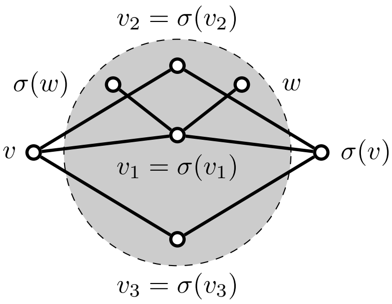


Figure 24: *If a graph G satisfies the conditions in Theorem 5.1 (ii), has no vertex of valence two, no vertex of valence three that is fixed by σ, and every 3-valent vertex v of G (except possibly the vertices that are incident with the edge that is fixed by σ) has the property that $\sigma(u) = u$ for all $u \in N_G(v)$, then there exists $v \in V(G)$ with $N_G(v) = \{v_1, v_2, v_3\}$, $\sigma(v_i) = v_i$ for all $i = 1, 2, 3$, and $val_G(v_i) = 4$ for some $i \in \{1, 2, 3\}$.*

$\{v_1\}$ and $E(\widehat{H}) = E(H) \cup \big\{\{v_1, v\}, \{v_1, \sigma(v)\}, \{v_1, w\}, \{v_1, \sigma(w)\}\big\}$ satisfies

$$|E(\widehat{H})| = |E(H)| + 4 \geq 2|V(H)| = 2|V(\widehat{H})| - 2,$$

contradicting the fact that G satisfies the Laman conditions.

Thus, every subgraph H of G' with $v, w, \sigma(v), \sigma(w) \in V(H)$ satisfies the count $|E(H)| \leq 2|V(H)| - 5$.

Suppose there exists a subgraph H of G' with $v, w \in V(H)$ that satisfies $|E(H)| = 2|V(H)| - 3$. Then $|V(H)| \geq 3$ since $\{v, w\} \notin E(H)$. Since G' is invariant under σ, $\sigma(H)$ is also a subgraph of G' and $\sigma(H)$ satisfies $\sigma(v), \sigma(w) \in V\big(\sigma(H)\big)$ and $|E\big(\sigma(H)\big)| = 2|E\big(\sigma(H)\big)| - 3$. Let $H' = H \cup \sigma(H)$. Then

$$|E(H')| = |E(H)| + |E\big(\sigma(H)\big)| - |E\big(H \cap \sigma(H)\big)|.$$

Suppose first that $E\big(H \cap \sigma(H)\big) = \emptyset$. Then $v_2, v_3 \notin V(H)$. Thus, v is an isolated vertex in H, and hence

$$|E(H - \{v\})| = |E(H)| = 2|V(H)| - 3 = 2|V(H - \{v\})| - 1.$$

This contradicts the fact that G satisfies the Laman conditions, because $|V(H - \{v\})| \geq 2$.

Suppose now that $|V\big(H \cap \sigma(H)\big)| \geq 1$. If $|V\big(H \cap \sigma(H)\big)| = 1$, then $|E\big(H \cap \sigma(H)\big)| = 2|V\big(H \cap \sigma(H)\big)| - 2$, and if $|V\big(H \cap \sigma(H)\big)| \geq 2$, then $|E\big(H \cap \sigma(H)\big)| \leq 2|V\big(H \cap \sigma(H)\big)| - 3$, because $H \cap \sigma(H)$ is a subgraph of G. Thus,

$$|E\big(H \cap \sigma(H)\big)| \leq 2|V\big(H \cap \sigma(H)\big)| - 2,$$

and hence

$$\begin{aligned} |E(H')| &= |E(H)| + |E\big(\sigma(H)\big)| - |E\big(H \cap \sigma(H)\big)| \\ &\geq 2|V(H)| - 3 + 2|V\big(\sigma(H)\big)| - 3 - (2|V\big(H \cap \sigma(H)\big)| - 2) \\ &= 2|V(H')| - 4. \end{aligned}$$

This is a contradiction, because $v, w, \sigma(v), \sigma(w) \in V(H')$.

It follows that every subgraph H of G' with $v, w \in V(H)$ or $\sigma(v), \sigma(w) \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 4$.

Therefore, as claimed, $\widetilde{G}$ satisfies the Laman conditions and if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(x) = \Phi(x)|_{V(\widetilde{G})}$ for all $x \in \mathcal{C}_s$, then $\widetilde{\Phi}(x) \in \mathrm{Aut}(\widetilde{G})$ for all $x \in \mathcal{C}_s$ and $\widetilde{\Phi} : \mathcal{C}_s \to \mathrm{Aut}(\widetilde{G})$ is a homomorphism. Since we also have $b_{\widetilde{\Phi}(s)} = 1$, it follows from the induction hypothesis that there exists a sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\widetilde{G}, \widetilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (iii). Since G is a $(\mathcal{C}_s, \widetilde{\Phi})$ X-replacement of $\widetilde{G}$ with $V(G) = V(\widetilde{G}) \cup \{v_1\}$,

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\widetilde{G}, \widetilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties. $\square$

Lemma 5.4 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to Aut(G)$ be a homomorphism. If there exists a $(\mathcal{C}_s, \Phi)$ construction sequence for G, then G has a proper $(\mathcal{C}_s, \Phi)$ 3Tree2 $\perp$ partition or a proper $(\mathcal{C}_s, \Phi)$ 3Tree2 $\|$ partition.*

Proof. We proceed by induction on $|V(G)|$. Let $V(K_2) = \{v_1, v_2\}$ and let $\Phi : \mathcal{C}_s \to K_2$ be the homomorphism defined by $\Phi(s) = (v_1\, v_2)$. Then K_2 has the proper $(\mathcal{C}_s, \Phi)$ 3Tree2 $\perp$ partition $\{E(T_0), E(T_1), E(T_2)\}$, where $T_0 = \langle\{v_1, v_2\}\rangle$, $T_1 = \langle\{v_1\}\rangle$ and $T_2 = \langle\{v_2\}\rangle$. Let $\Psi : \mathcal{C}_s \to K_2$ be the homomorphism defined by $\Psi(s) = id$. Then K_2 has the proper $(\mathcal{C}_s, \Psi)$ 3Tree2 $\|$ partition $\{E(T_0), E(T_1), E(T_2)\}$, where $T_0 = \langle\{v_1\}\rangle$, $T_1 = \langle\{v_1, v_2\}\rangle$ and $T_2 = \langle\{v_2\}\rangle$. This proves the base case.

Assume, then, that the result holds for all graphs with n or fewer than n vertices, where $n > 2$.

Let G be a graph with $|V(G)| = n + 1$ and let $\Phi : \mathcal{C}_s \to \mathrm{Aut}(G)$ be a homomorphism such that there exists a $(\mathcal{C}_s, \Phi)$ construction sequence

$$(K_2, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G, \Phi)$$

satisfying the conditions in Theorem 5.1 (iii). By Remark 5.1, G satisfies the Laman conditions, and hence, by Remark 2.2, any 3Tree2 partition of G must be proper. Therefore, it suffices to show that G has some $(\mathcal{C}_s, \Phi)$ $\perp$ or $(\mathcal{C}_s, \Phi)$ $\|$ 3Tree2 partition. In the following, we denote $\Phi(s)$ by σ.

By the induction hypothesis, G_{k-1} has a $(\mathcal{C}_s, \Phi_{k-1})$ $\perp$ or $(\mathcal{C}_s, \Phi_{k-1})$ $\|$ 3Tree2 partition $\{E(T_0^{(k-1)}), E(T_1^{(k-1)}), E(T_2^{(k-1)})\}$.

