MATH143 Calculus

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Syllabus

Differentiation

Introduction to differentiation using graphical methods. Differentiation from first principles (up to third-order polynomials only). Basic differentiation using formulae. Chain, product and quotient rules. Second-order and higher derivatives. Stationary points and their classification. Parametric, implicit and logarithmic differentiation.

Integration

The definite integral (as signed area). Indefinite integration (as the process inverse to differentiation). Integration of trigonometric functions using product and double-angle formulae. Integration of rational functions by completing the square and by partial fractions. Integration by parts and by substitution. Applications: arc length of a plane curve; area and centroid of a plane region; surface area, volume and centre of mass of a solid of revolution.

Numerical methods

The trapezium rule. Simpson's rule (the derivation is not examinable). The Newton-Raphson method.

Further topics

Taylor and Maclaurin series; estimation of integrals. L'Hôpital's rule.

Recommended text books

Either *Engineering Mathematics* by A. Croft, R. Davison and M. Hargreaves, third edition, 2001, Prentice Hall, ISBN 0 130 26858 5, £39·99.

Or *Modern Engineering Mathematics* by G. James, fourth edition, 2008, Pearson Education Ltd, ISBN 978-0-13-239144-3, £37·99.

Lectures and class exercises

The notes for this course are complete and, together with the workshop questions and assessed exercises, provide all the information you need – however, you should annotate them with your own notes taken during lectures. Examples are given in the notes but further practice will be gained by working through the class exercises, given on separate sheets, during lecture time. You are expected to attend lectures so that you can **participate in** and **learn from** the solution of these additional exercises.

Learning outcomes

At the end of this module, students should be able to:

(a) use the notation of differentiation and integration correctly;

(b) differentiate simple functions, including powers, the exponential function, trigonometric functions and their inverses;

(c) differentiate sums, products and quotients of functions;

(d) differentiate composite functions;

(e) differentiate implicitly and use this method to differentiate functions of the form $f(x) = g(x)^{h(x)}$;

(f) differentiate functions related through a parameter;

(g) integrate basic functions and evaluate these integrals;

(h) integrate simple trigonometric functions using product and double-angle formulae;

(i) integrate rational functions using partial fractions or by completing the square;

(j) integrate functions of the form $\dfrac{af'(x)}{f(x)}$ and $af'(x)f(x)$;

(k) integrate functions using the 'by parts' technique;

(l) integrate functions 'by substitution', using a change of variable;

(m) recognise the integration technique required for a given function;

(n) find lengths, areas, volumes, centres of mass and similar quantities by using integration techniques;

(o) find Taylor series and Maclaurin series for functions;

(p) estimate the values of integrals using the trapezium and Simpson's rules and using Taylor series;

(q) find numerical solutions of equations using the Newton-Raphson method;

(r) use L'Hôpital's rule to determine limits.

Assessment

20% weekly coursework

40% end-of-module progress test

40% end-of-year examination

You will receive the weekly coursework assignment from your tutor at the Friday workshop. These workshops run from 11am until 1pm, with the first hour being **compulsory** for all students.

Please note that MATH143 coursework must be put in the first-year coursework pigeonhole no later than 12 NOON WEDNESDAY.

This will be then be given to your tutor. Work received later than this time will not normally be accepted. Under no circumstances can coursework be accepted after the model answers are given out at the following workshop. As stated in the first-year handbook, if a student has submitted less than 80% of the coursework required, he or she will be deemed not to be in good standing with his or her tutor.

Progress test

The progress test will take place on **Thursday in Week 15** in normal lecture time. If a different room is to be used, you will be informed during lectures in Week 15 and the information will be put on the LUVLE webpage. It is the student's responsibility to arrive promptly for this test. If you are ill, a medical certificate must be submitted to the Teaching Office (Engineering A38). **If written confirmation of the reason for your absence is not received, you will be given a mark of zero.**

Examination

The examination will take place towards the end of the Summer Term. You will be informed by Student Registry, in writing, of the examination times and places. It is the student's responsibility to attend these examinations on time unless prevented by ill health.

1 Differentiation

1.1 Introduction

Calculus is used in many situations in order to analyse varying quantities. It involves two basic operations, differentiation and integration, which are opposite sides of the same coin. The Greeks developed the rudimentary ideas of integration, Archimedes using his 'method of exhaustion' to obtain an exact formula for the area of a circle. Differentiation was probably first used by Fermat in the 17th century, to determine the maxima and minima of certain functions. The connection between the two techniques was developed by Newton and Leibniz, working independently of each other, as a way of dealing with change and motion; it is the notation developed by Leibniz that is used today. Calculus has many applications throughout science and engineering and provides an essential language: for example, the laws of physics are formulated used calculus.

1.2 Limits of functions

Limits have already appeared in MATH142 for sequences. As these ideas will be used to develop some of the basic formulas for differentiation, we will revisit this topic briefly.

Definition

A function $f(x)$ is said to *approach* (or *tend to*) a limit l as x approaches the value a if we can make the value of $f(x)$ as close as we please to l by taking x sufficiently close to a. This is written as

$$\lim_{x\to a} f(x) = l \qquad \text{or} \qquad f(x) \to l \text{ as } x \to a.$$

Note that we do **not** require that $f(a) = l$ or even that $f(a)$ exists.

1.3 Rules for the manipulation of limits

If $f(x) \to l$ and $g(x) \to m$ as $x \to a$ then

$$\text{(i)} \qquad f(x) + g(x) \to l + m \text{ as } x \to a,$$

$$\text{(ii)} \qquad f(x)g(x) \to lm \text{ as } x \to a$$

$$\text{and (iii)} \qquad \text{if } m \neq 0 \text{ then } \frac{f(x)}{g(x)} \to \frac{l}{m} \text{ as } x \to a.$$

Examples

1. Evaluate $\lim_{x\to 0} \dfrac{3x^2+5x^3-2x^4}{2x^2+x^3}$.

   The rules for limits show that the denominator $2x^2+x^3 \to 0$ as $x \to 0$, so rule (iii) cannot be applied directly to this quotient. However, if we divide the numerator and denominator by the *lowest* power of x which appears (here, x^2) then, as $x \to 0$, the rules for limits apply and

$$\lim_{x\to 0} \frac{3x^2+5x^3-2x^4}{2x^2+x^3} = \lim_{x\to 0} \frac{3+5x-2x^2}{2+x} = \frac{3+0-0}{2+0} = \frac{3}{2}.$$

2. Evaluate $\lim_{x\to\infty} \dfrac{3x^2+4x-2}{5x^2-x+6}$.

   Here, neither the numerator nor denominator converge as $x \to \infty$; they both grow without limit. However, if we divide the numerator and denominator by the *highest* power of x which appears then, as $x \to \infty$, the individual terms will each tend to a limit. In this case, dividing by x^2 gives

$$\lim_{x\to\infty} \frac{3x^2+4x-2}{5x^2-x+6} = \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{2}{x^2}}{5-\frac{1}{x}+\frac{6}{x^2}} = \frac{3+0-0}{5-0+0} = \frac{3}{5}.$$

3. Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$. (As always, x is in **radians**.)

   The series expansion for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ,$$

   so

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \to 1$$

   as $x \to 0$. Strictly speaking, this working isn't covered by the rules for limits, because there are infinitely many terms in the series. However, this argument can be made valid; you should remember this limit as a fact.

1.4 Continuity

Definition

The function $f(x)$, defined at and around the point a, is said to be *continuous at a* if $f(x) \to f(a)$ as $x \to a$.

(Informally, the function $f(x)$ is continuous at a if we can draw its graph around a without lifting the pencil from the paper.)

A function is *continuous* if it is continuous at every point where it is defined.

Continuous functions include polynomials, such as $x^3 - 3x + 1$, the exponential function and the sine and cosine functions.

The function

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous (i.e., not continuous) at $x = 0$. (See Figure 1.)

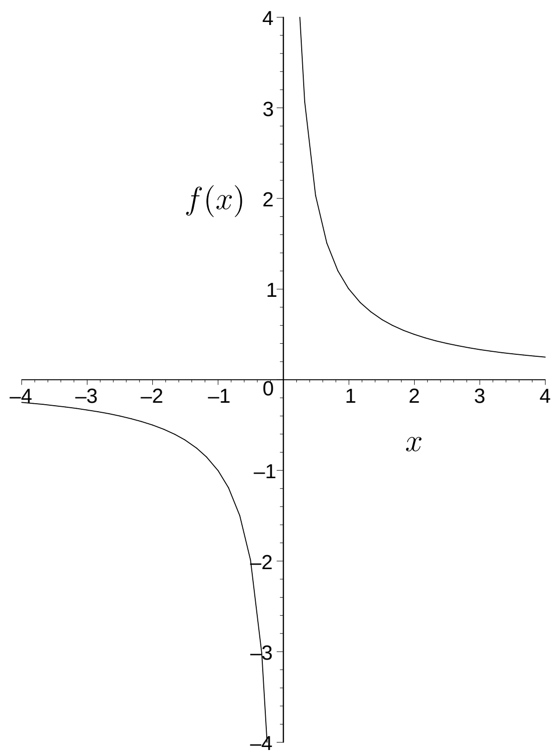

Figure 1: The graph of the function $f(x)$

1.5 Rules for manipulating continuous functions

If $f(x)$ and $g(x)$ are continuous at a then so are

(i) $kf(x)$, where k is a constant,

(ii) the sum $f(x) + g(x)$

(iii) the product $f(x)g(x)$ and

(iv) the quotient $f(x)/g(x)$, as long as $g(a) \neq 0$.

If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$ then

(v) the composite function $f(g(x))$ is continuous at a.

1.6 Basic ideas of differentiation

Differentiation is a means of measuring a 'rate of change'. This can be the rate of change of y with respect to x, i.e., the 'slope' of the graph $y = f(x)$, or the rate of change of a variable with respect to time, temperature et cetera.

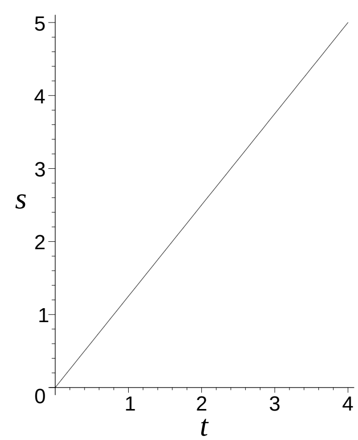

Figure 2: $s = ut$

One way to think about the meaning of differentiation is to consider an object moving in a straight line, with constant velocity $u\,\text{ms}^{-1}$. The distance s metres travelled by the object in t seconds is given by the formula

$$s = ut.$$

The graph of distance s against time t is a straight line (see Figure 2) and u is the gradient of this line.

