Mobility in symmetry-regular bar-and-joint frameworks

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Abstract

In a *symmetry-regular* bar-and-joint framework of given point-group symmetry, all bars and joints occupy general positions with respect to the symmetry elements. The symmetry-extended form of Maxwell's Rule is applied to this simplest type of framework and is used to derive counts within irreducible representations for infinitesimal mechanisms and states of self stress. In particular, conditions are given for symmetry-regular frameworks to have at least one infinitesimal mechanism (respectively, state of self stress) within each irreducible representation of the point group of the framework. Similar conditions are found for symmetry-regular body-and-joint frameworks.

1 Introduction

In 1864, James Clerk Maxwell published a counting rule that set out a necessary condition for a bar-and-joint framework to possess static and kinematic determinacy (Maxwell, 1864). In the modern form due to Pellegrino and Calladine (1986), the rule is

$$m - s = 3j - b - 6, \tag{1}$$

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where m is the number of mechanisms, s the number of states of self stress, b is the number of bars and j the number of joints in the framework. The form (1) applies to an unsupported three-dimensional framework, but is easily altered for supported frameworks and/or frameworks with other dimensionality.

A symmetry-extended form of the rule has been stated by Fowler and Guest (2000) in terms of point-group representations. The aim was to incorporate information not only on numbers of structural components and motifs, but also on the symmetries spanned by them in the point group of the framework, $\mathcal{G}$. In the language of point-group representations, the Maxwell Rule (1) becomes

$$\Gamma(m) - \Gamma(s) = \Gamma(j) \times \Gamma_T - \Gamma(b) - \Gamma_T - \Gamma_R, \tag{2}$$

where each Γ is the vector (or ordered set) of the traces of the corresponding representation matrices. Each such Γ is known in applied group theory as a representation of $\mathcal{G}$ (Bishop, 1973), or in mathematical group theory as a character (James and Liebeck, 2001). In this paper, the term character is reserved to denote an entry of a representation, i.e., the trace of a representation matrix. $\Gamma(m)$, $\Gamma(s)$, $\Gamma(j)$ and $\Gamma(b)$ are respectively the representations of mechanisms and states of self stress, and permutation representations of joints and bars. Γ_T and Γ_R are representations of the rigid-body translations and rotations. An equivalent statement in terms of the behaviour of the different objects under individual symmetry operations S is

$$\chi_m(S) - \chi_s(S) = \chi_j(S)\chi_T(S) - \chi_b(S) - \chi_T(S) - \chi_R(S), \tag{3}$$

where the various χ denote characters (i.e., traces of representation matrices) under operation S. For a permutation representation of a set of structureless objects, the character $\chi(S)$ is the number of objects left unshifted by S. For other types of object, the effects of the operation on signs and phases of the object must be taken into account. Techniques for calculation and manipulation of representations are described in many chemistry and physics texts, e.g., Bishop (1973), and comprehensive sets of character tables are available (Atkins et al., 1970; Altmann and Herzig, 1994).

Equations (2) and (3) deal with the full set of mechanisms and states of self stress, and lead to a larger set of necessary conditions, row by row or column by column of the character table, which can often lead to the detection of mechanisms, states of self stress, or both, that may have escaped the pure counting approach embodied in (1) (Kangwai and Guest, 1999; Kangwai et al., 1999; Kangwai and Guest, 2000; Guest, 2000; Fowler and Guest, 2002b,a, 2005; Kovács et al., 2004; Guest and Fowler, 2007, 2010; Fowler et al., 2008; Connelly et al., 2009; Schulze, 2010a,b). However, it has also proved fruitful in the study of, for example, protein mobility, to confine attention to mechanisms that are totally symmetric within the point group of the framework (Schulze and Whiteley, 2011). In such large systems, few if any structural elements occupy positions of non-trivial site symmetry, and useful global conclusions can be drawn from the study of frameworks under the restriction that all bars and joints are in general position. These restrictions, in particular the requirement that joints be in general position, also facilitate the construction of *orbit rigidity matrices* (Schulze and Whiteley, 2011) with rows and columns indexed by the orbits of bars and joints, respectively, leading to practical and theoretical advantages for prediction of mechanisms and states of self stress belonging to given irreducible representations.

The present note shows how this restriction to components in general positions, when applied to the symmetry-extended Maxwell Rule (2), leads to general consequences for the distributions of mechanisms and states of self stress across the symmetries available within the point group.