Case 1: Suppose G is a $(\mathcal{C}_s, \Phi_{k-1})$ single vertex addition by v of G_{k-1} with $N_G(v) = \{v_0, \sigma(v_0)\}$. Note that v_0 is a vertex of $T_1^{(k-1)}$ or $T_2^{(k-1)}$. Wlog, we assume $v_0 \in V(T_1^{(k-1)})$. Then $\sigma(v_0) \in V(T_2^{(k-1)})$. So, if we define

$$T_0^{(k)} = T_0^{(k-1)},$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V(T_1^{(k)}) &= V(T_1^{(k-1)}) \cup \{v\} \\ E(T_1^{(k)}) &= E(T_1^{(k-1)}) \cup \{\{v, v_0\}\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{array}{rcl} V\big(T_2^{(k)}\big) & = & V\big(T_2^{(k-1)}\big) \cup \{v\} \\ E\big(T_2^{(k)}\big) & = & E\big(T_2^{(k-1)}\big) \cup \big\{\{v, \sigma(v_0)\}\big\}, \end{array}$$

then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

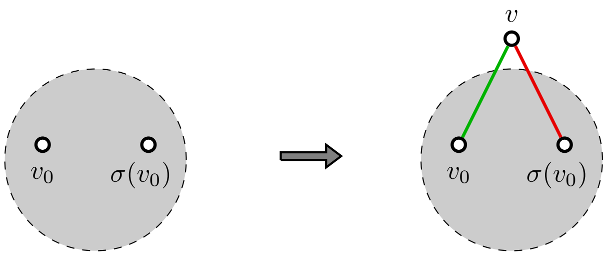


Figure 25: *Construction of a* $(\mathcal{C}_s, \Phi) \perp$ *or* $(\mathcal{C}_s, \Phi_{k-1}) \parallel$ *3Tree2 partition of* G *in the case where* G *is a* $(\mathcal{C}_s, \Phi_{k-1})$ *single vertex addition of* G_{k-1}.

Case 2: Suppose G is a $(\mathcal{C}_s, \Phi_{k-1})$ single edge split on $\{v_1, v_2\}; v$ of G_{k-1} with $N_G(v) = \{v_1, v_2, v_3\}$. Then $\Phi_{k-1}(s)(v_1) = \sigma(v_1) = v_2$ and $\Phi_{k-1}(s)(v_3) = \sigma(v_3) = v_3$. By Remark 5.4, $\big\{E\big(T_0^{(k-1)}\big), E\big(T_1^{(k-1)}\big), E\big(T_2^{(k-1)}\big)\big\}$ must be a $(\mathcal{C}_s, \Phi_{k-1}) \perp$ 3Tree2 partition of G_{k-1}. Clearly, $\{v_1, v_2\} \in E\big(T_0^{(k-1)}\big)$ and, by Remark 5.2, $v_3 \notin V\big(T_0^{(k-1)}\big)$. Therefore, $v_3 \in V\big(T_1^{(k-1)}\big)$. So, if we define $T_0^{(k)}$ to be the tree with

$$\begin{array}{rcl} V\big(T_0^{(k)}\big) & = & V\big(T_0^{(k-1)}\big) \cup \{v\} \\ E\big(T_0^{(k)}\big) & = & \big(E\big(T_0^{(k-1)}\big) \setminus \{v_1, v_2\}\big) \cup \big\{\{v, v_1\}, \{v, v_2\}\big\}, \end{array}$$

$T_1^{(k)}$ to be the tree with

$$\begin{array}{rcl} V\big(T_1^{(k)}\big) & = & V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) & = & E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_3\}\big\}, \end{array}$$

and

$$T_2^{(k)} = T_2^{(k-1)},$$

then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 3: Suppose G is a $(\mathcal{C}_s, \Phi_{k-1})$ double vertex addition by (v, w) of G_{k-1} with $N_G(v) = \{v_1, v_2\}$. Since $\Phi_{k-1}(s) = \sigma|_{V(G_{k-1})}$, we have $N_G(w) = \{\sigma(v_1), \sigma(v_2)\}$.

Case 3.1: If $v_1, v_2 \notin V\big(T_0^{(k-1)}\big)$, then $v_1, v_2, \sigma(v_1), \sigma(v_2) \in V\big(T_i^{(k-1)}\big)$ for $i = 1, 2$. In this case, we define

$$T_0^{(k)} = T_0^{(k-1)},$$

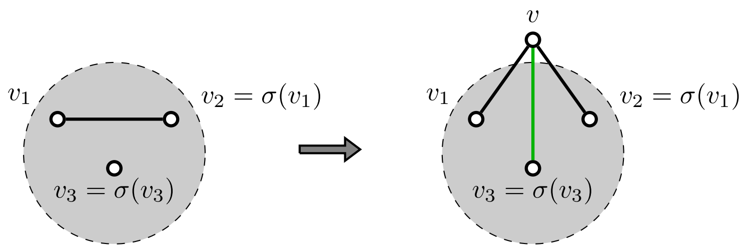


Figure 26: *Construction of a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ single edge split of G_{k-1}. The edges in black color represent edges of the invariant trees.*

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_1\}, \{w, \sigma(v_2)\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{v, v_2\}, \{w, \sigma(v_1)\}\big\}. \end{aligned}$$

Then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

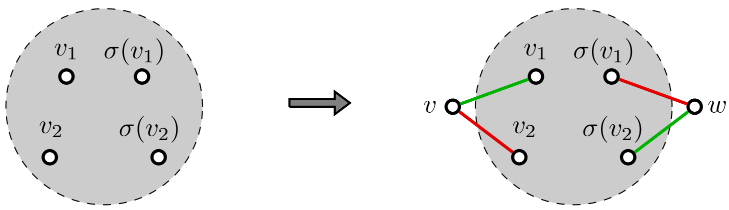


Figure 27: *Construction of a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ double vertex addition of G_{k-1} and $v_1, v_2 \notin V\big(T_0^{(k-1)}\big)$.*

Case 3.2: If $v_1 \in V\big(T_0^{(k-1)}\big)$ and $v_2 \notin V\big(T_0^{(k-1)}\big)$, then $\sigma(v_1) \in V\big(T_0^{(k-1)}\big)$ and $v_2, \sigma(v_2) \in V\big(T_i^{(k-1)}\big)$ for $i = 1, 2$. So, if we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_0^{(k)}\big) &= E\big(T_0^{(k-1)}\big) \cup \big\{\{v, v_1\}, \{w, \sigma(v_1)\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_2\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{w\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{w, \sigma(v_2)\}\big\}, \end{aligned}$$

then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 3.3: If both v_1 and v_2 are vertices of $T_0^{(k-1)}$ and $v_i \neq \sigma(v_i)$ for $i = 1, 2$, then wlog $v_2 \in V\big(T_1^{(k-1)}\big)$, and hence $\sigma(v_2) \in V\big(T_2^{(k-1)}\big)$, so that the previous construction in Case 3.2 can be used to obtain a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 3.4: If both v_1 and v_2 are vertices of $T_0^{(k-1)}$ and $v_i = \sigma(v_i)$ for some $i \in \{1, 2\}$, say wlog $v_1 = \sigma(v_1)$, then, by Remarks 5.2 and 5.3, $\big\{E\big(T_0^{(k-1)}\big), E\big(T_1^{(k-1)}\big), E\big(T_2^{(k-1)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi_{k-1}) \parallel$ 3Tree2 partition of G and $v_2 \neq \sigma(v_2)$. Suppose wlog that $v_2 \in V\big(T_1^{(k-1)}\big)$. Then $\sigma(v_2) \in V\big(T_2^{(k-1)}\big)$ and the construction in Case 3.2 can again be used to obtain a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

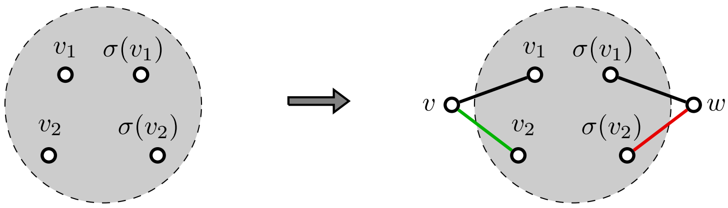


Figure 28: *Construction of a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ double vertex addition of G_{k-1} and at least one of v_1 or v_2 is a vertex of $T_0^{(k-1)}$. The edges in black color represent edges of the invariant tree.*

Case 4: Suppose G is a $(\mathcal{C}_s, \Phi_{k-1})$ double edge split on $(\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}); (v, w)$ of G_{k-1} with $E(G_k) = \big(E(G_{k-1}) \setminus \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}\big) \cup \big\{\{v, v_i\} | \, i = 1, 2, 3\big\} \cup \big\{\{w, \sigma(v_i)\} | \, i = 1, 2, 3\big\}$.