If the velocity varies with time (as in Figure 3) then the average velocity over the distance travelled between time t_0 and time t_1 is

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0},$$

which is the gradient of the straight line between $\big(t_0, s(t_0)\big)$ and $\big(t_1, s(t_1)\big)$.

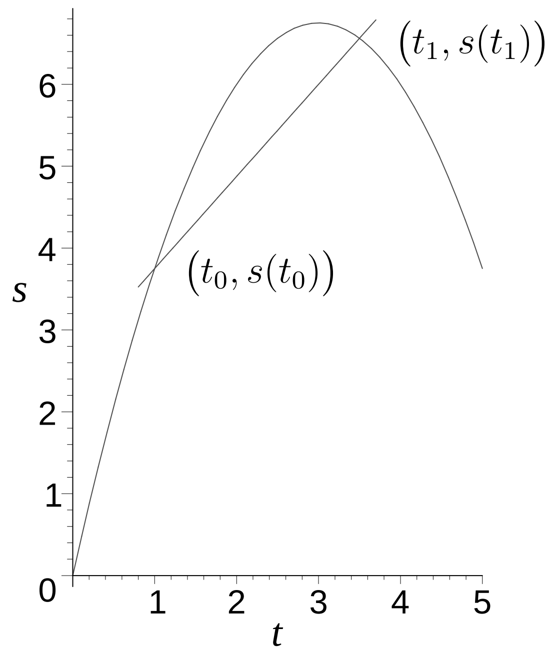

Figure 3: $s = f(t)$

If we put $t_1 = t_0 + \delta t$ and $s(t_1) = s(t_0) + \delta s$ then the gradient equals

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{\delta s}{\delta t}.$$

If we now let $t_1 \to t_0$ (equivalently, $\delta t \to 0$) then this quotient tends to the gradient of the tangent line to the graph at the point $\big(t_0, s(t_0)\big)$.

This process of passing from the function $s(t)$ to the gradient of the tangent is called *differentiation* and it measures the *rate of change* of $s(t)$ with respect to t. The gradient function is called the *derivative* of the function $s(t)$.

Definition

The *derivative* of a function $f(x)$ at the point x, written $f'(x)$ or $\dfrac{\mathrm{d}f}{\mathrm{d}x}$, is

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta f}{\delta x}.$$

(Informally, the derivative of a function $f(x)$ at the point x is the gradient of the tangent to the curve $y = f(x)$ at the point $\big(x, f(x)\big)$.)

This limit will **not** exist at x if

(a) $f(x)$ is not continuous at x or

(b) the graph of $f(x)$ has a 'corner' at x.

(Informally, a function is differentiable at x if its graph is continuous and 'smooth' at and near to x.) We are not going to dwell on this possibility.

1.7 Differentiation from first principles

Examples

1. If $f(x) = c$, where c is a constant, then $f(x + \delta x) = c$ and

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{c - c}{\delta x} = \lim_{\delta x \to 0} 0 = 0.$$

   As the derivative is the gradient of the tangent to the curve, this result should seem fairly obvious: the graph is, in this case, a horizontal line. Alternatively, it should be clear that the rate of change of a constant is zero.

2. If $f(x) = bx$, where b is a constant, then

$$f(x + \delta x) = b(x + \delta x) = bx + b\,\delta x$$

   and

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{bx + b\,\delta x - bx}{\delta x} \qquad = \lim_{\delta x \to 0} \frac{b\,\delta x}{\delta x} = \lim_{\delta x \to 0} b = b. \end{aligned}$$

   Again, this makes sense: the equation of a straight line through the origin is $y = mx$, with m the gradient of that line. The derivative of a function is the gradient of the tangent to the graph of that function, so for a straight line the derivative is a constant, the line's gradient.

3. If $f(x) = ax^2$, where a is a constant, then

$$f(x + \delta x) = a(x + \delta x)^2 = ax^2 + 2ax\,\delta x + a(\delta x)^2$$

so

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{ax^2 + 2ax\,\delta x + a(\delta x)^2 - ax^2}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{2ax\,\delta x + a(\delta x)^2}{\delta x} \\ &= \lim_{\delta x \to 0} (2ax + a\,\delta x) \\ &= 2ax. \end{aligned}$$

4. Combining the working from Examples 1, 2 and 3, we can see that

$$\frac{\mathrm{d}}{\mathrm{d}x}(ax^2 + bx + c) = 2ax + b.$$

In general, if $y = x^n$ (where n can be any number, integer or otherwise) then, using the binomial theorem, it may be shown that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}.$$

5. Find, from first principles, the derivative of $f(x) = e^x$.

From the definition,

$$f'(x) = \lim_{\delta x \to 0} \frac{e^{(x+\delta x)} - e^x}{\delta x} = \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}.$$

The series expansion of $e^{\delta x}$ is

$$e^{\delta x} = 1 + \delta x + \frac{(\delta x)^2}{2!} + \frac{(\delta x)^3}{3!} + \frac{(\delta x)^4}{4!} + \cdots$$

and therefore

$$\begin{aligned} f'(x) &= e^x \lim_{\delta x \to 0} \frac{(1 + \delta x + \frac{(\delta x)^2}{2!} + \frac{(\delta x)^3}{3!} + \cdots - 1)}{\delta x} \\ &= e^x \lim_{\delta x \to 0} \left(1 + \frac{\delta x}{2!} + \frac{(\delta x)^2}{3!} + \cdots\right) \\ &= e^x. \end{aligned}$$

(As for Example 3 in Section 1.3, this last step isn't properly justified by the rules for limits above, but this argument can be made rigorous and gives the right answer.)

6. Find the derivatives of the functions $f(x) = \sin x$ and $g(x) = \cos x$ from first principles.

   Using the trigonometric identity

   $$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

   with $A = x + \delta x$ and $B = x$, we have

   $$f'(x) = \lim_{\delta x\to 0}\frac{\sin(x+\delta x) - \sin x}{\delta x} = \lim_{\delta x\to 0}\frac{\cos(x+\frac{1}{2}\delta x)\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x}.$$

   By Example 3 in Section 1.3 (and the fact that $\cos x$ is continuous) it follows that

   $$f'(x) = \lim_{\delta x\to 0}\cos(x+\frac{1}{2}\delta x)\lim_{\delta x\to 0}\frac{\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x} = (\cos x)\times 1 = \cos x.$$

   To differentiate the function $g(x) = \cos x$ from first principles, we use the trigonometric identity

   $$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

   with $A = x + \delta x$ and $B = x$, which implies that

   $$\begin{aligned} g'(x) &= \lim_{\delta x\to 0}\frac{\cos(x+\delta x) - \cos x}{\delta x} \\ &= \lim_{\delta x\to 0}\frac{-\sin(x+\frac{1}{2}\delta x)\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x} \\ &= -\lim_{\delta x\to 0}\sin(x+\frac{1}{2}\delta x)\lim_{\delta x\to 0}\frac{\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x} = -\sin x. \end{aligned}$$

Note

There are three ways of thinking of the derivative: (i) using its definition, as a limit; (ii) graphically, as the gradient of the tangent to the graph or (iii) kinematically, as the rate of change of one variable with respect to another. Each of these should be kept in mind; choosing the most appropriate one may help when solving a problem.

1.8 Rules for differentiation

Scalar-multiplication rule

If $y = f(x)$ and k is a constant then

$$\frac{\mathrm{d}}{\mathrm{d}x}(ky) = k\frac{\mathrm{d}y}{\mathrm{d}x} = kf'(x).$$

Sum rule

If $y = f(x)$ and $z = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y+z) = \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} = f'(x) + g'(x).$$

Composite ('chain') rule

If $y = f(z)$ and $z = g(x)$ then $y = f(g(x))$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = f'(z)g'(x).$$

Examples

1. If $y = x^6 + 3x^4 - x^2 + 1$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 + 12x^3 - 2x.$$

The first two rules (for scalar multiples and sums) are summed up by saying that differentiation is *linear*. Linearity, together with Example 4 in Section 1.7, means that we can differentiate any polynomial.

2. If $y = e^{\sqrt{x}}$ then $y = e^z$, where $z = x^{1/2}$. The chain rule states that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x},$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}z}e^z\ \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2} = e^z \times \frac{1}{2}x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

3. If $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$ then, by linearity and the chain rule,

$$f'(x) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

4. If $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$ then

$$f'(x) = \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2} = \sinh x.$$

So far we have been dealing with functions involving x and y. Whatever letters the variables are labelled with, the rules remain the same.

5. Differentiate the following functions with respect to the appropriate variable.

$$\text{(i) } 15t - 10t^2 \qquad \text{(ii) } \sin\Big(5\theta + \frac{\pi}{3}\Big) \qquad \text{(iii) } e^{\cosh\omega}$$

(i) $\dfrac{\mathrm{d}}{\mathrm{d}t}(15t - 10t^2) = 15 - 20t.$

(ii) If $s = \sin\big(5\theta + \frac{\pi}{3}\big)$ and $z = 5\theta + \frac{\pi}{3}$ then $s = \sin z$ and

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{\mathrm{d}s}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\theta} = (\cos z) \times 5 = 5\cos\Big(5\theta + \frac{\pi}{3}\Big).$$

(iii) Let $p = e^{\cosh\omega}$ and $q = \cosh\omega$, so that $p = e^q$ and

$$\frac{\mathrm{d}p}{\mathrm{d}\omega} = \frac{\mathrm{d}p}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}\omega} = e^q \times \sinh\omega = e^{\cosh\omega}\sinh\omega.$$

6. Evaluate the derivative of $\ln\cos\theta$ when $\theta = \pi/4$.

Let $y = \ln\cos\theta$ and $z = \cos\theta$, so that $y = \ln z$ and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\ln\cos\theta) = \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{z}(-\sin\theta) = -\frac{\sin\theta}{\cos\theta} = -\tan\theta.$$

(We will show in Example 1 on p.16 that $\dfrac{\mathrm{d}}{\mathrm{d}z}\ln z = \dfrac{1}{z}$.) Hence the derivative of $\ln\cos\theta$ at $\theta = \pi/4$ is $-\tan(\pi/4) = -1$.