2 Mobility of a symmetry-regular framework

We call a bar-and-joint framework *symmetry-regular* if all joints and bars lie in general position with respect to the symmetry elements of the point group $\mathcal{G}$ that fixes the framework as a whole. In other words, no joint or bar in a symmetry-regular framework lies on any symmetry element of $\mathcal{G}$, and all joints and bars fall into (typically multiple) sets of equivalent objects permuted by all $|\mathcal{G}|$ operations of $\mathcal{G}$. These sets are *regular orbits* (Quinn et al., 1984) of $\mathcal{G}$. If the framework has j joints and b bars, then

$$j = j_0|\mathcal{G}|, \tag{4}$$

$$b = b_0|\mathcal{G}|, \tag{5}$$

where j_0 and b_0 are the respective numbers of regular orbits of joints and bars, respectively. Realisation of some point groups requires the presence of multiple orbits and hence $b_0 + j_0 > 1$ (Jahn and Teller, 1937), but this condition is in fact trivially satisified for the symmetry-regular frameworks of physical interest. The permutation representation of any single regular orbit of objects is Γ_{reg}, which has character $\chi_{\text{reg}}(E) = |\mathcal{G}|$ under the identity and $\chi_{\text{reg}}(S) = 0$ under all other operations. For all but the trivial point group $\mathcal{G} = C_1$, Γ_{reg} is reducible, and is the sum

$$\Gamma_{\text{reg}} = \sum_i g_i \Gamma_i, \tag{6}$$

where the summation runs over all the irreducible representations of the group, and g_i is the dimension of irreducible representation Γ_i, i.e., $\chi_i(E)$. Thus, Γ_{reg} contains one copy each of representations of types A and B, two of those of type E (or only one if E is a separably degenerate representation in an Abelian point group with complex characters), and three, four and five for those of types T, G and H, respectively (Bishop, 1973; Quinn et al., 1984).

From the expression for $\chi_{\text{reg}}(S)$, it is clear that the product of Γ_{reg} with any reducible representation Γ is simply $g\Gamma_{\text{reg}}$, where g is the dimension of Γ (equal to $\chi(E)$ for Γ).

For symmetry-regular frameworks, we have

$$\Gamma(j) = j_0\Gamma_{\text{reg}}, \tag{7}$$

$$\Gamma(b) = b_0\Gamma_{\text{reg}}, \tag{8}$$

and hence the symmetry-extended Maxwell Rule (2) reduces to

$$\Gamma(m) - \Gamma(s) = (3j_0 - b_0)\Gamma_{\text{reg}} - \Gamma_T - \Gamma_R. \tag{9}$$

Three cases can be distinguished, according to the mobility $m - s$ computed with the scalar counting version of the Maxwell Rule. A symmetry-regular framework may have:

Case(i) $3j_0 - b_0 < 0$, and hence (by equations (1), (4) and (5))

$$m - s = 3j - b - 6 = (3j_0 - b_0)|\mathcal{G}| - 6 < -6;$$

Case(ii) $3j_0 - b_0 = 0$, and hence

$$m - s = 3j - b - 6 = -6;$$

Case(iii) $3j_0 - b_0 > 0$, and hence

$$m - s = 3j - b - 6 = (3j_0 - b_0)|\mathcal{G}| - 6 > -6.$$

In Case (i), $|(3j_0 - b_0)||\mathcal{G}| + 6$ states of self stress are detectable by symmetry. Since $3j_0 - b_0 < 0$, it follows from equations (6) and (9) that there are at least g_i states of self stress for each irreducible representation Γ_i, augmented by a further six states of self stress that match the symmetries of the translations and rotations in the point group. Thus, every irreducible representation of the point group occurs as the symmetry of a state of self stress in this case.

In Case (ii), equation (9) becomes $\Gamma(m) - \Gamma(s) = -\Gamma_T - \Gamma_R$, and hence the only states of self stress detectable by symmetry are six that match the translations and rotations in the point group, and again no mechanisms are detectable by symmetry counting alone.

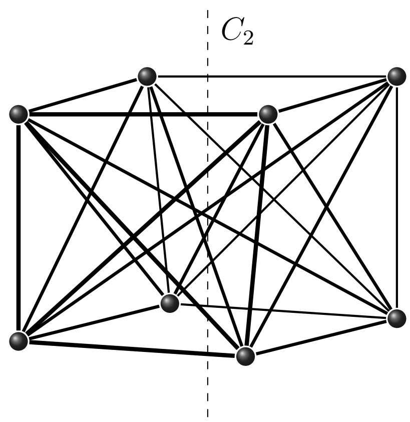

Figure 1: A symmetry-regular framework with point group $\mathcal{C}_2$.