Case 4.1: Suppose $\{v_1, v_2\} \in E\big(T_0^{(k-1)}\big)$. Then we also have $\{\sigma(v_1), \sigma(v_2)\} \in E\big(T_0^{(k-1)}\big)$.

Case 4.1a: If $v_3 \notin V\big(T_0^{(k-1)}\big)$, then $v_3, \sigma(v_3) \in V\big(T_i^{(k-1)}\big)$ for $i = 1, 2$. In this case we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_0^{(k)}\big) &= \big(E\big(T_0^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}\big) \\ &\quad \cup \big\{\{v, v_1\}, \{v, v_2\}, \{w, \sigma(v_1)\}, \{w, \sigma(v_2)\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_3\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{w\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{w, \sigma(v_3)\}\big\}. \end{aligned}$$

Then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

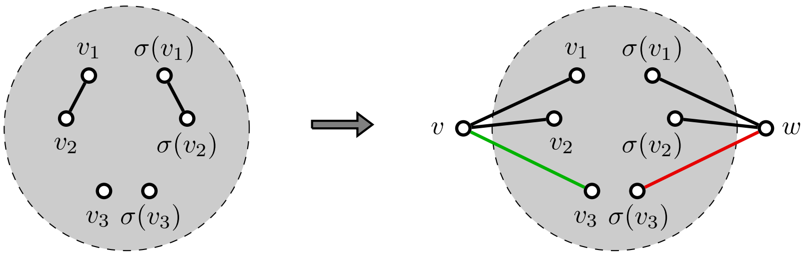


Figure 29: *Construction of a* $(\mathcal{C}_s, \Phi) \perp$ *or* $(\mathcal{C}_s, \Phi) \parallel$ *3Tree2 partition of* G *in the case where* G *is a* $(\mathcal{C}_s, \Phi_{k-1})$ *double edge split of* G_{k-1}, $\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\} \in E\big(T_0^{(k-1)}\big)$ *and either* $v_3 \notin V\big(T_0^{(k-1)}\big)$ *or* $v_3 \in V\big(T_0^{(k-1)}\big)$ *and* $\sigma(v_3) \neq v_3$. *The edges in black color represent edges of the invariant trees.*

Case 4.1b: If $\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\} \in E\big(T_0^{(k-1)}\big)$ and $v_3 \in V\big(T_0^{(k-1)}\big)$ with $\sigma(v_3) \neq v_3$, then wlog we have $v_3 \in V\big(T_1^{(k-1)}\big)$, and hence $\sigma(v_3) \in V\big(T_2^{(k-1)}\big)$, so that the previous construction in Case 4.1a can again be used to obtain a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 4.1c: Suppose $\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\} \in E\big(T_0^{(k-1)}\big)$ and $v_3 \in V\big(T_0^{(k-1)}\big)$ with $\sigma(v_3) = v_3$. Then, by Remarks 5.2 and 5.3, $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G and $\sigma(v_i) \neq v_i$ for $i = 1, 2$. Since $T_0^{(k-1)}$ is connected, we have that for $i = 1$ or $i = 2$, there exists a $v_3 - v_i$ path in $T_0^{(k-1)}$ that does not contain the edge $\{v_1, v_2\}$, say wlog $P = v_3, e_1, \dots, e_m, v_2$ is a $v_3 - v_2$ path in $T_0^{(k-1)}$ not containing the edge $\{v_1, v_2\}$. Then $\sigma(P)$ is a $v_3 - \sigma(v_2)$ path in $T_0^{(k-1)}$ not containing the edge $\{\sigma(v_1), \sigma(v_2)\}$ and P and $\sigma(P)$ do not share a common vertex other than v_3, for otherwise there exists a cycle in $T_0^{(k-1)}$. Assume wlog that $v_2 \in V\big(T_1^{(k)}\big)$, and hence $\sigma(v_2) \in V\big(T_2^{(k)}\big)$. Then we define $T_0^{(k)}$ to be the graph with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_0^{(k)}\big) &= \big(E\big(T_0^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\}\big\}\big) \\ &\quad \cup \big\{\{v, v_1\}, \{v, v_3\}, \{w, \sigma(v_1)\}, \{w, v_3\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the graph with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_2\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the graph with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{w\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{w, \sigma(v_2)\}\big\}. \end{aligned}$$

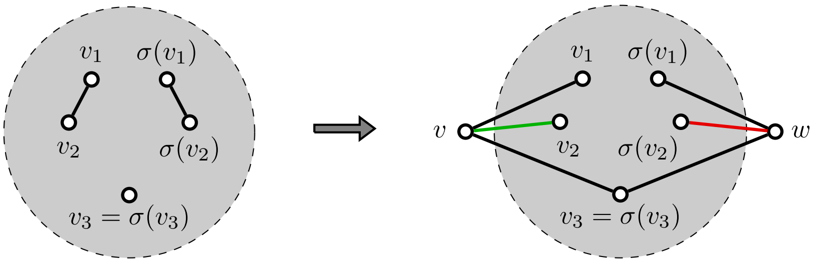


Figure 30: *Construction of a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ double edge split of G_{k-1}, $\{v_1, v_2\}, \{\sigma(v_1), \sigma(v_2)\} \in E\big(T_0^{(k-1)}\big)$, $v_3 \in V\big(T_0^{(k-1)}\big)$ and $\sigma(v_3) = v_3$. The edges in black color represent edges of the invariant trees.*

Clearly, the graphs $T_1^{(k)}$ and $T_2^{(k)}$ are trees and $T_0^{(k)}$ is connected. If there exists a cycle C in $T_0^{(k)}$, then C must contain at least one of the edges $\{v, v_3\}$ or $\{w, v_3\}$, for otherwise C does not contain any edge incident with v or w and there exists a cycle in $T_0^{(k-1)}$.

Suppose first that C contains only one of the two edges $\{v, v_3\}$ and $\{w, v_3\}$, say wlog C contains $\{v, v_3\}$, but not $\{w, v_3\}$. Then C contains the edge $\{v, v_1\}$, but not $\{w, \sigma(v_1)\}$. Thus, there exists a $v_3 - v_1$ path P' in $T_0^{(k-1)}$ that does not contain the edge $\{v_1, v_2\}$. This is a contradiction, because $v_3, e_1, \dots, e_m, v_2, \{v_1, v_2\}, v_1$ is also a $v_3 - v_1$ path in $T_0^{(k-1)}$ distinct from P'.

So, suppose C contains both edges $\{v, v_3\}$ and $\{w, v_3\}$. Then C also contains the edges $\{v, v_1\}$ and $\{w, \sigma(v_1)\}$. Thus, there exists a $v_1 - \sigma(v_1)$ path P'' in $T_0^{(k-1)}$ that does not contain the edges $\{v_1, v_2\}$ and $\{\sigma(v_1), \sigma(v_2)\}$. But $v_1, \{v_1, v_2\}, v_2, e_m, \dots, e_1, v_3, \sigma(e_1), \dots, \sigma(e_m), \sigma(v_2), \{\sigma(v_1), \sigma(v_2)\}, \sigma(v_1)$ is also a $v_1 - \sigma(v_1)$ path in $T_0^{(k-1)}$ distinct from P''.