7. Consider the function $f(x) = 3x^2 - 7x + 1$. Find

(i) the derivative of $f(x)$,

(ii) the gradient of $f(x)$ at $x = 2$ and

(iii) the equation of the tangent to $y = f(x)$ at the point where $x = 2$.

(i) The derivative $f'(x) = 6x - 7$.

(ii) The gradient of $f(x)$ at $x = 2$ is therefore $f'(2) = 6(2) - 7 = 5$.

(iii) The tangent is a straight line, so is described by the equation $y = mx + c$.
The constant m is the gradient at $x = 2$, which is 5.
The tangent passes through the point $\big(2, f(2)\big)$ and

$$f(2) = 3(2)^2 - 7(2) + 1 = -1.$$

Hence $y = -1$ when $x = 2$, so $-1 = 5(2) + c$ and $c = -11$.
Thus the equation of the tangent line at $x = 2$ is $y = 5x - 11$.

Product rule

If $y = f(x)$ and $z = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}(yz) = \frac{\mathrm{d}y}{\mathrm{d}x}z + y\frac{\mathrm{d}z}{\mathrm{d}x} = f'(x)g(x) + f(x)g'(x).$$

Quotient rule

If $y = f(x)$ and $z = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{z}\right) = \frac{\frac{\mathrm{d}y}{\mathrm{d}x}z - y\frac{\mathrm{d}z}{\mathrm{d}x}}{z^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Examples

1. If $y = x^3 \sin x$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(x^3)\sin x + x^3\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 3x^2\sin x + x^3\cos x.$$

2. If $s = e^t\sqrt{t}$ then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(e^t)t^{1/2} + e^t\frac{\mathrm{d}}{\mathrm{d}t}(t^{1/2}) = e^t t^{1/2} + e^t\left(\frac{1}{2}t^{-1/2}\right) = e^t\left(\sqrt{t} + \frac{1}{2\sqrt{t}}\right).$$

3. If $y = \ln x \sin x$ then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\ln x)\sin x + \ln x\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \frac{1}{x}\sin x + \ln x\cos x \\ &= \frac{\sin x}{x} + \ln x\cos x.\end{aligned}$$

4. If $y = \dfrac{x+1}{x^2+3}$ then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}(x+1)\right)(x^2+3) - (x+1)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)}{(x^2+3)^2} \\ &= \frac{(1)(x^2+3) - (x+1)(2x)}{(x^2+3)^2} \\ &= \frac{x^2+3-2x^2-2x}{(x^2+3)^2} \\ &= \frac{3-2x-x^2}{(x^2+3)^2}.\end{aligned}$$

5. If $T = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ then

$$\frac{\mathrm{d}T}{\mathrm{d}\theta} = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\right)\cos\theta - \sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

Inverse-function rule

If $y = f^{-1}(x)$ then $x = f\big(f^{-1}(x)\big) = f(y)$. From the chain rule we have

$$1 = \frac{\mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Big/ \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{f'(y)}.$$

Examples

1. If $y = \ln x$ then $x = e^y$ and $\dfrac{\mathrm{d}x}{\mathrm{d}y} = e^y$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

(This is one of the standard derivatives listed on p.6 of the Data Book.)

2. If $y = \sin^{-1} x$ then $x = \sin y$ and $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \cos y$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

(Since $\cos^2 x + \sin^2 x = 1$, we know that $\cos y = \pm\sqrt{1-\sin^2 y}$, but which sign do we take? As $\sin^{-1} x$ is increasing – draw a sketch to convince yourself of this – it has positive gradient, so we take the positive square root.)

3. If $y = \cos^{-1} x$ then $x = \cos y$ and $\dfrac{\mathrm{d}x}{\mathrm{d}y} = -\sin y$, so
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{-\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}.$$
(Again, the positive root is chosen to give the right sign: the function $\cos^{-1} x$ is decreasing.)

4. If $y = \tanh^{-1} x$ then $x = \tanh y$ and $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \operatorname{sech}^2 y$, so (using the fact that $\operatorname{sech}^2 y = 1 - \tanh^2 y$)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$

1.9 Second and higher derivatives

Definition

The *second derivative* of $y = f(x)$, denoted by $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ or $f''(x)$, is defined to be the derivative of the first derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$. In other words,
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right).$$
We can extend this to third, fourth and higher derivatives. In general, these are written as $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ or $f^{(n)}(x)$, where n is often written in Roman numerals in the latter case: we have
$$f(x), \quad f'(x), \quad f''(x), \quad f'''(x), \quad f^{(iv)}(x), \quad f^{(v)}(x) \quad \text{and so on.}$$

Examples

1. If $f(x) = x^3 - \sin x$ then
$$\begin{aligned} f'(x) &= 3x^2 - \cos x, \\ f''(x) &= 6x + \sin x, \\ f'''(x) &= 6 + \cos x, \\ f^{(iv)}(x) &= -\sin x, \\ f^{(v)}(x) &= -\cos x \end{aligned}$$
et cetera.

2. A projectile is thrown with an initial vertical component of velocity $30\,\mathrm{ms}^{-1}$. If the vertical height above the ground, s metres, is given by

$$s = 1{\cdot}7 + 30t - 4{\cdot}9t^2,$$

find the time at which the projectile is at a maximum height above the ground and hence determine this maximum height. Calculate the acceleration of the projectile.

The maximum height is achieved when the velocity $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$ is zero, that is, when the rate of change of distance with respect to time is zero. Thus

$$0 = \frac{\mathrm{d}s}{\mathrm{d}t} = 30 - 9{\cdot}8t$$

and the maximum height is achieved at time

$$t = \frac{30}{9{\cdot}8} = 3{\cdot}06 \qquad \text{(2 d.p.)}.$$

At this time,

$$s = 1{\cdot}7 + 30(3{\cdot}06) - 4{\cdot}9(3{\cdot}06)^2 = 47.6 \qquad \text{(1 d.p.)},$$

so the maximum height achieved by the projectile is 47·6 metres (to one decimal place).

Acceleration is the rate of change of velocity with respect to time, so this is

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -9{\cdot}8.$$

Note that the acceleration is the same for any time t here. This is acceleration due to the force of gravity acting toward the earth. Note also that, from the shape of the graph of s, it must have a maximum value and not a minimum.

1.10 Stationary points

If the derivative $f'(x) > 0$ then the gradient of the function $f(x)$ is positive, so the function is increasing; if $f'(x) < 0$ then f has negative gradient, so the function is decreasing.

Points where the derivative of a function is zero are called *stationary points*: they are where the rate of change of the function is zero.

The second derivative can give us information on stationary points. Let x_0 be a stationary point for the function $f(x)$, so that $f'(x_0) = 0$, and suppose $f''(x_0) > 0$.

As $f''(x_0)$ is the gradient of $f'(x)$ at x_0, the derivative $f'(x)$ has positive gradient, so is increasing, near x_0. Hence $f'(x)$ must be negative to the left of x_0 and positive to the right of x_0 (since $f'(x_0) = 0$). So the function $f(x)$ is decreasing to the left of x_0 and increasing to the right of x_0: the value $f(x_0)$ is a *local minimum.*

Similarly, if $f''(x_0) < 0$ then the derivative $f'(x)$ is decreasing near x_0, so the value $f(x_0)$ is a *local maximum.*

In Example 2 on p.18, $\dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -9{\cdot}8 < 0$, so the height found there is a maximum (as we have already seen).

If $f''(x_0) = 0$ then the second derivative tells us nothing and we need to look directly at the behaviour of $f'(x)$ for x near the stationary point.

Example

Find and classify the stationary points of the function $y = x^3$.

Since

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 = 0 \qquad \Longleftrightarrow \qquad x = 0,$$

the function has one stationary point, at $x = 0$. The second derivative

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x,$$

which is zero at $x = 0$, so this tells us nothing about the nature of the stationary point. Since $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ if $x \neq 0$, the function y is increasing and $x = 0$ is neither a local maximum nor a local minimum. (This should also be clear to you from a sketch of the function.)

1.11 Parametric differentiation

A curve may sometimes be expressed parametrically as $x = g(t)$, $y = h(t)$ rather than in the form $y = f(x)$. For example,

$$(x, y) = (t^2, 2t) \qquad (t \geqslant 0)$$

gives the curve $y^2 = 4x$: a parabola.

In these circumstances we can use the chain rule to find the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$: note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t}.$$

The second derivative needs more care; in general,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \frac{\mathrm{d}^2y}{\mathrm{d}t^2} \Big/ \frac{\mathrm{d}^2x}{\mathrm{d}t^2}.$$

Instead, using the chain rule again, we have that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right)\frac{\mathrm{d}t}{\mathrm{d}x}.$$

Examples

1. For the curve given parametrically by $x = t^2 - 1$, $y = t^3 - t$, we see that $\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 1$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2 - 1}{2t} = \frac{3}{2}t - \frac{1}{2t}.$$

The second derivative is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}t - \frac{1}{2}t^{-1}\right)\right)\frac{1}{2t} = \left(\frac{3}{2} + \frac{1}{2}t^{-2}\right)\frac{1}{2t} = \frac{3}{4t} + \frac{1}{4t^3}.$$

2. Consider the curve given parametrically by $x = \cos^3 t$, $y = \sin^3 t$; in this case,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3\cos^2 t \sin t \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3\sin^2 t \cos t.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\frac{\sin t}{\cos t} = -\tan t$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}}{\mathrm{d}t}(-\tan t)\right)\left(\frac{1}{-3\cos^2 t \sin t}\right) = \frac{-\sec^2 t}{-3\cos^2 t \sin t} = \frac{1}{3\cos^4 t \sin t}.$$

(Remember that $\sec t = 1/\cos t$.)

1.12 Implicit differentiation

Sometimes y is not defined explicitly as a function of x or in terms of a parameter, but is merely related to x by means of an equation. For example, the circle with radius 2 and centre $(0,0)$ is given by the familiar formula

$$x^2 + y^2 = 4.$$

In this situation we treat y as an unknown function of x, differentiate both sides of the equation with respect to x and then solve to find the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

To help with this we have the chain rule, which tells us that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(g(y)\big) = \frac{\mathrm{d}}{\mathrm{d}y}\big(g(y)\big)\frac{\mathrm{d}y}{\mathrm{d}x} = g'(y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Examples

1. If $x^2 + y^2 = 4$ then
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0,$$
so $2x + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x}{y}$.