Example 1 *Consider the symmetry-regular framework with point group* $\mathcal{C}_2$ *depicted in Fig. 1. This framework is a ring of four edge-sharing tetrahedra with four additional bars which correspond to the four diagonals of the cube formed by the eight vertices. For this framework, the scalar counting version of the Maxwell Rule detects six states of self stress, because* $3j - 6 - b = 3 \times 8 - 6 - 24 = -6$*. The character table for the group* $\mathcal{C}_2$ *is*

$\mathcal{C}_2$	E	C_2	*Symmetry of rigid motions*
A	*1*	*1*	$z,\ R_z$
B	*1*	*-1*	$x,\ y,\ R_x,\ R_y$

So, since $j_0 = 4$ *and* $b_0 = 12$*, we have* $3j_0 - b_0 = 0$ *and equation (9) becomes*

$$\Gamma(m) - \Gamma(s) = -\Gamma_T - \Gamma_R = -(A + 2B) - (A + 2B) = -2A - 4B.$$

Thus, there exist two states of self stress of symmetry A (fully symmetric self stresses) and four states of self stress of symmetry B (anti-symmetric self stresses). In particular, we detect a self stress for each irreducible representation of the group $\mathcal{C}_2$*. For frameworks satisfying the condition of Case (ii), this is not true in general (since for certain point groups,* $\Gamma_T + \Gamma_R$ *does not contain each of the irreducible representations of the group (Bishop, 1973)).*

In Case (iii), the precise prediction for the mobility depends on the numerical value of $m - s$ and on the point group.

If an irreducible representation Γ_i has a positive weight on the LHS of equation (9), then every framework which is symmetric with the given group has an infinitesimal motion of symmetry Γ_i.

If the infinitesimal motion is totally symmetric (i.e., if Γ_i is the trivial irreducible representation which assigns the scalar 1 to each symmetry operation of the group), then it also extends to a finite symmetry-preserving mechanism, provided that the framework is 'generic' modulo the given symmetry constraints (or equivalently, if the orbit rigidity matrix of the framework has maximal rank). See Schulze (2010b) and Schulze and Whiteley (2011) for details. Similarly, an infinitesimal motion which is symmetric with respect to Γ_i, where Γ_i is not the totally symmetric irreducible representation, extends to a finite mechanism, provided that the framework is at a 'regular point' for the algebraic variety of all Γ_i-symmetric configurations (see again Schulze (2010b) for details). Such a mechanism preserves the sub-symmetry described by the kernel of the representation Γ_i (see Guest and Fowler, 2007; Schulze, 2010b).

While it is in general difficult to check whether a framework is at a regular point for a given (not totally symmetric) irreducible representation Γ_i, there exist some special situations, where the presence of a finite mechanism can easily be deduced from the existence of an infinitesimal motion of symmetry Γ_i. As shown in Guest and Fowler (2007) a Γ_i-symmetric motion will be finite if, in the point group of the undisplaced framework, there is neither a Γ_i-symmetric nor a totally symmetric self stress. This will be the case in many of the following examples, which do not possess a state of self stress.

Clearly, if all irreducible representations have positive weight on the LHS of (9), then there are symmetry-detectable mechanisms of all symmetries; if some have zero or negative weight, there are gaps in the symmetries of the detectable mechanisms, and for those irreducible representations with negative weights, there are corresponding symmetry-detectable states of self stress. As the representation of the rigid-body motions, $\Gamma_T + \Gamma_R$, has fixed dimension 6, the detailed prediction for Case (iii) depends on the size and composition of Γ_{reg} in $\mathcal{G}$. Defining

$$\Gamma_{\text{rigid}} = \Gamma_T + \Gamma_R = \sum_i n_i \Gamma_i, \tag{10}$$

it is convenient to sub-divide Case (iii) according to the values in the sets $\{n_i\}$ *vs* $\{g_i\}$, as follows. In what follows, we will use 'detectable' as a synonym for 'detectable by symmetry'. The three sub-cases are:

Sub-case(iii)(a) $3j_0 - b_0 > 0$ and Γ_{rigid} is an exact multiple of Γ_{reg}, i.e., $\Gamma_{\text{rigid}} = k\Gamma_{\text{reg}}$, $k \in \mathbb{N}, k \geq 1$;

Sub-case(iii)(b) $3j_0 - b_0 > 0$ and Γ_{rigid} is contained in Γ_{reg}, i.e., $n_i \leq g_i$ for all irreducible representations Γ_i;

Sub-case(iii)(c) $3j_0 - b_0 > 0$ and Γ_{rigid} is contained in $k\Gamma_{\text{reg}}$, i.e., $n_i \leq kg_i$ for all i (where $k = 2$ or 4).

