Final Report on EPSRC grant EP/E002625/1 : Dilations and Higher Rank Operator Algebras — with Applications

1. Background

Many significant research areas of contemporary analysis lie in noncommutative generalisations of mathematical frameworks and with multi-variable and higher rank perspectives. One example of this, closely related to the project occurs in the theory of operator algebras of Cuntz-Krieger-Toeplitz type [45]. This theory is now being extended to the deeper setting of algebras associated with higher rank graphs.

The traditional single operator model has been and remains a vital perspective in the formulation, solution and understanding of many problems in analysis and applications; generalisations of interpolation theory (the problems of Carathéodory, Féjér and Pick) and robust control theory are two such application areas. The basic operator model, promoted in Sz-Nagy and Foias' celebrated book [50], has now been developed in several multi-variable and noncommutative directions. In particular, variants of dilation theory for contractions, von Neuman's inequality, and commutant lifting have been obtained by Arveson, Davidson, Popescu and others. See, for example, [3, 12, 38, 39, 40]. Concurrently Davidson, Pitts, Popescu and others have developed the operator algebra structure theory and also representation theory of the associated noncommutative H^∞ algebras generated by freely commuting isometries, the so-called free semigroup algebras $\mathcal{L}_n$. (See [11].) It is in this general area that David Kribs and the PI initiated in [26, 27] the analysis of graph based generalisations of the free semigroup algebras. The main classification result of [26] showed that the underlying countable directed graph is determined by the operator algebra. In addition they developed basic Fourier series techniques, determined the commutant algebra, obtained Beurling type invariant subspace theorems, and determined reflexivity and conditions for semisimplicity. Recent work inspired by these developments include [19, 21, 22, 47]. Moreover, these graph algebras provide new contexts for generalised interpolation and this has been developed in the recent paper of Dritschel et al [15].

The main project aims, (i) and (ii) below, were directed towards extending the classification and dilation theory above to higher rank graphs.

Noncommutative dynamics, in the form of semigroups of endomorphisms of a von Neumann algebra, has been an active area of research for some twenty years. The contemporaneous development of quantum stochastic dynamics, in the form of homomorphic processes on operator algebras driven by noncommutative analogues of Brownian motion and Poisson processes, has been summarised in recent lecture notes [1]. A common theme is the construction and analysis of dilations of completely positive semigroups [7, 35] and the analysis of cocycles of dynamics [2, 17, 34].

Associated with a semigroup of endomorphisms of a von Neumann algebra is a product system, which plays the role of a spectrum for the semigroup. For a full algebra $B(H)$ with infinite dimensional separable Hilbert space H, this is a continuous tensor product of Hilbert spaces [2], and such E_0-semigroups are classified up to cocycle conjugacy by the isomorphism class of their product system. Product systems having sufficiently many units (Type I systems) are isomorphic to the continuous tensor product structure of a symmetric

Fock space $\mathcal{F}$ over the L^2-space of k-valued functions, which is the product system for the shift semigroup on $B(\mathcal{F})$, the CCR flow of rank $\dim \mathsf{k}$. Families of examples of product systems of Type II and Type III have been found by Powers [43, 44], Tsirelson and Vershik [51, 52, 53].

Quantum computing and quantum information theory form a rapidly expanding field of considerable diversity [36]. Recent work of Kribs, Ruskai, and others has shown the techniques of operator theory and operator algebra are fundamental in this analysis. At the most basic level this includes both the unitary operator formulation of a quantum gate and in providing the basic mathematical model for quantum channels in terms of completely positive maps. As two specific examples, the important topic of quantum error correction has seen recent advances in work of Kribs, Laflamme and Poulin [25], which provides a unified framework and has led to increased efficiency in fault tolerant quantum computing through operator algebra methods. Moreover, other recent work of Choi, Kribs and Zyczkowski [10] introduced the notion of higher rank numerical ranges that has taken on a life of its own in linear algebra and matrix analysis [30], [54].

2. Stated Aims of the Project

The original aims listed below were made intentionally wide, it being uncertain at the time of application just which areas would prove fruitful and preoccupying. As we describe below, the project has fulfilled the overall aim and has achieved outstandingly significant advances in a number of directions. Also, unexpectedly, we have been able to obtain definitive classification results for an entirely new class of rank 2 analytic Toeplitz algebras, namely those associated with *matricial* commutation relations (See Section 3.1.)