Thus, $T_0^{(k)}$ is a tree and $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 4.2: Suppose $\{v_1, v_2\} \notin E\big(T_0^{(k-1)}\big)$, say wlog $\{v_1, v_2\} \in E\big(T_1^{(k-1)}\big)$. Then we also have $\{\sigma(v_1), \sigma(v_2)\} \in E\big(T_2^{(k-1)}\big)$.

Case 4.2a: If $v_3 \in V\big(T_0^{(k-1)}\big)$, then $\sigma(v_3) \in V\big(T_0^{(k-1)}\big)$. In this case we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_0^{(k)}\big) &= E\big(T_0^{(k-1)}\big) \cup \big\{\{v, v_3\}, \{w, \sigma(v_3)\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) &= \big(E\big(T_1^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}\big\}\big) \cup \big\{\{v, v_1\}, \{v, v_2\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{w\} \\ E\big(T_2^{(k)}\big) &= \big(E\big(T_2^{(k-1)}\big) \setminus \big\{\{\sigma(v_1), \sigma(v_2)\}\big\}\big) \\ &\quad \cup \big\{\{w, \sigma(v_1)\}, \{w, \sigma(v_2)\}\big\}. \end{aligned}$$

Then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

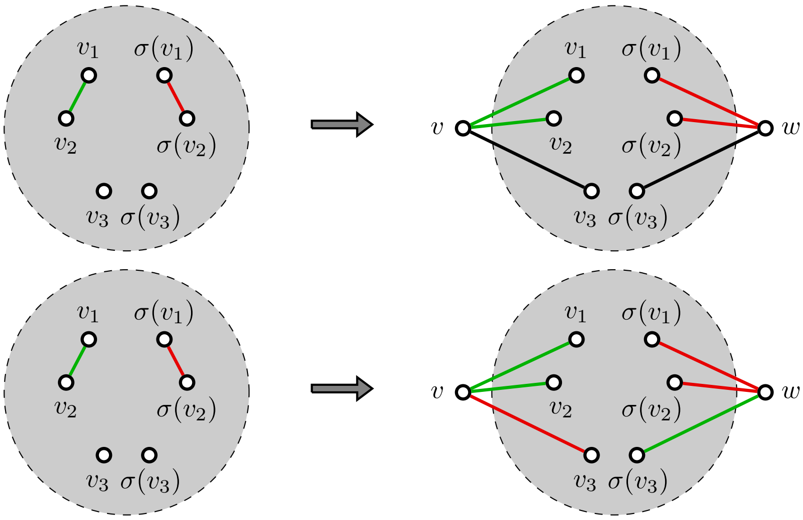


Figure 31: *Construction of a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ double edge split of G_{k-1}, $\{v_1, v_2\} \in E\big(T_1^{(k-1)}\big)$ and $\{\sigma(v_1), \sigma(v_2)\} \in E\big(T_2^{(k-1)}\big)$. The edges in black color represent edges of the invariant tree.*

Case 4.2b: If $v_3 \notin V\big(T_0^{(k-1)}\big)$, then $v_3 \in V\big(T_i^{(k-1)}\big)$ for $i = 1, 2$, and we define

$$T_0^{(k)} = T_0^{(k-1)},$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_1^{(k)}\big) &= \big(E\big(T_1^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}\big\}\big) \\ &\quad \cup \big\{\{v, v_1\}, \{v, v_2\}, \{w, \sigma(v_3)\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{v, w\} \\ E\big(T_2^{(k)}\big) &= \big(E\big(T_2^{(k-1)}\big) \setminus \big\{\{\sigma(v_1), \sigma(v_2)\}\big\}\big) \\ &\quad \cup \big\{\{w, \sigma(v_1)\}, \{w, \sigma(v_2)\}, \{v, v_3\}\big\}. \end{aligned}$$

Then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is again a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 5: Finally, suppose that G is a $(\mathcal{C}_s, \Phi_{k-1})$ X-replacement by v of G_{k-1} with $E(G) = \big(E(G_{k-1}) \setminus \big\{\{v_1, v_2\}, \{v_3, v_4\}\big\}\big) \cup \big\{\{v, v_i\} | \, i \in \{1, 2, 3, 4\}\big\}$. Then $\Phi_{k-1}(s)(\{v_1, v_2\}) = \{v_3, v_4\}$. Wlog we assume $\Phi_{k-1}(s)(v_1) = \sigma(v_1) = v_3$ and $\Phi_{k-1}(s)(v_2) = \sigma(v_2) = v_4$.

Case 5.1: Suppose $\{v_1, v_2\} \notin E\big(T_0^{(k-1)}\big)$, say wlog $\{v_1, v_2\} \in E\big(T_1^{(k-1)}\big)$. Then $\{v_3, v_4\} \in E\big(T_2^{(k-1)}\big)$. So, if we define

$$T_0^{(k)} = T_0^{(k-1)},$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v\} \\ E\big(T_1^{(k)}\big) &= \big(E\big(T_1^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}\big\}\big) \cup \big\{\{v, v_1\}, \{v, v_2\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{v\} \\ E\big(T_2^{(k)}\big) &= \big(E\big(T_2^{(k-1)}\big) \setminus \big\{\{v_3, v_4\}\big\}\big) \cup \big\{\{v, v_3\}, \{v, v_4\}\big\}, \end{aligned}$$

then $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

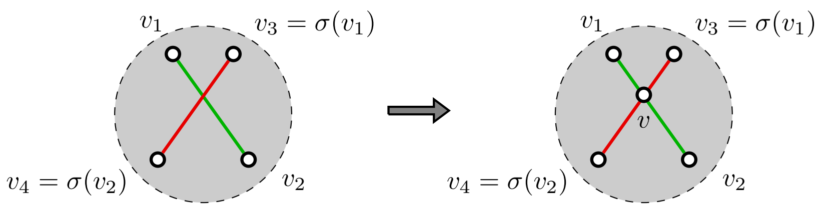


Figure 32: *Construction of a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ X-replacement of G_{k-1}, $\{v_1, v_2\} \in E\big(T_1^{(k-1)}\big)$ and $\{v_3, v_4\} \in E\big(T_2^{(k-1)}\big)$.*

Case 5.2: Suppose $\{v_1, v_2\}, \{v_3, v_4\} \in E\big(T_0^{(k-1)}\big)$. Since $T_0^{(k-1)}$ is a tree and $\Phi_{k-1}(s)\big(T_0^{(k-1)}\big) = T_0^{(k-1)}$, there either exists a $v_1 - v_3$ path that does not contain the vertices v_2 and v_4 or a $v_2 - v_4$ path that does not contain the vertices v_1 and v_3 in $T_0^{(k-1)}$. Suppose wlog that P is a $v_2 - v_4$ path in $T_0^{(k-1)}$ that does not contain the vertices v_1 and v_3. Wlog we may also assume that $v_2 \in V\big(T_2^{(k-1)}\big)$, and hence $v_4 \in V\big(T_1^{(k-1)}\big)$. If all the vertices and edges of P, as well as the edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$, are deleted from $T_0^{(k-1)}$, then the resulting subgraph of $T_0^{(k-1)}$ has at least two components, namely the components A with $v_1 \in V(A)$ and $\sigma(A) = B$ with $v_3 \in V(B)$.

Case 5.2.1: Suppose $V(A) = \{v_1\}$. Then we also have $V(B) = \{v_3\}$.