We next consider each case in turn.

Sub-case (iii)(a): Since Γ_{rigid} is an exact multiple of Γ_{reg}, it follows from the dimensions of the representations that $\Gamma_{\text{rigid}} = (6/|\mathcal{G}|)\Gamma_{\text{reg}}$, with $|\mathcal{G}| \leq 6$. The mobility representation obeys

$$\Gamma(m) - \Gamma(s) = (3j_0 - b_0 - \frac{6}{|\mathcal{G}|})\Gamma_{\text{reg}}, \tag{11}$$

with $|\mathcal{G}| = 1, 2, 3, 6$. The groups of this type are: $\mathcal{C}_1$ with $|\mathcal{G}| = 1$, $\mathcal{C}_\text{s}$ and $\mathcal{C}_i$, with $|\mathcal{G}| = 2$, $\mathcal{C}_3$ with $|\mathcal{G}| = 3$, and $\mathcal{C}_{3\text{v}}$, $\mathcal{C}_{3\text{h}}$ and $\mathcal{S}_6$ with $|\mathcal{G}| = 6$.

For $3j_0 - b_0 < 6/|\mathcal{G}|$ (and $|\mathcal{G}| < 6$), there are detectable states of self stress spanning all symmetries. If $3j_0 - b_0 = 6/|\mathcal{G}|$, symmetry detects neither states of self stress nor mechanisms. If $3j_0 - b_0 > 6/|\mathcal{G}|$, symmetry detects mechanisms in all irreducible representations.

In particular, note that detectable states of self stress occur only for the groups $\mathcal{C}_1, \mathcal{C}_\text{s}, \mathcal{C}_\text{i}$ and $\mathcal{C}_3$.

Example 2 *Consider the symmetry-regular $\mathcal{C}_s$-symmetric framework with $j_0 = 3$ illustrated in Fig. 2. Here, b_0 has been chosen as 6, to achieve the isostatic count of $3j_0 - b_0 - 3 = 0$. Successive removal of orbits of bars, one orbit at a time, adds two mechanisms at each stage. By equation (11), the extra pair of mechanisms spans $\Gamma_{reg} = A' + A''$, i.e., one of the extra mechanisms is symmetric and the other is anti-symmetric with respect to the mirror.*

Sub-case(iii)(b): If the inequalities are strict for all i, (i.e., if $n_i < g_i$ for all i) then it follows from equation (9) that the framework has detectable mechanisms belonging to every irreducible representation of the group. The groups of this type are: $\mathcal{T}$, $\mathcal{T}_\text{d}$, $\mathcal{T}_\text{h}$, $\mathcal{O}$, $\mathcal{O}_\text{h}$, $\mathcal{I}$, $\mathcal{I}_\text{h}$.

Example 3 *Consider a realisation of the cuboctahedron with point group symmetry $\mathcal{T}$ — the group of rotational symmetries of the regular tetrahedron (see also Fig. 3). This framework satisfies the condition of Case (iii) since $3j_0 - b_0 = 3 \times 1 - 2 = 1 > 0$. Moreover, from the character table of the group $\mathcal{T}$*

$\mathcal{T}$	E	$4C_3$	$4C_3^2$	$3C_2$	*Symmetry of rigid motions*
A	1	1	1	1	
E	2	-1	-1	2	
T	3	0	0	-1	$x,\ y,\ z,\ R_x,\ R_y,\ R_z$