(i) The classification of nonselfadjoint subalgebras of higher rank graph algebras, such as the analytic Toeplitz algebras of k-graphs.

(ii) The development of representation and dilation theory for intertwining tuples and higher rank analytic Toeplitz algebras.

(iii) The interplay of methods for the analysis of discrete and continuous completely positive semigroups.

(iv) Computation of the roots of product systems with application to the determination of Arveson indices.

(v) Investigation of the structure of adapted operator-valued additive cocycles of CCR flows.

(vi) Problems in the mathematical theory of quantum computation, quantum information and quantum state estimation.

3. Project Achievements

The main project outputs are the following papers and preprints.

O1. S.C. Power, *Classifying Higher Rank Analytic Toeplitz Algebras*, New York Journal of Mathematics Volume 13 (2007), 271-298.

O2. S.C. Power and B. Solel, *Operator algebras associated with unitary commutation relation*, ArXiv preprint no. 0704.0079, 39 pages.

O3. K. R. Davidson, S. C. Power, D. Yang, Atomic representations of rank 2 graph algebras, Journal of Functional Analysis, Volume 255, (2008), 819-853.

O4. K. R. Davidson, S. C. Power, D. Yang, Dilation theory for rank two graph algebras, J. Operator Th., to appear.

O5. S. Olphert and S.C. Power, Higher Rank Wavelets, ArXiv preprint no. 0808.3879, 35 pages.

O6. D. Kretschmann, D.W. Kribs, R.W. Spekkens, Complementarity of private and correctable subsystems in quantum cryptography and error correction, Physical Review A, to appear.

3.1. **Aim(i).** *Classifying higher rank analytic Toeplitz algebras.* (O1,O2.)

In output O1 we introduced methods for the classification of the higher rank analytic Toeplitz algebras $\mathcal{L}_\Lambda$ of higher rank graphs Λ. We focussed on the fundamental context of algebras of single vertex graphs, and classification up to isometric isomorphism, and we also considered the norm closed subalgebras $\mathcal{A}_\theta$. The latter algebras are higher rank generalisations of Popescu's noncommutative disc algebras $\mathcal{A}_n$, quotients of which obtain function algebras which are higher rank variants of Arveson's d-shift algebras. Here θ denotes either a single permutation, sufficient to encode the relations of a 2-graph, or a set of permutations in the case of a k-graph. In the formalism we identified a single vertex higher rank graph (Λ, d) with a unital multi-graded semigroup $\mathbb{F}_\theta^+$. In the 2-graph case this provides a semigroup with generators $e_1, \dots, e_n$ and $f_1, \dots, f_m$ subject only to the commutation relations $e_i f_j = f_{j'} e_{i'}$ where $\theta(i,j) = (i', j')$ for a permutation θ of the nm pairs (i, j).

We obtained classifications of the algebras $\mathcal{L}_\Lambda$ up to graded isometric isomorphism in full generality, and obtained classifications of special classes (such as the 9 algebras for the case $n = m = 2$) up to general isometric algebra isomorphism.

In O2, which was partly inspired by O1 and by the dilation theory of Solel [48], we defined an entirely new class of analytic Toeplitz algebras derived from matricial commutation relations. These operators have generators $L_{e_1}, \dots, L_{e_n}, L_{f_1}, \dots, L_{f_m}$ subject to the unitary commutation relations of the form

$$L_{e_i} L_{f_j} = \sum_{k,l} u_{i,j,k,l} L_{f_l} L_{e_k}$$

where $u = (u_{i,j,k,l})$ is an $nm \times nm$ unitary matrix. By determining the detailed structure of isometric automorphisms (employing some techniques of Voiculescu on O_n along the way) we followed a strategy in output O1 and obtained a definitive classification of algebra types in terms of an equivalence relation on the set of unitary matrices u. As a corollary we obtained a complete classification of the (permutation commutation relation) algebras A_θ, with n, m arbitrary, thus completing the classification in O1 for all rank 2 cases.

3.2. **Aim(ii).** : *The development of representation and dilation theory for intertwining tuples and higher rank analytic Toeplitz algebras.* (O3, O4, O5.)