2. Consider the ellipse given by the equation $x^2 + xy + y^2 = 3$. Find the equation of the tangent to the ellipse at the point $(1,1)$.

   The tangent at $(1,1)$ is a straight line with general equation $y = mx + c$; the gradient m equals the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = y = 1$. Differentiating implicitly, we have
$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^2 + xy + y^2\big) = \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0$$
and, noting that xy is a product, this implies that
$$2x + \left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad \text{so} \qquad (x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = -2x - y.$$
Hence $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2x+y}{x+2y}$ and $m = -\dfrac{2+1}{1+2} = -1$.

   To find c, note that the tangent passes through the point $(1,1)$, so $1 = -1 + c$ and $c = 2$. The equation of the tangent is therefore $y = 2 - x$.

3. For the ellipse of Example 2, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Differentiating both sides of the equation

$$2x + x\frac{\mathrm{d}y}{\mathrm{d}x} + y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

implicitly with respect to x, noting that $x\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ are products, we have

$$2 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + \frac{\mathrm{d}y}{\mathrm{d}x} + \left(2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0.$$

Hence

$$(x+2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2 - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\left(\frac{1 + \frac{\mathrm{d}y}{\mathrm{d}x} + (\frac{\mathrm{d}y}{\mathrm{d}x})^2}{x+2y}\right) = -2\left(\frac{1 - \frac{2x+y}{x+2y} + (\frac{2x+y}{x+2y})^2}{x+2y}\right).$$

Multiplying the numerator and denominator of this last quantity by $(x+2y)^2$, we see that

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -2\left(\frac{(x+2y)^2 - (2x+y)(x+2y) + (2x+y)^2}{(x+2y)^3}\right)\\ &= -2\left(\frac{x^2+4xy+4y^2-2x^2-4xy-xy-2y^2+4x^2+4xy+y^2}{(x+2y)^3}\right)\\ &= -2\left(\frac{3(x^2+xy+y^2)}{(x+2y)^3}\right).\end{aligned}$$

Finally, the original equation tells us that $x^2+xy+y^2=3$, so

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-18}{(x+2y)^3}.$$

1.13 Logarithmic differentiation

A neat trick, known as logarithmic differentiation, can be used to find the derivative of expressions of the form $y = f(x)^{g(x)}$. The idea is to take the natural logarithms of both sides and then differentiate with respect to x.

Note first that if $y = \ln\big(h(x)\big)$ then, by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{h'(x)}{h(x)}. \qquad (\star)$$

Example

If $y = (\cosh x)^x$ then

$$\ln y = \ln\big((\cosh x)^x\big) = x \ln\cosh x.$$

Differentiating both sides with respect to x (using the chain rule and noting that $x\ln\cosh x$ is a product) gives that

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln\cosh x + x\frac{\mathrm{d}}{\mathrm{d}x}(\ln\cosh x) = \ln\cosh x + x\left(\frac{\sinh x}{\cosh x}\right),$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(\ln\cosh x + x\tanh x) = (\cosh x)^x(\ln\cosh x + x\tanh x).$$

This technique can also be used to simplify expressions which contain products, quotients and composite functions.

Example

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \dfrac{4\sqrt{\sin 2x}}{(x+1)^2(3x-1)}$.

First, take logarithms of both sides:

$$\ln y = \ln 4 + \frac{1}{2}\ln\sin 2x - 2\ln(x+1) - \ln(3x-1).$$

Differentiating with respect to x, using the observation $(\star)$ above and noting that $\ln 4$ is a constant, we see that

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos 2x}{\sin 2x} - \frac{2}{x+1} - \frac{3}{3x-1}.$$

Rearranging and replacing y by its definition as a function of x, we see that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sqrt{\sin 2x}}{(x+1)^2(3x-1)}\left(\cot 2x - \frac{2}{x+1} - \frac{3}{3x-1}\right).$$

2 Integration

2.1 Definite integrals

Definition

Let $f(x)$ be a function on the interval $[a, b]$, where $a < b$; in other words, $f(x)$ is defined for all values of x from $x = a$ to $x = b$ inclusive. Informally, we define the *definite integral* of $f(x)$ from a to b, denoted by

$$\int_a^b f(x)\,\mathrm{d}x,$$

to be the area under the graph of $y = f(x)$ between $x = a$ and $x = b$. More precisely, we mean the area between $y = f(x)$ and the x-axis, with the **area below the x-axis counting as negative**. We also define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

and

$$\int_a^a f(x)\,\mathrm{d}x = 0.$$

We call the function being integrated, $f(x)$, the *integrand* of the integral, $\mathrm{d}x$ the *differential* of x and the points a and b the *limits of integration*. Think of the integral sign $\int$ (a stretched 'S', for 'sum') and the differential $\mathrm{d}x$ as brackets around the integrand – they must always appear together. The x in $\mathrm{d}x$ tells us the name of the variable we are integrating over.

Examples

1. Find $\displaystyle\int_a^b c\,\mathrm{d}x$, where c is constant.

   The graph of $y = c$ is a horizontal straight line, and the (positive) rectangular area between this line and the x-axis is either $(b - a)c$ (if $c \geqslant 0$) or $(b - a)(-c)$ (if $c < 0$). As area below the x-axis counts as negative, we see that, in both cases,

$$\int_a^b c\,\mathrm{d}x = (b - a)c.$$

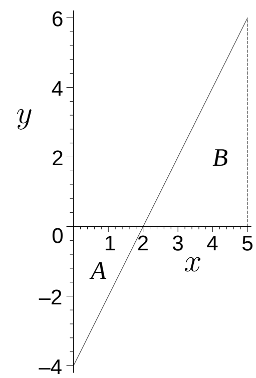

Figure 4: $y = 2x - 4$

2. Calculate $\displaystyle\int_0^5 (2x - 4)\,\mathrm{d}x$.

   In Figure 4, area $A = \frac{1}{2} \times 2 \times 4 = 4$ and area $B = \frac{1}{2} \times 3 \times 6 = 9$. Since A is below the x-axis, its area counts as negative, so

$$\int_0^5 (2x - 4)\,\mathrm{d}x = 9 - 4 = 5.$$

For more complicated functions, the area is found (and the integral is defined) by taking the limit of the approximations we obtain by dividing the area into rectangular strips, each of width δx, and then letting $\delta x \to 0$. (See Figure 5.)

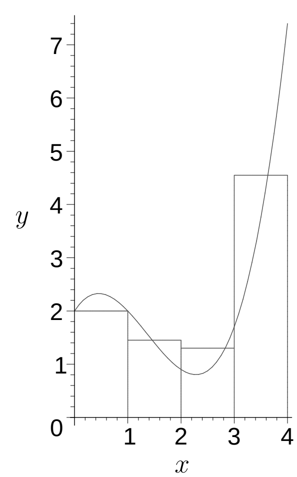

Figure 5: $y = \frac{1}{2}x(x - 1)(x - \frac{31}{10}) + 2$

If

$$a = x_0 < a + \delta x = x_1 < a + 2\delta x = x_2 < \cdots < x_n = b$$

and the height of each rectangle is $\frac{1}{2}\big(f(x_{i-1}) + f(x_i)\big)$ for $i = 1, 2, 3, \cdots, n$ (given n such rectangles) then the approximate area is equal to

$$\sum_{i=1}^{n} \frac{1}{2}\big(f(x_{i-1}) + f(x_i)\big)\delta x.$$

As $\delta x \to 0$,

$$\sum_{i=1}^{n} \frac{1}{2}\big(f(x_{i-1}) + f(x_i)\big)\delta x \to \int_a^b f(x)\,\mathrm{d}x.$$

This concept (the integral as the limit of certain sums) will come in useful later for some of the applications of integration. As with differentiation, however, we will generally not work from first principles.

2.2 Indefinite integrals

Definition

The function $F(x)$ is an *indefinite integral* for the continuous function $f(x)$ if $F(x)$ is differentiable and $F'(x) = f(x)$, in which case we write

$$\int f(x)\,\mathrm{d}x = F(x) + c,$$

where c is the *constant of integration*.

Here there are no limits of integration. A function, the indefinite integral $F(x)$, is the result of indefinite integration, rather than a number (the area) for definite integration.

Example

Since $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$, the function $\tan x$ is an indefinite integral for $\sec^2 x$ and

$$\int \sec^2 x\,\mathrm{d}x = \tan x + c.$$

The Fundamental Theorem of Calculus gives the link between indefinite and definite integrals. It is what allows us to calculate definite integrals simply by recognising derivatives, and without having to find the areas of lots of rectangles.

2.3 The fundamental theorem of calculus

Theorem (FTC)

If $F(x)$ is an indefinite integral for the continuous function $f(x)$, so that $F'(x) = f(x)$, then the definite integral

$$\int_a^b f(x)\,\mathrm{d}x = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

(The square-bracket notation is just shorthand for the final quantity.)

Put simply, this means that to integrate, we need only recognise the integrand as a derivative.

The constant of integration

The Fundamental Theorem of Calculus means that integration can be regarded as the inverse process to differentiation. We know that $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$, so x^2 is one choice for the indefinite integral

$$\int 2x\,\mathrm{d}x.$$

However, differentiating $x^2 + 3$ or $x^2 - 7$ also results in $2x$, since differentiating a constant gives zero, so these three functions are all equally good choices for an indefinite integral of $2x$. All we can say is

$$\int 2x\,\mathrm{d}x = x^2 + c;$$

a constant of integration **must** be included with the indefinite integral.

If $F(x)$ and $G(x)$ are both indefinite integrals for $f(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(F(x) - G(x)\big) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) - \frac{\mathrm{d}}{\mathrm{d}x}G(x) = f(x) - f(x) = 0,$$

so $F(x) - G(x)$ is an indefinite integral for the zero function. By the FTC,

$$0 = \int_a^b 0\,\mathrm{d}x = \big(G(b) - F(b)\big) - \big(G(a) - F(a)\big)$$

and rearranging this shows that $G(b) - G(a) = F(b) - F(a)$. Hence any two choices of indefinite integral for $f(x)$ differ at most by a constant, and all choices give the same value for the definite integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Examples

1. Find the indefinite integrals of (a) x^n (when $n \neq -1$), (b) $\dfrac{1}{x}$ and (c) $\sin x$.

   (a) We have $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^{n+1}) = (n+1)x^n$, so

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{n+1}x^{n+1}\right) = x^n.$$

   Hence $\displaystyle\int x^n \,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$.