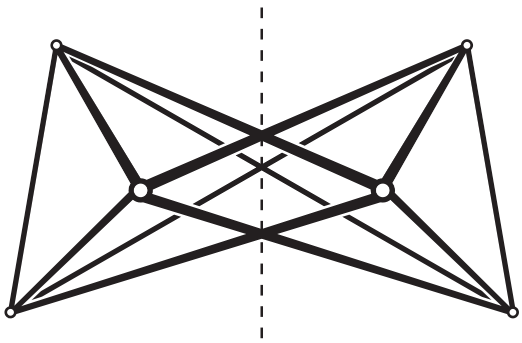

Figure 2: An example of an isostatic framework with reflection symmetry (the mirror plane is indicated by a dotted line). This $\mathcal{C}_s$-symmetric framework is non-planar. Larger and smaller circles indicate joints that lie respectively in front of, and behind the median plane of the framework.

it follows that $\Gamma_{reg} - \Gamma_{rigid} = (A + E + 3T) - 2T = A + E + T$. *Thus, we detect a mechanism for each of the irreducible representations of* $\mathcal{T}$*, one of symmetry* A*, two of symmetry* E*, and three of symmetry* T*. Each of these mechanisms is finite since the framework clearly does not have any self stress (it is obtained from a triangulated convex polyhedron, which is isostatic by Cauchy's theorem, by removing six bars).*

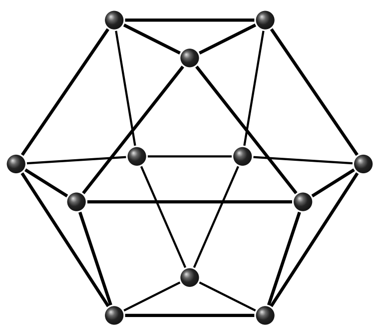

Figure 3: A bar-and-joint framework connected as the skeleton of the cuboctahedron.

If, instead, $n_i = g_i$ for some i, there are systematic absences in the list of detectable mechanisms for $(3j_0 - b_0) = 1$: where $n_i = g_i$, Γ_i is missing from the list. For $(3j_0 - b_0) \geq 2$, however, all irreducible representations are present in $\Gamma(m)$. The groups of this type, and the irreducible representations missing from $\Gamma(m)$ are presented in Table 1. [1]

Point group	Missing irreducible representations
$\mathcal{C}_{\mathrm{nv}}$, with $n \geq 4$	A_1, A_2, $E_{(1)}$
$\mathcal{C}_{\mathrm{nh}}$, with $n \geq 4$	A'/A_g, A''/A_u, E'/E_g, E''/E_u
$\mathcal{D}_{\mathrm{2h}}$	all except A_g and A_u
$\mathcal{D}_{\mathrm{nh}}$, with $n \geq 3$	A_2'/A_{2g}, A_2''/A_{2u}
$\mathcal{D}_{\mathrm{2d}}$	A_2, B_2 and E
$\mathcal{D}_{\mathrm{nd}}$, with $n \geq 3$	$A_{2(g)}$, $A_{2(u)}/B_2$
$\mathcal{S}_{\mathrm{4n}}$, with $n > 1$	A, B, E_1, $E_{(n/2-1)}$
$\mathcal{S}_{\mathrm{4n+2}}$, with $n \geq 1$	A_g, A_u, $E_{(1)g}$, $E_{(1)u}$

Table 1: List of groups with $n_i = g_i$, and hence some irreducible representations missing from $\Gamma(m)$ for $(3j_0 - b_0) = 1$.

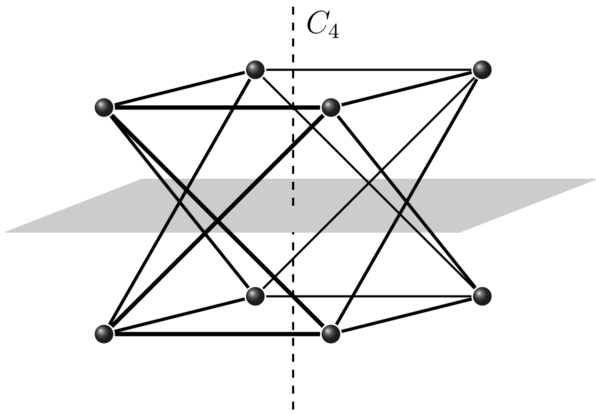

Figure 4: A framework which is symmetry-regular with respect to $\mathcal{C}_{4h}$.