Case 5.2.1a: If $v_1 \in V\big(T_1^{(k-1)}\big)$, then $v_3 \in V\big(T_2^{(k-1)}\big)$. In this case we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \setminus \{v_1, v_3\} \\ E\big(T_0^{(k)}\big) &= E\big(T_0^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}, \{v_3, v_4\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v, v_3\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_3\}, \{v, v_4\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{v, v_1\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{v, v_1\}, \{v, v_2\}\big\}. \end{aligned}$$

Case 5.2.1b: If $v_1 \in V\big(T_2^{(k-1)}\big)$, then $v_3 \in V\big(T_1^{(k-1)}\big)$. In this case we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_0^{(k)}\big) &= V\big(T_0^{(k-1)}\big) \setminus \{v_1, v_3\} \\ E\big(T_0^{(k)}\big) &= E\big(T_0^{(k-1)}\big) \setminus \big\{\{v_1, v_2\}, \{v_3, v_4\}\big\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_1^{(k)}\big) &= V\big(T_1^{(k-1)}\big) \cup \{v, v_1\} \\ E\big(T_1^{(k)}\big) &= E\big(T_1^{(k-1)}\big) \cup \big\{\{v, v_1\}, \{v, v_4\}\big\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V\big(T_2^{(k)}\big) &= V\big(T_2^{(k-1)}\big) \cup \{v, v_3\} \\ E\big(T_2^{(k)}\big) &= E\big(T_2^{(k-1)}\big) \cup \big\{\{v, v_2\}, \{v, v_3\}\big\}. \end{aligned}$$

In both cases, $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Case 5.2.2: Finally, suppose $|V(A)| = |V(B)| = m \geq 2$. Then we first carry out the same construction as in Case 5.2.1. Subsequently, we delete all the edges of A and B from $E\big(T_0^{(k)}\big)$, one edge from both A and B at a time, and add them to either $E\big(T_1^{(k)}\big)$ or $E\big(T_2^{(k)}\big)$ in the following way.

Let $\widetilde{A}$ be the subgraph of A that only contains the single vertex v_1 and let

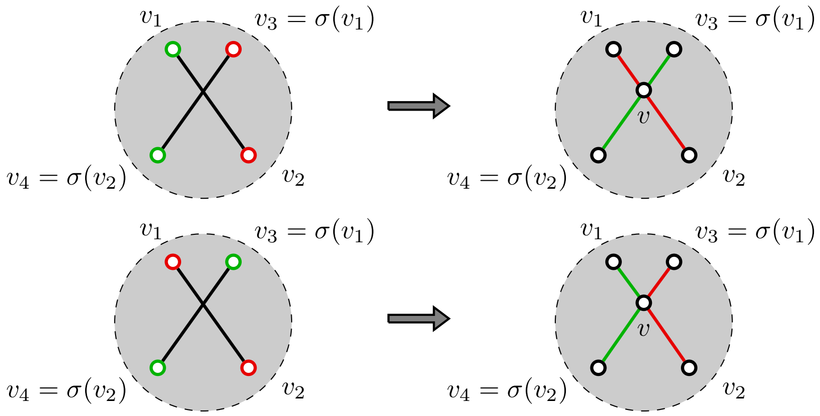


Figure 33: *Construction of a $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_s, \Phi_{k-1})$ X-replacement of G_{k-1} and $\{v_1, v_2\}, \{v_3, v_4\} \in E\big(T_0^{(k-1)}\big)$. The edges in black color represent edges of the invariant tree.*

$\widetilde{B}$ be the subgraph of B that only contains the single vertex $\sigma(v_1) = v_3$. Let $\{v_1, z\}$ be an edge of A. Then $\{v_3, \sigma(z)\}$ is an edge of B. By the construction in Case 5.2.1, $v_1, v_3 \in V\big(T_i^{(k)}\big)$ for $i = 1, 2$. Also, $\sigma(z) \neq z$ and $z, \sigma(z) \in V\big(T_0^{(k)}\big)$, which says that either $z \in V\big(T_1^{(k)}\big)$ and $\sigma(z) \in V\big(T_2^{(k)}\big)$ or $z \in V\big(T_2^{(k)}\big)$ and $\sigma(z) \in V\big(T_1^{(k)}\big)$.

We now delete the edges $\{v_1, z\}$ and $\{v_3, \sigma(z)\}$ from $E\big(T_0^{(k)}\big)$ and if $z \in V\big(T_1^{(k)}\big)$, then we add $\{v_1, z\}$ to $E\big(T_2^{(k)}\big)$ and $\{v_3, \sigma(z)\}$ to $E\big(T_1^{(k)}\big)$, and if $z \in V\big(T_2^{(k)}\big)$, then we add $\{v_1, z\}$ to $E\big(T_1^{(k)}\big)$ and $\{v_3, \sigma(z)\}$ to $E\big(T_2^{(k)}\big)$. Subsequently, we add the vertex z to $V(\widetilde{A})$, the vertex $\sigma(z)$ to $V(\widetilde{B})$, the edge $\{v_1, z\}$ to $E(\widetilde{A})$, and the edge $\{v_3, \sigma(z)\}$ to $E(\widetilde{B})$. If we then have $A = \widetilde{A}$, then $B = \widetilde{B}$ and $\big\{E\big(T_0^{(k)}\big), E\big(T_1^{(k)}\big), E\big(T_2^{(k)}\big)\big\}$ is a $(\mathcal{C}_s, \Phi) \perp$ or $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition of G.

Otherwise, there exists an edge $\{x, y\}$ in $E(A) \setminus E(\widetilde{A})$ with $x \in V(\widetilde{A})$ and $y \in V(A) \setminus V(\widetilde{A})$, and hence there also exists the edge $\{\sigma(x), \sigma(y)\}$ in $E(B) \setminus E(\widetilde{B})$ with $\sigma(x) \in V(\widetilde{B})$ and $\sigma(y) \in V(B) \setminus V(\widetilde{B})$. Note that since $x \in V(\widetilde{A})$ and $\sigma(x) \in V(\widetilde{B})$, we have $x, \sigma(x) \in V\big(T_i^{(k)}\big)$ for $i = 1, 2$. So, we can repeat the above construction step for the edges $\{x, y\}$ and $\{\sigma(x), \sigma(y)\}$. This process can be continued until $A = \widetilde{A}$ and $B = \widetilde{B}$. $\square$

Lemma 5.5 *Let G be a graph with $|V(G)| \geq 2$, $\mathcal{C}_s = \{Id, s\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_s \to Aut(G)$ be a homomorphism. If G has a proper $(\mathcal{C}_s, \Phi)$ 3Tree2 $\perp$ partition or a proper $(\mathcal{C}_s, \Phi)$ 3Tree2 $\parallel$ partition, then $\mathscr{R}_{(G, \mathcal{C}_s, \Phi)} \neq \emptyset$ and G is $(\mathcal{C}_s, \Phi)$-generically isostatic.*

Proof. **Case 1:** Suppose G has a proper $(\mathcal{C}_s, \Phi) \parallel$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$. In the following, we again denote $\Phi(s)$ by σ. There exists an edge $e = \{w, z\} \in E(T_1)$ such that $\sigma(w) = w$ and $\sigma(z) = z$ and, by Remark 5.3, $val_{T_1}(w) = 1$, $w \in E(T_0)$, and no other vertex of G that is fixed by σ is a vertex of T_0.

Since G has a 3Tree2 partition, G satisfies the count $|E(G)| = 2|V(G)| - 3$. Therefore, by Theorems 2.1 and 2.6, it suffices to find some framework $(G, p) \in \mathscr{R}_{(G,\mathcal{C}_s,\Phi)}$ that is independent.