   (b) For $x > 0$ we know that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \dfrac{1}{x}$. On the other hand, for $x < 0$ we have

   $$\frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(-x)\big) = -\frac{1}{-x} = \frac{1}{x}.$$

   Hence $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln x + c_1$ if $x > 0$ and $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln(-x) + c_2$ if $x < 0$. Thus

   $$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c.$$

   (c) We know that

   $$\frac{\mathrm{d}}{\mathrm{d}x}(-\cos x) = -\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -(-\sin x) = \sin x$$

   and therefore $\displaystyle\int \sin x \,\mathrm{d}x = -\cos x + c$.

2. Calculate $\displaystyle\int_2^4 x^2 \,\mathrm{d}x$.

   From 1(a), $\displaystyle\int x^2 \,\mathrm{d}x = \frac{1}{2+1}x^{2+1} + c = \frac{1}{3}x^3 + c$. Therefore, by the FTC,

   $$\int_2^4 x^2 \,\mathrm{d}x = \left[\frac{1}{3}x^3 + c\right]_2^4 = \left(\frac{1}{3}4^3 + c\right) - \left(\frac{1}{3}2^3 + c\right) = \frac{64-8}{3} = \frac{56}{3}.$$

   Note that the constant of integration always cancels out, so we can ignore c when finding a definite integral.

2.4 Elementary integration

Rules of integration

(a) **Scalar-multiplication rule**

If k is a constant then $\displaystyle\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x$.

(b) **Sum rule**

$$\int \big(f(x)+g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

Hence integration, like differentiation, is *linear*.

(c) **Linear-composite rule**

If a and b are constants and $F'(x) = f(x)$ then

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + c.$$

This is a special case of integration by substitution, which we will meet later. To see why it is true, we use the chain rule: let $y = ax + b$ and note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{a}F(ax+b)\right) = \frac{1}{a}\frac{\mathrm{d}F}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a}F'(y)a = f(ax+b),$$

as required.

Examples

1. Find $\displaystyle\int (6x^3 + 15x^2 + 2\sin x)\,\mathrm{d}x$.

We have that

$$\begin{aligned}
&\int (6x^3 + 15x^2 + 2\sin x)\,\mathrm{d}x \\
&\quad = 6\int x^3\,\mathrm{d}x + 15\int x^2\,\mathrm{d}x + 2\int \sin x\,\mathrm{d}x && \text{by (a) and (b)} \\
&\quad = 6\frac{1}{4}x^4 + 15\frac{1}{3}x^3 + 2\int \sin x\,\mathrm{d}x && \text{by Example 1 on p.28} \\
&\quad = \frac{3}{2}x^4 + 5x^3 - 2\cos x + c && \text{by the FTC.}
\end{aligned}$$

Note that we only need one constant of integration.

2. Find $\displaystyle\int (1+7x)^{3/2}\,\mathrm{d}x$.

   Since $\displaystyle\int y^{3/2}\,\mathrm{d}y = \frac{2}{5}y^{5/2} + c$, the linear-composite rule (c) shows that

$$\int (1+7x)^{3/2}\,\mathrm{d}x = \frac{1}{7} \times \frac{2}{5}(1+7x)^{5/2} + c = \frac{2}{35}(1+7x)^{5/2} + c.$$

3. Find $\displaystyle\int (2\sin 3\theta - 3\cos 5\theta)\,\mathrm{d}\theta$.

   The FTC implies that

$$\int \sin\psi\,\mathrm{d}\psi = -\cos\psi + c_1 \qquad \text{and} \qquad \int \cos\psi\,\mathrm{d}\psi = -\sin\psi + c_2.$$

   Hence

$$\begin{aligned}\int (2\sin 3\theta - 3\cos 5\theta)\,\mathrm{d}\theta &= 2\int \sin 3\theta\,\mathrm{d}\theta - 3\int \cos 5\theta\,\mathrm{d}\theta && \text{by (a) and (b)}\\ &= 2\times\frac{1}{3}(-\cos 3\theta) - 3\times\frac{1}{5}(\sin 5\theta) + c && \text{by (c)}\\ &= -\frac{2}{3}\cos 3\theta - \frac{3}{5}\sin 5\theta + c.\end{aligned}$$

Integration is inherently more difficult than differentiation. Whereas methodically applying various rules allows us to differentiate most reasonable functions, the same is not true for integration. Indeed, some fairly elementary functions such as $\sqrt{\sin x}$ have no elementary function as their integral, and numerical methods are required. Much of the practice of integration involves using certain techniques, some of which are given below.

2.5 Trigonometric identities

Examples

1. Since $\sin\alpha\sin\beta = \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big)$,

$$\int \sin 4x \sin 3x\,\mathrm{d}x = \frac{1}{2}\int (\cos x - \cos 7x)\,\mathrm{d}x = \frac{1}{2}\sin x - \frac{1}{14}\sin 7x + c.$$

2. Since $\cos 2x = 2\cos^2 x - 1$, it follows that

$$\begin{aligned}\int_0^{\pi/2} \cos^2 x \,\mathrm{d}x &= \int_0^{\pi/2} \frac{1}{2}(1+\cos 2x)\,\mathrm{d}x \\ &= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right) - \left(0 + \frac{1}{4}\sin 0\right) = \frac{\pi}{4}.\end{aligned}$$

2.6 Partial fractions

This method is used to integrate rational functions: these are functions that can be written as the quotient of two polynomials.

1. Find $\displaystyle\int \frac{5x-4}{x^2-8x+12}\,\mathrm{d}x$.

   Note first that $\displaystyle\frac{5x-4}{x^2-8x+12} = \frac{5x-4}{(x-6)(x-2)} = \frac{13}{2(x-6)} - \frac{3}{2(x-2)}$, so

$$\begin{aligned}\int \frac{5x-4}{x^2-8x+12}\,\mathrm{d}x &= \frac{13}{2}\int \frac{1}{x-6}\,\mathrm{d}x - \frac{3}{2}\int \frac{1}{x-2}\,\mathrm{d}x \\ &= \frac{13}{2}\ln|x-6| - \frac{3}{2}\ln|x-2| + c.\end{aligned}$$

2. Find $\displaystyle\int \frac{4x+3}{(x-3)^2}\,\mathrm{d}x$.

   Since

$$\frac{4x+3}{(x-3)^2} = \frac{15}{(x-3)^2} + \frac{4}{x-3},$$

   it follows that

$$\int \frac{4x+3}{(x-3)^2}\,\mathrm{d}x = 15\int \frac{1}{(x-3)^2}\,\mathrm{d}x + 4\int \frac{1}{x-3}\,\mathrm{d}x = \frac{-15}{x-3} + 4\ln|x-3| + c.$$

3. Find $\displaystyle\int \frac{1}{1-x^2}\,\mathrm{d}x$.

   As $\displaystyle\frac{1}{1-x^2} = \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$, we see that

$$\begin{aligned}\int \frac{1}{1-x^2}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{1-x}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{1+x}\,\mathrm{d}x \\ &= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + c = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c.\end{aligned}$$

2.7 Completing the square

Any integral of the form

$$\int \frac{1}{ax^2 + bx + c}\,\mathrm{d}x \qquad \text{or} \qquad \int \frac{1}{\sqrt{ax^2 + bx + c}}\,\mathrm{d}x,$$

where $a \neq 0$, can, by completing the square, be transformed into one of the forms given in the table of standard integrals on p.6 of the Data Book.

Examples

1. Find $\displaystyle\int \frac{1}{x^2 + 4x + 7}\,\mathrm{d}x$.

   Completing the square gives

   $$x^2 + 4x + 7 = (x+2)^2 + 3 = 3\left(1 + \left(\frac{x+2}{\sqrt{3}}\right)^2\right),$$

   so

   $$\begin{aligned}\int \frac{1}{x^2 + 4x + 7}\,\mathrm{d}x &= \frac{1}{3}\int \frac{1}{1 + \left((x+2)/\sqrt{3}\right)^2}\,\mathrm{d}x \\ &= \frac{1}{3} \times \sqrt{3}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + c = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + c.\end{aligned}$$

2. Find $\displaystyle\int \frac{1}{\sqrt{x(1+x)}}\,\mathrm{d}x$.

   Completing the square gives

   $$x(1+x) = x + x^2 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4}\left((2x+1)^2 - 1\right).$$

   Hence

   $$\begin{aligned}\int \frac{1}{\sqrt{x(1+x)}}\,\mathrm{d}x &= \sqrt{4}\int \frac{1}{\sqrt{(2x+1)^2 - 1}}\,\mathrm{d}x \\ &= 2 \times \frac{1}{2}\cosh^{-1}(2x+1) + c = \cosh^{-1}(2x+1) + c.\end{aligned}$$

2.8 Piecewise-standard functions

The observation that

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

whenever $a < c < b$ allows us to integrate piecewise-standard functions, i.e., functions which are made up by 'gluing together' standard functions on subintervals.

Example

Evaluate $\displaystyle\int_{-1}^{2} |x^3|\,\mathrm{d}x$.

Noting that $|x^3| = x^3$ if $x \geqslant 0$ and $|x^3| = -x^3$ if $x \leqslant 0$, it follows that

$$\begin{aligned}\int_{-1}^{2} |x^3|\,\mathrm{d}x &= \int_{-1}^{0} (-x^3)\,\mathrm{d}x + \int_{0}^{2} x^3\,\mathrm{d}x \\ &= \frac{1}{4}\Big(\Big[-x^4\Big]_{-1}^{0} + \Big[x^4\Big]_{0}^{2}\Big) = \frac{1}{4}(0-(-1)+2^4-0) = \frac{17}{4}.\end{aligned}$$

This idea will be useful when studying Fourier Series in MATH145.

2.9 Further methods of integration

Integration by parts

There is no simple 'product rule' for integration. However, the product rule for differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

can be rearranged to give

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Integrating with respect to x and using the FTC, we see that

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

If the function to be integrated can be expressed as $u\dfrac{\mathrm{d}v}{\mathrm{d}x}$ for suitable functions u and v then we can apply this formula, called 'integration by parts'. The trick is to choose a function u which gets simpler when differentiated.