Example 4 *The framework in Figure 4 is symmetry-regular with respect to the symmetry group $\mathcal{C}_{4h}$ and satisfies $j_0 = 1$ and $b_0 = 2$, so that $(3j_0 - b_0) = 1$. (Note that the point group of the framework is actually the group $\mathcal{D}_{4h}$. However, the framework is not symmetry-regular with respect to $\mathcal{D}_{4h}$.) Since $\Gamma_{reg} = A_g + B_g + E_g + A_u + B_u + E_u$ and $\Gamma_{rigid} = A_g + E_g + A_u + E_u$, we*

[1]Many published character tables for $\mathcal{S}_8$ correctly assign $\Gamma(x, y)$ to E_1 but incorrectly $\Gamma(R_x, R_y)$ to E_1 instead of E_3 (Shirts, 2007). The problem extends to some tables for $\mathcal{S}_{12}$, $\mathcal{S}_{16}$ and $\mathcal{S}_{20}$ (Altmann and Herzig, 1994), where the correct assignment is $\Gamma(R_x, R_y) = E_{(n/2-1)}$.

detect two motions, one of symmetry B_g (which preserves the sub-group $\mathcal{C}_{2h}$) and one of symmetry B_u (which preserves the sub-group $\mathcal{S}_4$). Both of these mechanisms can be shown to be finite, although the framework also has a self stress (and hence an additional infinitesimal motion).

Sub-case(iii)(c): The groups with $n_i \leq kg_i$ for all i (where $k = 2$ or 4) are: $\mathcal{C}_2$ ($k = 4$), $\mathcal{C}_{2\text{v}}$ ($k = 2$), $\mathcal{C}_{2\text{h}}$ ($k = 2$), $\mathcal{D}_\text{n}$ with $n \geq 2$ ($k = 2$), $\mathcal{S}_4$ ($k = 2$), and $\mathcal{C}_\text{n}$, with $n \geq 4$ ($k = 2$).

When $n_i = kg_i$, the representation Γ_i is the symmetry of a detectable state of self stress, a detectable mechanism, or absent from both lists, depending on whether $(3j_0 - b_0) < k$, $(3j_0 - b_0) > k$, or $(3j_0 - b_0) = k$, respectively.

When $n_i = (k/2)g_i$, Γ_i is either the symmetry of a detectable mechanism, or is absent from the lists of both mechanisms and self stresses, depending on whether $(3j_0 - b_0) > k/2$ or $(3j_0 - b_0) = k/2$.

In the groups $\mathcal{C}_\text{n}$, with $n \geq 4$, $n_i = 0$ for all but A and $E_{(1)}$. Hence for these groups, all irreducible representations, except A and $E_{(1)}$, are present in the list of detectable mechanisms for all positive $(3j_0 - b_0)$.

Example 5 *The symmetry-regular framework with $\mathcal{C}_{2v}$ symmetry shown in Fig. 5 is a three-dimensional realisation of the complete bipartite graph, $K_{4,4}$. When considered in the 3D setting, this framework is underbraced by two bars, and hence has two finite mechanisms if realised generically without symmetry. For $\mathcal{C}_{2v}$, we have $k = 2$, $\Gamma_{reg} = A_1 + A_2 + B_1 + B_2$, and $\Gamma_{rigid} = A_1 + A_2 + 2B_1 + 2B_2$, so that*

$$\begin{aligned}\Gamma(m) - \Gamma(s) &= (3j_0 - b_0)\Gamma_{reg} - \Gamma_{rigid} \\ &= (3j_0 - b_0 - 1)\Gamma_{reg} - B_1 - B_2.\end{aligned}$$

For the example shown, we have $j_0 = 2$, $b_0 = 4$ (and hence $(3j_0 - b_0) = 2 = k$). Thus, $\Gamma(m) - \Gamma(s) = A_1 + A_2$, i.e., the framework has two infinitesimal mechanisms, one totally symmetric and one preserving only the $\mathcal{C}_2$ rotational symmetry. Except at specific singular geometric configurations (Schulze and Whiteley, 2011), the framework does not have a state of self stress, and hence these infinitesimal mechanisms are in fact finite. Projected into a horizontal plane, the A_1 motion corresponds to the Bottema mechanism (Bottema, 1960). The A_2 motion has quadrupolar character, and displacement along the A_2 path reduces the overall symmetry to $\mathcal{C}_2$, where $k = 4$, $\Gamma_{reg} = A + B$, and $\Gamma_{rigid} = 2A + 4B$, so that

$$\Gamma(m) - \Gamma(s) = (3j_0 - b_0 - 2)\Gamma_{reg} - 2B,$$

where now $j_0 = 4$, $b_0 = 8$, and $(3j_0 - b_0) = 4 = k$, as $|\mathcal{G}|$ has fallen to 2. Hence, in the lower symmetry group, there are two totally symmetric mechanisms.