Let V_i be the set of vertices of G that are not in $V(T_i)$ for $i = 0, 1, 2$ and let (G, p, q) be the frame with $p : V(G) \to \mathbb{R}^2$ and $q : E(G) \to \mathbb{R}^2$ defined by

$$p(v) = \begin{cases} (0,1) & \text{if } v \in V_0 \\ (-1,0) & \text{if } v \in V_1 \\ (1,0) & \text{if } v \in V_2 \setminus \{v\} \\ (0,0) & \text{if } v = w \end{cases}$$

$$q(b) = \begin{cases} (1,0) & \text{if } b \in E_{V_1,\{w\}} \text{ or } b \in E_{V_2\setminus\{w\},\{w\}} \\ (2,0) & \text{if } b \in E_{V_1,V_2\setminus\{w\}} \\ (-1,1) & \text{if } b \in E(T_1) \setminus \big\{\{w,z\}\big\} \\ (1,1) & \text{if } b \in E(T_2) \\ (0,1) & \text{if } b = \{w,z\} \end{cases},$$

where for disjoint sets $X, Y \in V(G)$, $E_{X,Y}$ denotes the set of edges of G incident with a vertex in X and a vertex in Y.

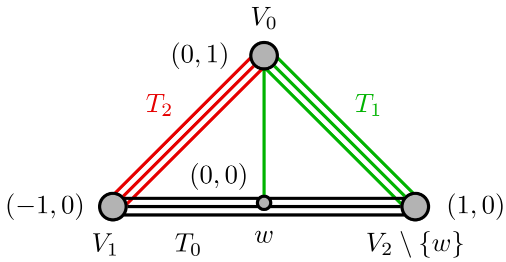


Figure 34: *The frame (G, p, q) in Case 1 of the proof of Lemma 5.5.*

We claim that the generalized rigidity matrix $\mathbf{R}(G, p, q)$ has linearly independent rows. To see this, we first rearrange the columns of $\mathbf{R}(G, p, q)$ in such a way that we obtain the matrix $\mathbf{R}'(G, p, q)$ which has the $(2i-1)^{st}$ column of $\mathbf{R}(G, p, q)$ in its i^{th} column and the $(2i)^{th}$ column of $\mathbf{R}(G, p, q)$ in its $(|V(G)| + i)^{th}$ column for $i = 1, 2, \dots, |V(G)|$. Let F_b denote the row vector of $\mathbf{R}'(G, p, q)$ that corresponds to the edge $b \in E(G)$. We then rearrange the rows of $\mathbf{R}'(G, p, q)$ in such a way that we obtain the matrix $\mathbf{R}''(G, p, q)$ which has the vectors F_b with $b \in E(T_0)$ in the rows $1, 2, \dots, |E(T_0)|$, the vectors F_b with $b \in E(T_1) \setminus \big\{\{w,z\}\big\}$ in the following $|E(T_1)| - 1$ rows, the vector $F_{\{w,z\}}$ in the next row, and the vectors F_b with $b \in E(T_2)$ in the last $|E(T_2)|$ rows. So

$\mathbf{R}''(G,p,q)$ is of the form

$$\left(\begin{array}{ccccccc|cccccccc}
1 & & & & -1 & & & & & & & & & & \\
 & & & \vdots & & & & & & & & \mathbf{0} & & & \\
 & & 2 & & & -2 & & & & & & & & & \\
\hline
-1 & & & 1 & & & & 1 & & & & -1 & & & \\
 & & & \vdots & & & & & & & & \vdots & & & \\
 & & -1 & & & & 1 & & & & 1 & & & & -1 \\
\hline
 & & & \mathbf{0} & & & & & 1 & & & & -1 & & \\
\hline
 & 1 & & & -1 & & & & & 1 & & & -1 & & \\
 & & & \vdots & & & & & & & & \vdots & & & \\
 & & & 1 & & -1 & & & & & & 1 & & -1 &
\end{array}\right).$$

Clearly, $\mathbf{R}(G,p,q)$ has a row dependency if and only if $\mathbf{R}''(G,p,q)$ does. Suppose $\mathbf{R}''(G,p,q)$ has a row dependency of the form

$$\sum_{b\in E(G)} \alpha_b F_b = 0,$$

where $\alpha_b \neq 0$ for some $b \in E(T_0)$. Since T_0 is a tree, it follows that

$$\sum_{b\in E(T_0)} \alpha_b F_b \neq 0.$$

Thus, there exists a vertex $v_s \in V(T_0)$, $s \in \{1,2,\ldots,|V(G)|\}$, such that

$$\sum_{b\in E(T_0)} \alpha_b (F_b)_s = C \neq 0.$$

Since $v_s \in V(T_0)$, v_s belongs to either T_1 or T_2.

Suppose first that $v_s \in V(T_2)$ and $v_s \notin V(T_1)$. Then $(F_b)_s = 0$ and $(F_b)_{|V(G)|+s} = 0$ for all $b \in E(T_1)$ and we have

$$\sum_{b\in E(T_2)} \alpha_b (F_b)_s = -C.$$

This says that

$$\sum_{b\in E(T_2)} \alpha_b (F_b)_{|V(G)|+s} = \sum_{b\in E(G)} \alpha_b (F_b)_{|V(G)|+s} = -C \neq 0,$$

a contradiction.

So, suppose that $v_s \in V(T_1)$ and $v_s \notin V(T_2)$. Then $(F_b)_s = 0$ and $(F_b)_{|V(G)|+s} = 0$ for all $b \in E(T_2)$ and we have

$$\sum_{b\in E(T_1)} \alpha_b (F_b)_s = -C.$$

Note that $v_s \neq w$, because $val_{T_1}(w) = 1$ and $(F_{\{w,z\}})_s = 0$ for all $s = 1,2,\ldots,|V(G)|$. Also, $v_s \neq z$, since $z \notin V(T_0)$. Therefore,

$$\sum_{b\in E(T_1)} \alpha_b (F_b)_{|V(G)|+s} = \sum_{b\in E(G)} \alpha_b (F_b)_{|V(G)|+s} = C \neq 0,$$

which is again a contradiction. So, if $\sum_{b\in E(G)} \alpha_b F_b = 0$ is a row dependency of $\mathbf{R}''(G,p,q)$, then $\alpha_b = 0$ for all $b \in E(T_0)$.

It is now only left to show that the matrix $\widetilde{\mathbf{R}}(G,p,q)$ which is obtained from $\mathbf{R}''(G,p,q)$ by deleting those rows of $\mathbf{R}''(G,p,q)$ that correspond to the edges of T_0 has linearly independent rows. Clearly, $\widetilde{\mathbf{R}}(G,p,q)$ has linearly independent rows if and only if the matrix $\widehat{\mathbf{R}}(G,p,q)$ has linearly independent rows, where $\widehat{\mathbf{R}}(G,p,q)$ is obtained by deleting the row $F_{\{w,z\}}$ from $\widetilde{\mathbf{R}}(G,p,q)$. In order to show that $\widehat{\mathbf{R}}(G,p,q)$ has linearly independent rows we may multiply $\widehat{\mathbf{R}}(G,p,q)$ by appropriate matrices of basis transformation and then use arguments analogous to above. So, as claimed, the frame (G,p,q) is independent.

Now, if (G,p) is not a framework, then we need to symmetrically pull apart those joints of (G,p,q) that have the same location in $\mathbb{R}^2$ and whose vertices are adjacent. So suppose $|V_1| \geq 2$. Then it follows that $|V_1| = |V_2 \setminus \{w\}| \geq 2$, because $\sigma(V_1) = V_2 \setminus \{w\}$. Since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, one of $\langle V_1\rangle \cap T_i$, $i = 0, 2$, is not connected.