Examples

1. Find $\displaystyle\int xe^x \, \mathrm{d}x$.

   Put $u = x$ and $v = e^x$, so that $\dfrac{\mathrm{d}v}{\mathrm{d}x} = e^x$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$. Then

   $$\begin{aligned}\int xe^x \, \mathrm{d}x &= \int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x \\ &= uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x = xe^x - \int e^x \, \mathrm{d}x = xe^x - e^x + c.\end{aligned}$$

   By choosing $u = x$, the integrand $v\dfrac{\mathrm{d}u}{\mathrm{d}x} = e^x$ is a function we know how to integrate. Sometimes getting to this stage requires more than one application of the formula.

2. Find $\displaystyle\int x^2 \sin x \, \mathrm{d}x$.

   Here we take $u = x^2$ and $v = -\cos x$. (If we choose $u = \sin x$ then the integrand $v\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{3}x^3 \cos x$ is more complicated.) Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and

   $$\int x^2 \sin x \, \mathrm{d}x = -x^2 \cos x + 2 \int x \cos x \, \mathrm{d}x.$$

   Using integration by parts again, on this new integral (taking $u = x$ and $v = \sin x$) we see that

   $$\begin{aligned}\int x^2 \sin x \, \mathrm{d}x &= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, \mathrm{d}x \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c.\end{aligned}$$

3. Find $\displaystyle\int x^3 \ln x \, \mathrm{d}x$.

   Put $u = \ln x$ and $v = \dfrac{1}{4}x^4$, so that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = x^3$. Then

   $$\begin{aligned}\int x^3 \ln x \, \mathrm{d}x &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^4 \frac{1}{x} \, \mathrm{d}x \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 \, \mathrm{d}x \quad = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c.\end{aligned}$$

In the examples above, the integrand was made simpler until it could be recognised as the derivative of something. When dealing with a product involving functions such as e^{ax}, $\sin x$ and $\cos x$, the integrand may not simplify. However, we may be able to integrate by parts twice, return to something very like the original and hence determine our integral. An example will explain this idea.

Example

Evaluate $I = \displaystyle\int_0^{\pi/4} e^{5x} \sin 2x \, \mathrm{d}x$.

If $u = e^{5x}$ and $v = -\frac{1}{2}\cos 2x$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 5e^{5x}$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x$, so integration by parts shows that

$$\begin{aligned} I = \int_0^{\pi/4} e^{5x} \sin 2x \, \mathrm{d}x &= \left[-\frac{1}{2}e^{5x}\cos 2x\right]_0^{\pi/4} + \frac{5}{2}\int_0^{\pi/4} e^{5x}\cos 2x \, \mathrm{d}x \\ &= \left(-\frac{1}{2}e^{5\pi/4}\cos\frac{\pi}{2} + \frac{1}{2}e^0 \cos 0\right) + \frac{5}{2}\int_0^{\pi/4} e^{5x}\cos 2x \, \mathrm{d}x \\ &= \frac{1}{2} + \frac{5}{2}\int_0^{\pi/4} e^{5x}\cos 2x \, \mathrm{d}x. \end{aligned}$$

Now put $u = e^{5x}$ and $v = \frac{1}{2}\sin 2x$, so $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 5e^{5x}$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$ and integrating by parts once more gives that

$$\begin{aligned} I &= \frac{1}{2} + \frac{5}{2}\left(\left[\frac{1}{2}e^{5x}\sin 2x\right]_0^{\pi/4} - \frac{5}{2}\int_0^{\pi/4} e^{5x}\sin 2x \, \mathrm{d}x\right) \\ &= \frac{1}{2} + \frac{5}{2}\left(\frac{1}{2}e^{5\pi/4}\sin\frac{\pi}{2} - \frac{1}{2}e^0\sin 0 - \frac{5}{2}I\right) \\ &= \frac{1}{2} + \frac{5}{4}e^{5\pi/4} - \frac{25}{4}I. \end{aligned}$$

Hence

$$\frac{29}{4}I = \frac{1}{2} + \frac{5}{4}e^{5\pi/4}$$

and therefore

$$I = \frac{4}{29}\left(\frac{1}{2} + \frac{5}{4}e^{5\pi/4}\right) = 8{\cdot}8197 \qquad \text{(4 d.p.).}$$

2.10 Integration by substitution

This technique is a consequence of the chain rule for differentiation. Let $F(x)$ be an indefinite integral for $f(x)$ and let $y = g(x)$. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} F\big(g(x)\big) = F'\big(g(x)\big) g'(x) = f(y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The FTC implies that

$$\int f(y)\,\mathrm{d}y = F(y) + c = F\big(g(x)\big) + c = \int f(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x$$

and we have the following change-of-variable formula:

$$\int f(y)\,\mathrm{d}y = \int f(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x.$$

Some examples will show how this works in practice; recall that $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 1 \Big/ \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Examples

1. To find $I_1 = \displaystyle\int x^2\sqrt{5+x^3}\,\mathrm{d}x$, put $t = 5 + x^3$, so that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = 3x^2$ and

$$I_1 = \int x^2\sqrt{t}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{3}\int \sqrt{t}\,\mathrm{d}t = \frac{2}{9}t^{3/2} + c = \frac{2}{9}(5+x^3)^{3/2} + c.$$

2. To find $I_2 = \displaystyle\int \frac{x+2}{x^2+4x+7}\,\mathrm{d}x$, put $t = x^2 + 4x + 7$, so that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = 2x + 4$ and

$$I_2 = \int \frac{x+2}{t}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln|t| + c = \frac{1}{2}\ln|x^2+4x+7| + c.$$

3. To find $I_3 = \displaystyle\int \tan^3 x\,\mathrm{d}x$, put $t = \tan x$, so that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \sec^2 x = 1 + t^2$ and

$$\begin{aligned} I_3 = \int t^3\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= \int \frac{t^3}{1+t^2}\,\mathrm{d}t \\ &= \int \Big(t - \frac{t}{1+t^2}\Big)\,\mathrm{d}t \\ &= \frac{1}{2}t^2 - \frac{1}{2}\ln(1+t^2) + c = \frac{1}{2}\tan^2 x + \ln|\cos x| + c. \end{aligned}$$

Often the integrand is not of a form which permits a straightforward use of the technique. A substitution is made which we hope will lead to a simplification.

Example

To find $\displaystyle\int \frac{1}{x+\sqrt{2-x}}\,\mathrm{d}x$, we try to get rid of the square root by putting $t=\sqrt{2-x}$. Then $t^2 = 2-x$, so $x = 2-t^2$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -2t$. Hence

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x+\sqrt{2-x}} &= \int \frac{1}{2-t^2+t}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int \frac{2t}{(t-2)(t+1)}\,\mathrm{d}t \\
&= \int \frac{4}{3(t-2)} + \frac{2}{3(t+1)}\,\mathrm{d}t \\
&= \frac{4}{3}\ln|t-2| + \frac{2}{3}\ln|t+1| + c \\
&= \frac{4}{3}\ln|\sqrt{2-x}-2| + \frac{2}{3}\ln|\sqrt{2-x}+1| + c.
\end{aligned}$$

When applying substitution to definite integrals we do not need to substitute back to get the answer in terms of the original variable. Instead we modify the limits of integration.

Examples

1. Show that $I_1 = \displaystyle\int_2^7 \frac{1}{(x+1)\sqrt{x+2}}\,\mathrm{d}x = \ln(3/2)$.

   Put $t = \sqrt{x+2}$ and note that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \dfrac{1}{2\sqrt{x+2}} = \dfrac{1}{2t}$. As x increases from 2 to 7, t increases from 2 to 3, so

$$\begin{aligned}
I_1 = \int_2^3 \frac{1}{(t^2-1)t}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = 2\int_2^3 \frac{1}{t^2-1}\,\mathrm{d}t = \int_2^3 \Big(\frac{1}{t-1} - \frac{1}{t+1}\Big)\,\mathrm{d}t
&= \Big[\ln(t-1) - \ln(t+1)\Big]_2^3 \\
&= \ln 2 - \ln 4 - \ln 1 + \ln 3 \\
&= \ln(3/2) \\
&= 0{\cdot}4055 \quad \text{(4 d.p.).}
\end{aligned}$$

2. Evaluate $I_2 = \int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x$.

   Put $x = 3\sin\theta$, so that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\cos\theta$ and

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = \pm 3\cos\theta.$$

   To get the range of integration from $x = 0$ to $x = 3/2$, we take $\theta = 0$ to $\theta = \sin^{-1}(1/2) = \pi/6$. Then $\cos\theta$ is positive on this interval, so

$$I_2 = \int_0^{\pi/6} \frac{9\sin^2\theta}{3\cos\theta}\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int_0^{\pi/6} \frac{3\sin^2\theta}{\cos\theta}\,3\cos\theta\,\mathrm{d}\theta = 9\int_0^{\pi/6} \sin^2\theta\,\mathrm{d}\theta.$$

   Now,

$$\begin{aligned}\int_0^{\pi/6} \sin^2\theta\,\mathrm{d}\theta &= \frac{1}{2}\int_0^{\pi/6}(1-\cos 2\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} \\ &= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3} + \frac{1}{2}\sin 0\right) = \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right)\end{aligned}$$

   and therefore $I_2 = \frac{3\pi}{4} - \frac{9\sqrt{3}}{8} = 0{\cdot}4076$ (4 d.p.).

3. Evaluate $I_3 = \int_0^{\pi/4} \frac{1}{\cos^2 x + 9\sin^2 x}\,\mathrm{d}x$.

   Put $t = \tan x$, so that $\frac{\mathrm{d}t}{\mathrm{d}x} = \sec^2 x$ and as x goes from 0 to $\pi/4$, the variable t goes from 0 to 1. Then

$$\begin{aligned}I_3 &= \int_0^1 \frac{1}{\cos^2 x + 9\sin^2 x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_0^1 \frac{\cos^2 x}{\cos^2 x + 9\sin^2 x}\,\mathrm{d}t = \int_0^1 \frac{1}{1+9t^2}\,\mathrm{d}t = \left[\frac{1}{3}\tan^{-1}(3t)\right]_0^1 = \frac{1}{3}\tan^{-1} 3,\end{aligned}$$

   which equals $0{\cdot}4163$ (4 d.p.). (The penultimate equality follows from the linear-composite rule.)

Inverse-function rule

Integration by parts and by substitution combine to give the following formula.

If $y = f^{-1}(x)$, so that $x = f(y)$, then

$$\int f^{-1}(x)\,\mathrm{d}x = xy - \int f(y)\,\mathrm{d}y.$$

(To see where this comes from, note that

$$\int y\,\mathrm{d}x = \int 1 \times y\,\mathrm{d}x = xy - \int x\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = xy - \int x\,\mathrm{d}y.$$

The trick of writing the integrand $g(x)$ as $1 \times g(x)$ and then integrating by parts can be useful in other situations.)