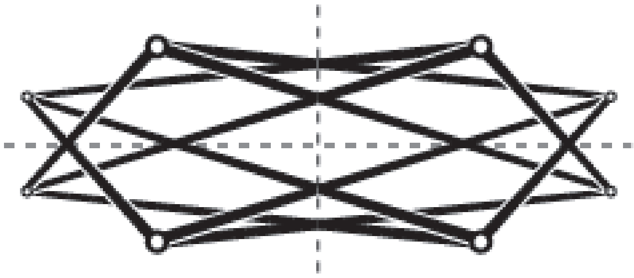

Figure 5: An example of an over-constrained framework with $\mathcal{C}_{2\mathrm{v}}$ symmetry based on a non-planar realisation of the complete bipartite graph $K_{4,4}$. As in Fig. 2, dotted lines indicate mirror planes and larger and smaller circles indicate joints that lie respectively in front of, and behind the median plane of the framework.

3 Restriction to the plane

The discussion has concentrated on the Maxwell Rule for bar-and-joint frameworks in 3D, but similar conclusions are readily obtained for frameworks restricted to 2D.

The 2D restriction is made by deletion of terms from Γ_T and Γ_R. The counting rule, symmetry theorem and the theorem for symmetry-regular frameworks equivalent to (1), (2) and (9) are:

$$m - s = 2j - b - 3, \tag{12}$$

$$\Gamma(m) - \Gamma(s) = \Gamma(j) \times \Gamma_T(x, y) - \Gamma(b) - \Gamma_T(x, y) - \Gamma_R(xy), \tag{13}$$

$$\Gamma(m) - \Gamma(s) = (2j_0 - b_0)\Gamma_{\mathrm{reg}} - \Gamma_T(x, y) - \Gamma_R(xy), \tag{14}$$

where the framework is supposed to be confined to the xy plane, $\Gamma_T(x, y)$ is the representation of the two translations in the framework plane, and $\Gamma_R(xy)$ is the representation of the rotation in that plane.

In 2D, the possible point groups are $\mathcal{C}_{\mathrm{n}}$ and $\mathcal{C}_{\mathrm{nv}}$ (with $\mathcal{C}_{1\mathrm{v}} \equiv \mathcal{C}_{\mathrm{s}}$). It follows from the character tables of these groups that the representations of the rigid-body motions in the xy plane have the following form:

2D point group	$\Gamma_{\text{rigid}}(x, y)$
$\mathcal{C}_1$	3A
$\mathcal{C}_2$	$A + 2B$
$\mathcal{C}_n$, $n \geq 3$	$A + E_{(1)}$
$\mathcal{C}_s$	$A' + 2A''$
$\mathcal{C}_{2v}$	$A_2 + B_1 + B_2$
$\mathcal{C}_{nv}$, $n \geq 3$	$A_2 + E_{(1)}$

In a direct analogy with the $3D$ mobility analysis, the $2D$ analysis falls into the following three cases:

Case (i) $2j_0 - b_0 < 0$, and hence

$$m - s = 2j - b - 3 = (2j_0 - b_0)|\mathcal{G}| - 3 < -3;$$

Case (ii) $2j_0 - b_0 = 0$, and hence

$$m - s = 2j - b - 3 = -3;$$

Case (iii) $2j_0 - b_0 > 0$, and hence

$$m - s = 2j - b - 3 = (2j_0 - b_0)|\mathcal{G}| - 3 > -3.$$

As in $3D$, mechanisms can be detected only in Case (iii). Note that for the groups $\mathcal{C}_1$ and $\mathcal{C}_3$, $\Gamma_{\text{rigid}}(x, y)$ is an exact multiple of Γ_{reg} (for $\mathcal{C}_3$, we even have $\Gamma_{\text{rigid}}(x, y) = \Gamma_{\text{reg}}$). The groups for which $\Gamma_{\text{rigid}}(x, y)$ is contained in Γ_{reg} are $\mathcal{C}_n$, $n > 3$, and $\mathcal{C}_{nv}$, $n \geq 2$, and the groups for which $\Gamma_{\text{rigid}}(x, y)$ is contained in $k\Gamma_{\text{reg}}$, where $k = 2$, are $\mathcal{C}_2$ and $\mathcal{C}_s$.