Suppose first that $\langle V_1\rangle \cap T_0$ is not connected. Then $\langle V_2 \setminus \{w\}\rangle \cap T_0$ is also not

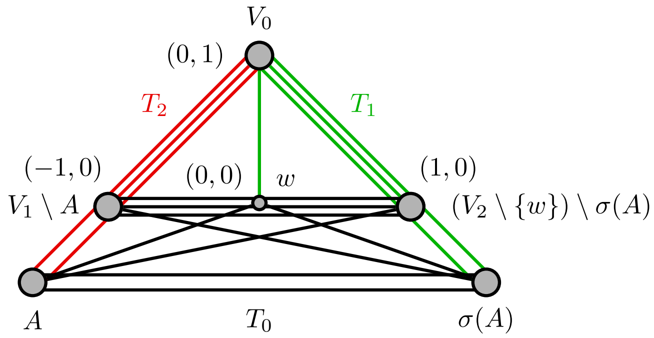


Figure 35: *The frame (G,p_t,q_t) in the case where $\langle V_1\rangle \cap T_0$ is not connected.*

connected. Let A be the set of vertices in one of the components of $\langle V_1\rangle \cap T_0$ and $\sigma(A)$ be the set of vertices in the corresponding component of $\langle V_2 \setminus \{w\}\rangle \cap T_0$. For $t \in \mathbb{R}$, we define $p_t : V(G) \to \mathbb{R}^2$ and $q_t : E(G) \to \mathbb{R}^2$ by

$$p_t(v) = \begin{cases} (-1-t, -t) & \text{if } v \in A \\ (1+t, -t) & \text{if } v \in \sigma(A) \\ p(v) & \text{otherwise} \end{cases}$$

$$q_t(b) = \begin{cases} (1+t, t) & \text{if } b \in E_{A,\{w\}} \\ (2+t, t) & \text{if } b \in E_{A,(V_2\setminus\{w\})\setminus\sigma(A)} \\ (1+t, -t) & \text{if } b \in E_{\sigma(A),\{w\}} \\ (2+t, -t) & \text{if } b \in E_{\sigma(A),V_1\setminus A} \\ q(b) & \text{otherwise} \end{cases}.$$

Suppose now that $\langle V_1\rangle \cap T_2$ is not connected. Then $\langle V_2 \setminus \{w\}\rangle \cap T_1$ is also not connected. Let B and $\sigma(B)$ be the vertex sets of components of $\langle V_1\rangle \cap T_2$ and $\langle V_2 \setminus \{w\}\rangle \cap T_1$, respectively. In this case, for $t \in \mathbb{R}$, we define $p_t : V(G) \to \mathbb{R}^2$ and $q_t : E(G) \to \mathbb{R}^2$ by

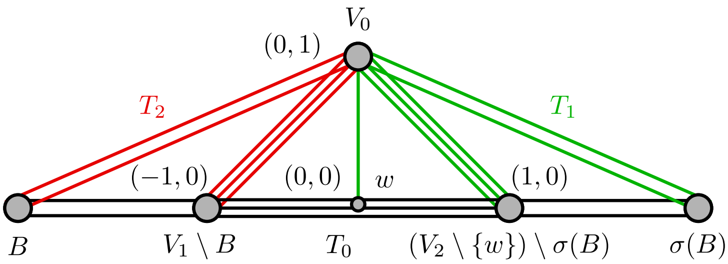


Figure 36: *The frame (G, p_t, q_t) in the case where $\langle V_1 \rangle \cap T_2$ is not connected.*

$$
\begin{aligned}
p_t(v) &= \begin{cases} (-1-t, 0) & \text{if } v \in B \\ (1+t, 0) & \text{if } v \in \sigma(B) \\ p(v) & \text{otherwise} \end{cases} \\
q_t(b) &= \begin{cases} (1+t, 1) & \text{if } b \in E_{B,V_0} \\ (-1-t, 1) & \text{if } b \in E_{\sigma(B),V_0} \\ q(b) & \text{otherwise} \end{cases} .
\end{aligned}
$$

In both cases, we have $(G, p_t, q_t) = (G, p, q)$ if $t = 0$. Therefore, by Lemma 3.3, there exists a $t_0 \in \mathbb{R}$, $t_0 \neq 0$, such that the frame (G, p_{t_0}, q_{t_0}) is independent. This process can be continued until we obtain an independent frame $(G, \hat{p}, \hat{q})$ which has the property that if $\hat{p}(u) = \hat{p}(v)$ for some $\{u, v\} \in E(G)$, then $u, v \in V_0$.

Suppose $(G, \hat{p})$ is still not a framework. Then $|V_0| \geq 2$ and since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, $\langle V_0 \rangle \cap T_1$ or $\langle V_0 \rangle \cap T_2$ is not connected. In fact, since $\sigma(\langle V_0 \rangle \cap T_1) = \langle V_0 \rangle \cap T_2$, both $\langle V_0 \rangle \cap T_1$ and $\langle V_0 \rangle \cap T_2$ are not connected. Let A be the set of vertices in one of the components of $\langle V_0 \rangle \cap T_2$ and $\sigma(A)$ be the set of vertices in the corresponding component of $\langle V_0 \rangle \cap T_1$. We denote $A \cap \sigma(A)$ by D and $A \cup \sigma(A)$ by F. Clearly, $E_{D, V_0 \setminus F} = \emptyset$, $E_{A \setminus D, V_0 \setminus F} \subseteq E(T_1)$ and $E_{\sigma(A) \setminus D, V_0 \setminus F} \subseteq E(T_2)$. Further, we have $E_{A \setminus D, \sigma(A) \setminus D} = \emptyset$ as the following argument shows.

Suppose to the contrary that there exists $\{x, y\} \in E(G)$ with $x \in A \setminus D$ and $y \in \sigma(A) \setminus D$. Then $\{x, y\} \in E(T_1)$ or $\{x, y\} \in E(T_2)$, say wlog $\{x, y\} \in E(T_2)$. Since $\{x, y\} \in E(\langle V_0 \rangle)$, it follows that $\{x, y\} \in E(\langle V_0 \rangle \cap T_2)$. Therefore, since $x \in A$, y must also be a vertex of A, contradicting the fact that $y \in \sigma(A) \setminus D$.

Finally, note that $E_{A \setminus D, D} \subseteq E(T_2)$, because if $\{x, y\} \in E(T_1)$, where $x \in A \setminus D$ and $y \in D$, then we must have $x \in \sigma(A)$, contradicting $x \in A \setminus D$. Similarly, we have $E_{\sigma(A) \setminus D, D} \subseteq E(T_1)$.

So, for $t \in \mathbb{R}$, we define $\hat{p}_t : V(G) \to \mathbb{R}^2$ and $\hat{q}_t : E(G) \to \mathbb{R}^2$ by

$$
\begin{aligned}
\hat{p}_t(v) &= \begin{cases} (-t, 1+t) & \text{if } v \in A \setminus D \\ (t, 1+t) & \text{if } v \in \sigma(A) \setminus D \\ (0, 1+2t) & \text{if } v \in D \\ \hat{p}(v) & \text{otherwise} \end{cases} \\
\hat{q}_t(b) &= \begin{cases} \hat{q}(b) + (t, t) & \text{if } b \in E_{V_2 \setminus \{w\}, \sigma(A) \setminus D} \\ \hat{q}(b) + (-t, t) & \text{if } b \in E_{V_1, A \setminus D} \\ \hat{q}(b) + (0, 2t) & \text{if } b \in E_{V_2 \setminus \{w\}, D} \\ \hat{q}(b) + (0, 2t) & \text{if } b \in E_{V_1, D} \\ \hat{q}(b) & \text{otherwise} \end{cases} .
\end{aligned}
$$

Then $(G, \hat{p}_t, \hat{q}_t) = (G, \hat{p}, \hat{q})$ if $t = 0$. Therefore, by Lemma 3.3, there exists a

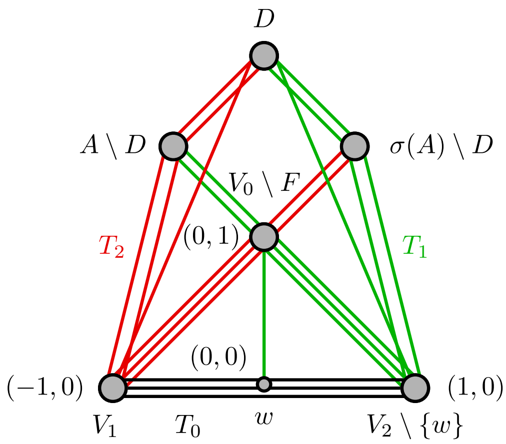


Figure 37: *The frame* $(G, \hat{p}_t, \hat{q}_t)$.