The algebras studied in output O1 also formed part of the motivation for the work of outputs O3 and O4 (joint work with Davidson and Yang at Waterloo) the other part of the motivation deriving from earlier work on multivariable dilation theory and on the atomic representation theory for free semigroup algebras of Davidson and Pitts.

In O4 we developed a dilation theory for (row contractive) representations of the permutation algebras $\mathcal{A}_\theta$, and their unitary relation variants $\mathcal{A}_u$. In particular, we showed that row contractive defect free representations have unique minimal dilations to $*$-representations, and that these are precisely the completely contractive representations of $\mathcal{A}_\theta$. This in turn enabled us to obtain an identification of the C*-envelope of these algebras. Furthermore, in O4 we obtained a simple (correspondence-free) proof of a remarkable result of Solel, namely his generalised Ando theorem for row contractions satisfying unitary relations.

The motivation for studying atomic representations, which was carried out in output O3, came from the case of the free semigroup generated by $e_1, \ldots, e_m$ with no relations. Davidson and Pitts classified the atomic $*$-representations of $\mathbb{F}_m^+$ and showed that the irreducibles fall into two types, known as ring representations and infinite tail representations. The former representations also provided interesting classes of C*-algebra representations of $\mathcal{O}_m$ which are amenable to analysis and which are closely related to wavelets.

However, the 2-graph situation turned out to be considerably more complicated than the case of the free semigroup. In O3 we showed that the irreducible atomic $*$-representations of $\mathbb{F}_\theta^+$ fall into six types. These types are analysed in great detail and related to group symmetries latent in the representation and identified via quotients of the discrete group $C_n \times C_m$. These irreducibles also provided a rich class of $*$-representations with interesting combinatorial and analytic structure. Having identified the irreducibles further analysis to obtain in output O3 a complete classification of all atomic representations on separable Hilbert space.

Amongst the developments emerging from the project is the connection between higher rank wavelets and representation theory. This is current work between Kribs, Olphert and Power which seems very promising in view of the well developed connections in the rank one setting. (Jorgensen-Kribs [20] for example.) As a prelude to this, and of considerable independent interest, in output O5, which also benefitted from major contributions from the PI's research student. In that paper we have delineated a new theory of Higher Rank Wavelets. For example nonseparable rank two "Latin square" wavelets, with threefold scalings, are constructed and these lead to interesting families representations of subalgebras of O_9.

3.3. **Aim(vi).** *Problems in the mathematical theory of quantum computation, quantum information and quantum state estimation.* (O6)

The operator algebraic approach to quantum error correction [4, 5] was formulated during the tenure of this grant. A specific project enabled by this grant was the joint quantum error correction and quantum cryptography work [23], which arose through a collaboration between Kribs and Spekkens (then a post-doc at Cambridge) during one of Kribs' visits to England. The project establishes a formal link between two basic notions in quantum cryptography and error correction: private codes and error-correcting codes are complementary to each other, and there is an algebraic bridge that allows cross-fertilization between the two perspectives. Moreover, an approximate version of the result was obtained in terms of completely bounded norms applied to the differences of completely positive maps. An important technical tool in the analysis was the Stinespring dilation continuity theorem of [24], that shows two completely positive maps are close to each other if and only if they have unitary dilations that are close. The Stinespring dilation theorem is a basic classical result in operator theory. There are now many dilation theoretic results in operator theory, including Sz.-Nagy's minimal isometric dilation of a contraction operator and its non-commutative multi-variable extension by Frazho-Bunce-Popescu. It is quite natural to ask whether analogues of the continuity theorem of [24] can be found for other dilations. For instance, is there a "meta theorem" that applies to all dilations? In fact, this is a fundamental question that has not been addressed previously in operator theory. At the end of this grant, preliminary discussions were initiated between the PI, Kribs, and Davidson on tackling this set of problems.

An additional development in Aim(vi) took place in Kribs' last collaborative visit to Lancaster supported by the grant, namely the dissemination of a set of lectures on quantum information to coincide with a workshop on Phenomena in High Dimensions. Discussions with the PI, Lindsay, Blower, and other participants helped to formalise a presentation of the material and Kribs intends to complete a book in this area by 2010.

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