4 Extension to body-and-joint frameworks

Similar reasoning can be applied to the analysis of mobility of body-and-joint frameworks, where the joints may be of any type, e.g. revolute hinges, screw joints or spherical joints. The mobility criterion for a linkage consisting of v bodies connected by e joints, where joint i permits f_i relative freedoms, is (Hunt, 1978)

$$m - s = 6(v - 1) - 6e + \sum_{i=1}^{e} f_i. \tag{15}$$

The symmetry-extended version of the mobility rule is then (Guest and Fowler, 2005)

$$\Gamma(m) - \Gamma(s) = \Gamma(v, C) \times (\Gamma_T + \Gamma_R) - \Gamma_T - \Gamma_R - \Gamma_{\parallel}(e, C) \times (\Gamma_T + \Gamma_R) + \Gamma_{\text{f}}, \tag{16}$$

where the notation is motivated by the association of the bodies with the vertices of a contact polyhedron C and the hinges with the edges of that polyhedron. $\Gamma_{\|}(e, C)$ is the representation of a set of vectors along the edges of C, and Γ_{f} is the representation of the total set of freedoms allowed by the joints. Calculation of Γ_{f} requires specification of the types of hinges, but is straightforwardly calculated for each type.

In the case of a symmetry-regular body-and-joint framework with a contact polyhedron belonging to point group $\mathcal{G}$, the bodies and hinges are all in general position, and both $\Gamma(v, C)$ and $\Gamma_{\|}(e, C)$ consist of sets of copies of the regular representation. The hinges may admit different types and numbers of freedoms, but again Γ_{f} consists of a number of complete copies of the regular representation. The form of (16) applicable to symmetry-regular frameworks is therefore

$$\Gamma(m) - \Gamma(s) = (6v_0 - 6e_0 + F_0) \times \Gamma_{\mathrm{reg}} - \Gamma_T - \Gamma_R, \tag{17}$$

where the orbit counts are $v_0 = v/|\mathcal{G}|$ for bodies, $e_0 = e/|\mathcal{G}|$ for joints, and F_0 for total freedoms, where

$$F_0 = \sum_{i=1}^{5} i\, f_{0,i} \tag{18}$$

and $f_{0,i}$ is the number of orbits of hinges that admit i freedoms. Given this equation, the analysis follows the same course as for pin-jointed frameworks, with $6(v_0 - e_0) + F_0$ playing the role of $3j_0 - b_0$ in the arguments.

Example 6 *Consider the body-and-joint framework shown in Fig. 6, which is a representative of the twist-boat conformation of cyclohexane, or the* 6*-loop (Guest and Fowler, 2010). The point-group symmetry is* $\mathcal{C}_2$*, where* $\Gamma_T - \Gamma_R = 2A + 4B$ *and all bodies and joints lie in general position with respect to the* C_2 *axis. We have 6 bodies, 6 joints and 6 freedoms spanning* $v_0 = 3$*,* $e_0 = 3$ *and* $F_0 = 3$ *copies of the regular representation* $A + B$*. Thus, the symmetry-extended mobility rule, (17), gives* $\Gamma(m) - \Gamma(s) = (6v_0 - 6e_0 + F_0) \times \Gamma_{reg} - \Gamma_T - \Gamma_R = A - B$*. Hence,* $\Gamma(m)$ *is* Γ_{reg} *minus the* B *representation, and* $\Gamma(m)$ *is* Γ_{reg} *minus the* A *representation. Simple scalar counting is consistent with an isostatic framework, but symmetry has revealed a totally symmetric (and hence finite) mechanism, and an antisymmetric state of self stress.*

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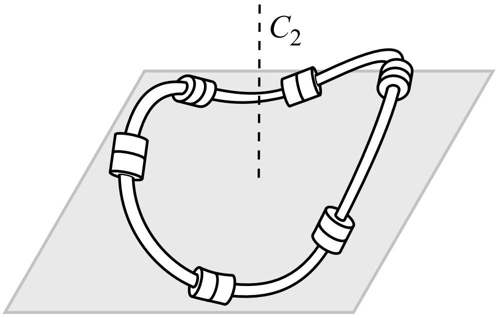

Figure 6: A schematic view of a 6-loop, adapted from Guest and Fowler (2010). Six curved bodies are connected by six in-line revolute joints, each of which allows a single, twisting, degree of freedom.

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