Examples

1. If $y = \ln x$ then $x = e^y$ and

$$\int \ln x\,\mathrm{d}x = xy - \int e^y\,\mathrm{d}y = xy - e^y + c = x\ln x - x + c.$$

2. If $y = \sin^{-1} x$ then $x = \sin y$ and

$$\begin{aligned}\int \sin^{-1} x\,\mathrm{d}x &= xy - \int \sin y\,\mathrm{d}y \\ &= x\sin^{-1} x + \cos y + c \\ &= x\sin^{-1} x + \sqrt{1 - \sin^2 y} + c \\ &= x\sin^{-1} x + \sqrt{1 - x^2} + c.\end{aligned}$$

   (We take the positive square root because $\cos y \geqslant 0$ when $-\pi/2 \leqslant y \leqslant \pi/2$, which is the range of values taken by $\sin^{-1}$.)

3. If $y = \sinh^{-1} x$ then $x = \sinh y$ and

$$\begin{aligned}\int \sinh^{-1} x\,\mathrm{d}x &= xy - \int \sinh y\,\mathrm{d}y \\ &= x\sinh^{-1} x - \cosh y + c \\ &= x\sinh^{-1} x - \sqrt{1 + \sinh^2 y} + c = x\sinh^{-1} x - \sqrt{1 + x^2} + c.\end{aligned}$$

   (Since $\cosh y > 0$ for all y, we take the positive square root.)

2.11 Applications of integration

Arc length of a curve

The length along the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \approx \sum \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \sum \sqrt{(\delta x)^2 + (\delta y)^2}.$$

Example

Find the length of the parabola $y = x^2$ between $x = 0$ and $x = 1$.

From the formula given above,

$$s = \int_0^1 \sqrt{1 + (2x)^2}\,\mathrm{d}x = \int_0^1 \sqrt{1 + 4x^2}\,\mathrm{d}x.$$

Putting $x = \frac{1}{2}\sinh u$, we have $\dfrac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}\cosh u$ and

$$\sqrt{1 + 4x^2} = \sqrt{1 + \sinh^2 u} = \cosh u.$$

When $x = 0$, $u = \sinh^{-1} 0 = 0$ and when $x = 1$, $u = \sinh^{-1} 2$, so

$$\begin{aligned}
s &= \int_0^{\sinh^{-1} 2} \cosh u \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^{\sinh^{-1} 2} \cosh^2 u\,\mathrm{d}u \\
&= \frac{1}{4}\int_0^{\sinh^{-1} 2} (1 + \cosh 2u)\,\mathrm{d}u \\
&= \frac{1}{4}\left[u + \frac{\sinh 2u}{2}\right]_0^{\sinh^{-1} 2} \qquad (\star) \\
&= \frac{1}{4}\Big[u + \sinh u \cosh u\Big]_0^{\sinh^{-1} 2} \\
&= \frac{1}{4}\left[u + \sinh u\sqrt{1 + \sinh^2 u}\right]_0^{\sinh^{-1} 2} \\
&= \frac{1}{4}\Big(\sinh^{-1} 2 + 2\sqrt{1 + 2^2} - 0 - 0\Big) = \frac{\sqrt{5}}{2} + \frac{1}{4}\sinh^{-1} 2 = 1{\cdot}4789 \quad \text{(4 d.p.)}.
\end{aligned}$$

(You could use your calculator to find the answer once you reach $(\star)$.)

Area of a plane region

Suppose $f(x)$ and $g(x)$ are continuous functions and $f(x) \geqslant g(x)$ as x ranges over the interval $[a, b]$, i.e., for all x such that $a \leqslant x \leqslant b$. The area of the plane region between the graphs of $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$ is equal to

$$A = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=f(x)} \mathrm{d}y\,\mathrm{d}x \approx \sum\sum \delta x\,\delta y.$$

(Draw a picture to convince yourself of this.)

Example

Find the area enclosed by the curves $y = \sec^2 x$ and $y = \sin x$ between $x = 0$ and $x = \pi/4$.

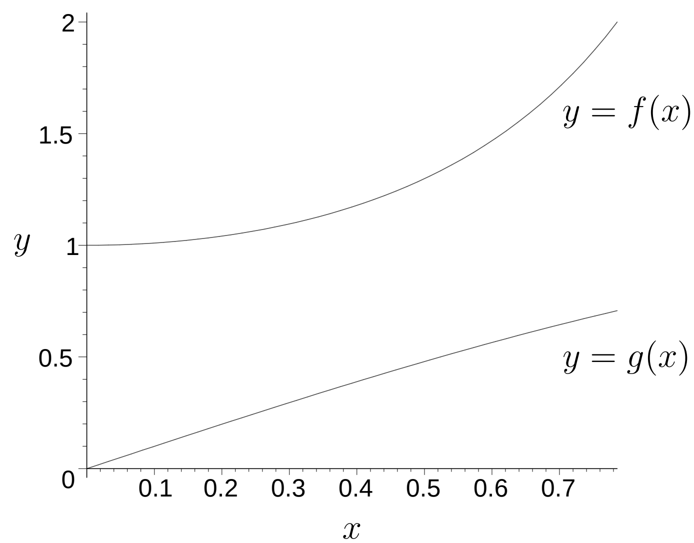

Figure 6: $f(x) = \sec^2 x$, $g(x) = \sin x$

Note that $\sec^2 x \geqslant 1 \geqslant \sin x$ for $0 \leqslant x \leqslant \pi/4$, as required, so the area

$$\begin{aligned} A = \int_0^{\pi/4} (\sec^2 x - \sin x)\,\mathrm{d}x &= \Big[\tan x + \cos x\Big]_0^{\pi/4} \\ &= \tan\frac{\pi}{4} + \cos\frac{\pi}{4} - \tan 0 - \cos 0 \\ &= 1 + \frac{1}{\sqrt{2}} - 0 - 1 \\ &= \frac{1}{\sqrt{2}} = 0{\cdot}7071 \quad \text{(4 d.p.)}. \end{aligned}$$

Centroid of a plane area

As above, let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geqslant g(x)$ on $[a, b]$ and consider the same plane region, that bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, which has area A.

The centroid of this region has coordinates $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{1}{A}\int_a^b x\big(f(x) - g(x)\big)\,\mathrm{d}x = \frac{1}{A}\int_{x=a}^{x=b}\int_{y=g(x)}^{y=f(x)} x\,\mathrm{d}y\,\mathrm{d}x \approx \frac{1}{A}\sum\sum x\,\delta x\,\delta y$$

and

$$\bar{y} = \frac{1}{2A}\int_a^b (f(x)^2 - g(x)^2)\,\mathrm{d}x = \frac{1}{A}\int_{x=a}^{x=b}\int_{y=g(x)}^{y=f(x)} y\,\mathrm{d}y\,\mathrm{d}x \approx \frac{1}{A}\sum\sum y\,\delta x\,\delta y.$$

In the special case where the lower curve is the x-axis (i.e., $g(x) = 0$) we have

$$A = \int_a^b f(x)\,\mathrm{d}x, \quad \bar{x} = \frac{1}{A}\int_a^b xf(x)\,\mathrm{d}x \quad \text{and} \quad \bar{y} = \frac{1}{2A}\int_a^b f(x)^2\,\mathrm{d}x.$$

If the region is made out of a thin material of uniform density then the centroid is its centre of mass. The centre of mass $\bar{\mathbf{r}}$ of a finite number of particles is equal to their average position, weighted by their masses: $\bar{\mathbf{r}} = \sum_i m_i\mathbf{r}_i / \sum_i m_i$; analogously, the coordinates of the centroid are given by the integrals of position times density.

Example

Find the coordinates of the centroid of the region enclosed by the parabola $y = x^2 - x$ and the line $y = x$. (See Figure 7.)

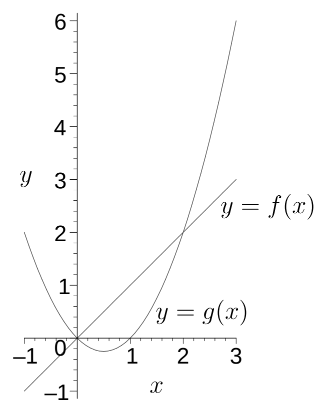

Figure 7: $f(x) = x$, $g(x) = x^2 - x$

The two curves intersect when

$$x^2 - x = x \iff x^2 - 2x = 0 \iff x(x-2) = 0 \iff x = 0 \text{ or } 2.$$

In between $x = 0$ and $x = 2$ we have $x > x^2 - x$, so we take

$$f(x) = x \qquad \text{and} \qquad g(x) = x^2 - x.$$

Then

$$A = \int_0^2 (2x - x^2)\,\mathrm{d}x = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} - 0 + 0 = \frac{4}{3},$$

so

$$\bar{x} = \frac{1}{A}\int_0^2 x(2x - x^2)\,\mathrm{d}x = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left(\frac{16}{3} - 4 - 0 + 0\right) = 1$$

and

$$\begin{aligned}\bar{y} = \frac{1}{2A}\int_0^2 (x^2 - (x^2 - x)^2)\,\mathrm{d}x &= \frac{3}{8}\int_0^2 (x^2 - x^4 + 2x^3 - x^2)\,\mathrm{d}x \\ &= \frac{3}{8}\left[-\frac{1}{5}x^5 + \frac{1}{2}x^4\right]_0^2 \\ &= \frac{3}{8}\left(-\frac{32}{5} + 8 - 0 + 0\right) = \frac{3}{5}.\end{aligned}$$

The coordinates of the centroid are therefore $(1, 3/5)$.

Area of a surface of revolution

A *surface of revolution* is obtained by rotating the curve $y = f(x)$ (from $x = a$ to $x = b$) through 2π radians about the x-axis, where $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$. Its surface area S is given by the following formula:

$$\begin{aligned}S &= 2\pi \int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{x=a}^{x=b} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x\,\mathrm{d}\theta \approx \sum\sum \sqrt{(\delta x)^2 + (\delta y)^2}\,y\,\delta\theta.\end{aligned}$$

If θ is the angle through which the curve is rotated then a small patch of the surface will have width $y\,\delta\theta$ and length $\delta s = \sqrt{(\delta x)^2 + (\delta y)^2}$, so area $\delta s\,y\,\delta\theta$.