$t_0 \in \mathbb{R}$, $t_0 \neq 0$, such that the frame $(G, \hat{p}_{t_0}, \hat{q}_{t_0})$ is independent.

Now, if $|A \setminus D| \geq 2$, then $|\sigma(A) \setminus D| = |A \setminus D| \geq 2$. Since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, $\langle A \setminus D\rangle \cap T_1$ or $\langle A \setminus D\rangle \cap T_2$ is not connected, say wlog $\langle A \setminus D\rangle \cap T_2$ is not connected. Then $\langle \sigma(A) \setminus D\rangle \cap T_1$ is also not connected. Let B be the set of vertices in one of the components of $\langle A \setminus D\rangle \cap T_2$ and $\sigma(B)$ be the set of vertices in the corresponding component of $\langle \sigma(A) \setminus D\rangle \cap T_1$. Then, by using arguments analogous to above, we can pull apart the vertices of B from $(A \setminus D) \setminus B$ in the direction of the vector $(-t, t)$ and the vertices of $\sigma(B)$ from $(\sigma(A) \setminus D) \setminus \sigma(B)$ in the direction of the vector (t, t) in order to obtain a new independent frame.

This process can be continued until we obtain an independent frame $(G, \tilde{p}, \tilde{q})$ with $\tilde{p}(u) \neq \tilde{p}(v)$ for all $\{u, v\} \in E(G)$. Then, by Remark 3.1, $(G, \tilde{p})$ is an independent framework and, if necessary, an appropriate rotation of $(G, \tilde{p})$ about the origin yields an independent framework in the set $\mathscr{R}_{(G,\mathcal{C}_s,\Phi)}$.

Case 2: Suppose G has a proper $(\mathcal{C}_s, \Phi) \perp$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$. Let V_i be the set of vertices of G that are not in $V(T_i)$ for $i = 0, 1, 2$ and let $e_0 = (0, 1)$, $e_1 = (-1, 0)$, and $e_2 = (1, 0)$. We let (G, p, q) be the frame with $p : V(G) \to \mathbb{R}^2$ and $q : E(G) \to \mathbb{R}^2$ defined by

$$\begin{aligned} p(v) &= e_i \quad \text{if } v \in V_i \\ q(b) &= \begin{cases} (2,0) & \text{if } b \in E(T_0) \\ (-1,1) & \text{if } b \in E(T_1) \\ (1,1) & \text{if } b \in E(T_2) \end{cases} . \end{aligned}$$

The proof that (G, p, q) is independent and that we can construct an independent framework $(G, \tilde{p}) \in \mathscr{R}_{(G,\mathcal{C}_s,\Phi)}$ is analogous to the proof of Case 1. □

Lemmas 5.2, 5.3, 5.4, and 5.5 provide a complete proof for Theorem 5.1.

Remark 5.5 Theorem 5.1 still holds if we omit $(\mathcal{C}_s, \Phi_i)$ single edge splits in condition (iii). However, all the other inductive construction techniques, including

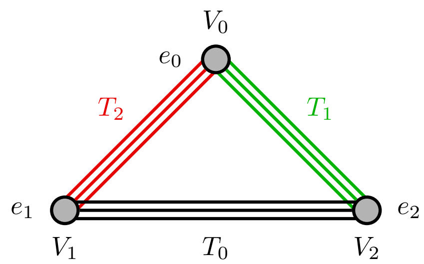


Figure 38: *The frame* (G, p, q) *in Case 2 of the proof of Lemma 5.5.*

the $(\mathcal{C}_s, \Phi_i)$ X-replacement, are necessary to characterize all $(\mathcal{C}_s, \Phi)$-generically isostatic graphs in terms of an inductive construction sequence [16].

Remark 5.6 The geometric proofs of Lemmas 5.2.6 and 5.2.7 in [16] can easily be adapted to also give direct geometric proofs that condition (iii) implies condition (i) in Theorems 4.1 and 5.1, i.e., that the existence of an (S, Φ) construction sequence for G implies that $\mathscr{R}_{(G,S,\Phi)} \neq \emptyset$ and that G is (S, Φ)-generically isostatic, for $S = \mathcal{C}_2$ or $S = \mathcal{C}_s$.

6 Further work

If one wants to prove the Laman-type conjectures for the dihedral groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ of respective orders 4 and 6 (see [3, 16]) in the analogous way as the symmetrized Laman Theorems for $\mathcal{C}_3$, $\mathcal{C}_2$, and $\mathcal{C}_s$, one has to consider two basic cases: first, the case where the given graph G has a vertex of valence 2, and secondly, the case where G has a vertex of valence 3 and no vertex of valence 2.

For each of the groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$, the first case can be treated in a straightforward fashion by using appropriate symmetrized versions of a vertex 2-addition.

We have seen in Section 5.2 that for vertices of valence 3, the presence of a single reflection s in the symmetry group S gives rise to a large number of subcases that need to be treated separately, where each subcase corresponds to a particular allocation of the 3-valent vertex and its three neighbors to the permutation cycles of the graph automorphism $\Phi(s)$. Since the symmetry groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ contain more than just one reflection (namely two and three, respectively), the number of subcases that need to be considered for these groups is even larger than it was in the case of $\mathcal{C}_s$. So, while we suspect that the Laman-type conjectures in [3] for the groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ can be proven in this way, the number of cases that need to be treated in these proofs becomes extremely large.

For the dihedral group $\mathcal{C}_{3v}$ of order 6, we conjecture that $(\mathcal{C}_{3v}, \Phi)$-generically isostatic graphs can also be characterized by means of suitably defined symmetrized 3Tree2 partitions. For further details on this conjecture we refer the reader to [16].

Due to the structure of the dihedral group $\mathcal{C}_{2v}$ of order 4, however, there does not seem to exist an analogous characterization of $(\mathcal{C}_{2v}, \Phi)$-generically isostatic graphs in terms of symmetrized 3Tree2 partitions.

Note that it is an immediate consequence of the symmetrized Laman Theorems for $\mathcal{C}_3$ (see [19]), $\mathcal{C}_2$, and $\mathcal{C}_s$ (and the analogous conjectures for $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$) that there is (would be) a polynomial time algorithm to determine whether a given graph G is (S, Φ)-generically isostatic. In fact, although the Laman conditions involve an exponential number of subgraphs of G, there are several algorithms that determine whether they hold in $c|V(G)||E(G)|$ steps, where c is a constant. The pebble game ([10]) is an example for such an algorithm. The additional symmetry conditions for the number of fixed structural elements can trivially be checked in constant time, from the graph automorphisms.

The results and methods presented in this paper open up a very wide range of new questions and directions concerning the rigidity and flexibility of various geometric constraint systems that possess non-trivial symmetries, ranging from the rigidity of symmetric pinned bar-and-joint frameworks (see [20, 16], for example), and symmetric body-bar, body-hinge and molecular structures (see [21, 24, 29] for some background on these structures and [8, 16, 21, 24, 29] for a number of relevant conjectures as well as some initial results), to various geometric constraint systems with symmetries appearing in CAD (see [15], for example). For some of these framework systems, such as 2-dimensional pinned bar-and-joint frameworks and body-bar structures with half-turn or reflectional symmetry, for instance, the results of this paper have already been transferred to necessary and sufficient conditions for these structures to be generically isostatic modulo the given symmetry [8, 16].

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