Examples

1. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = r^2$ which is enclosed by the planes $x = a$ and $x = b$, where $-r \leqslant a < b \leqslant r$.

   The sphere may be obtained by rotating $y = \sqrt{r^2 - x^2}$ about the x-axis. On this curve we have $y^2 = r^2 - x^2$, and differentiating with respect to x gives that
   $$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$
   Hence
   $$S = 2\pi \int_a^b y\sqrt{1 + \frac{x^2}{y^2}}\,\mathrm{d}x = 2\pi \int_a^b \sqrt{y^2 + x^2}\,\mathrm{d}x = 2\pi \int_a^b r\,\mathrm{d}x = 2\pi r(b - a).$$

2. Find the surface area of the surface of revolution obtained by rotating about the x-axis the curve $y = e^x$ from $x = 0$ to $x = \ln 3$.

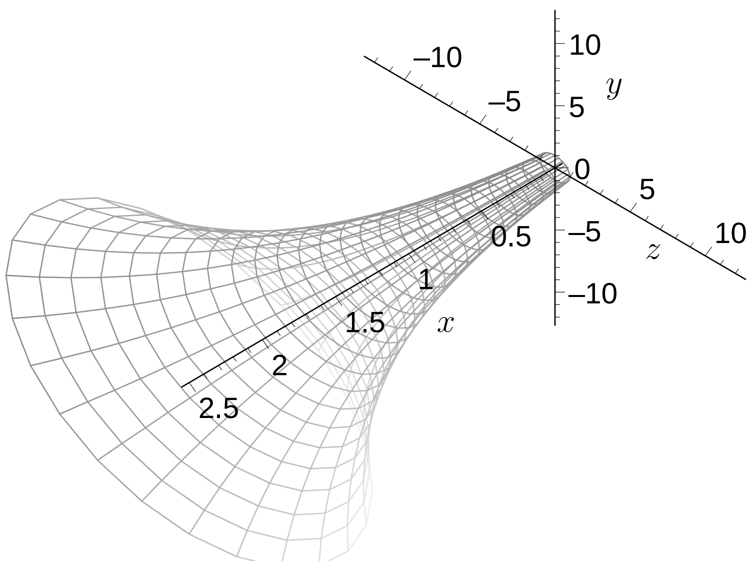

Figure 8: The surface of revolution obtained by rotating $y = e^x$

From the formula,
$$S = 2\pi \int_0^{\ln 3} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 2\pi \int_0^{\ln 3} y\sqrt{1 + y^2}\,\mathrm{d}x$$
$$= 2\pi \int_1^3 y\sqrt{1 + y^2}\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y = 2\pi \int_1^3 \sqrt{1 + y^2}\,\mathrm{d}y.$$

Using the substitution $u = \sinh^{-1} y$ and noting that $\dfrac{\mathrm{d}y}{\mathrm{d}u} = \cosh u$, we have

$$\begin{aligned} S = 2\pi \int_{\sinh^{-1} 1}^{\sinh^{-1} 3} \sqrt{1 + \sinh^2 u}\frac{\mathrm{d}y}{\mathrm{d}u}\,\mathrm{d}u &= 2\pi \int_{\sinh^{-1} 1}^{\sinh^{-1} 3} \cosh^2 u\,\mathrm{d}u \\ &= \pi \int_{\sinh^{-1} 1}^{\sinh^{-1} 3} (1 + \cosh 2u)\,\mathrm{d}u \\ &= \pi \left[u + \frac{1}{2}\sinh 2u \right]_{\sinh^{-1} 1}^{\sinh^{-1} 3} \\ &= 28{\cdot}3048 \quad \text{(4 d.p.)}. \end{aligned}$$

Volume of a solid of revolution

A *solid of revolution* is obtained by rotating about the x-axis the area beneath the curve $y = f(x)$, where x ranges over the interval $[a, b]$ and $f(x) \geqslant 0$ for all such x. Its volume V is given by the formula

$$V = \pi \int_a^b f(x)^2\,\mathrm{d}x = \int_{\theta=0}^{2\pi} \int_{x=a}^{x=b} \int_{y=0}^{f(x)} y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta \approx \sum\sum\sum \delta x\,\delta y\,y\,\delta\theta.$$

If θ is the angle through which the curve is rotated, a small piece of V will have length δx, height δy and width $y\,\delta\theta$.

Examples

1. Find the volume of the solid of revolution obtained by rotating the area below the parabola $y = 2 - x^2$ and above the x-axis.

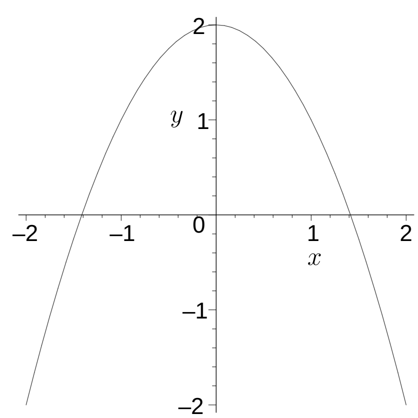

Figure 9: The parabola $y = 2 - x^2$

The parabola $y = 2 - x^2$ crosses the x-axis at $2 - x^2 = 0$, so when $x = \pm\sqrt{2}$, and lies above the x-axis in between. Hence

$$\begin{aligned} V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^2 \,\mathrm{d}x = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4x^2 + x^4) \,\mathrm{d}x \\ &= \pi \left[4x - \frac{4}{3}x^3 + \frac{x^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \pi \left(4\sqrt{2} - \frac{4}{3}2\sqrt{2} + \frac{4\sqrt{2}}{5} + 4\sqrt{2} - \frac{4}{3}2\sqrt{2} + \frac{4\sqrt{2}}{5} \right) \\ &= \frac{64\sqrt{2}\pi}{15} \\ &= 18{\cdot}9563 \quad \text{(4 d.p.)}. \end{aligned}$$

2. A sphere of radius r is obtained by rotating the curve $y = \sqrt{r^2 - x^2}$ about the x-axis, where $-r \leqslant x \leqslant r$. Show that its volume $V = 4\pi r^3/3$.

   From the formula above,

$$\begin{aligned} V = \pi \int_{-r}^{r} (r^2 - x^2) \,\mathrm{d}x &= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\ &= \pi \left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3} \right) \\ &= \frac{4\pi r^3}{3}, \end{aligned}$$

   as required.

Centre of mass of a solid of revolution

A solid of revolution is obtained as above, by rotating about the x-axis the area beneath the curve $y = f(x)$, with x ranging from a to b and $f(x) \geqslant 0$. Its centre of mass lies on the x-axis (i.e., has y and z coordinates $\bar{y} = \bar{z} = 0$) and has x coordinate

$$\bar{x} = \frac{\pi}{V} \int_a^b x f(x)^2 \,\mathrm{d}x = \frac{1}{V} \int_{\theta=0}^{2\pi} \int_{x=a}^{b} x \int_{y=0}^{f(x)} y \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}\theta \approx \frac{1}{V} \sum\sum\sum x \,\delta x \,\delta y \, y \,\delta\theta.$$

As for the centroid, the coordinates of the centre of mass are given by integrating position times density. As the solid is symmetrical about the x-z and x-y planes, the integrals for $\bar{y}$ and $\bar{z}$ will both be zero.

Example

The area enclosed by the curve $y = (3x+1)^{1/4}$, the x-axis, the y-axis and the line $x = 5$ is rotated through 2π radians about the x-axis. Calculate the x coordinate of the centre of mass of the resulting solid of revolution.

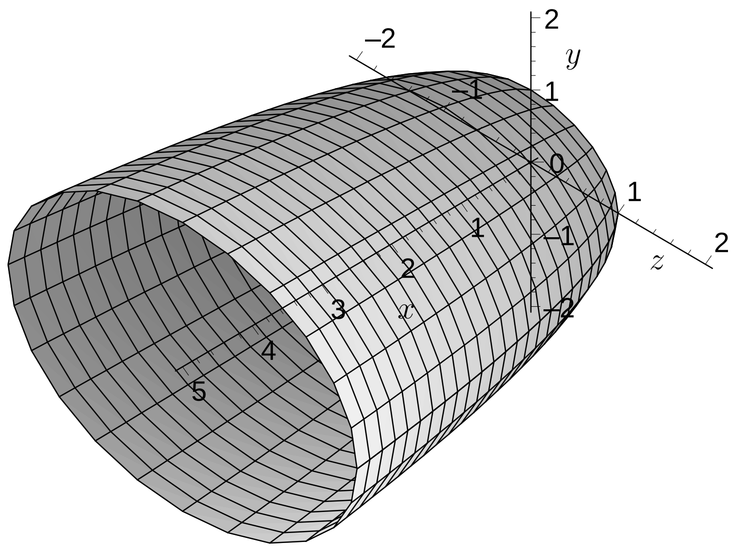

Figure 10: The solid of revolution obtained from $y = (3x+1)^{1/4}$

From the formula above,

$$V = \pi \int_0^5 \left((3x+1)^{1/4}\right)^2 \mathrm{d}x = \pi \int_0^5 (3x+1)^{1/2}\,\mathrm{d}x$$

$$= \pi \left[\frac{2}{9}(3x+1)^{3/2}\right]_0^5 = \frac{2\pi}{9}(16^{3/2} - 1^{3/2}) = 14\pi.$$

Hence

$$\bar{x} = \frac{\pi}{V}\int_0^5 x f(x)^2\,\mathrm{d}x = \frac{1}{14}\int_0^5 x(3x+1)^{1/2}\,\mathrm{d}x.$$

Using integration by parts with $u = x$ and $v = \dfrac{2}{9}(3x+1)^{3/2}$, we have

$$\bar{x} = \frac{1}{14}\left(\left[\frac{2}{9}x(3x+1)^{3/2}\right]_0^5 - \frac{2}{9}\int_0^5 (3x+1)^{3/2}\,\mathrm{d}x\right)$$

$$= \frac{1}{63}\left(320 - \left[\frac{2}{15}(3x+1)^{5/2}\right]_0^5\right) = \frac{320}{63} - \frac{2}{63}\frac{1023}{15} = \frac{102}{35} = 2{\cdot}9143 \quad \text{(4 d.p.)}.$$

3 Numerical methods

3.1 The trapezium rule

In practical situations, functions often cannot be integrated analytically (in terms of familiar functions, as above). We have to resort to numerical methods to obtain answers with the desired degree of accuracy.

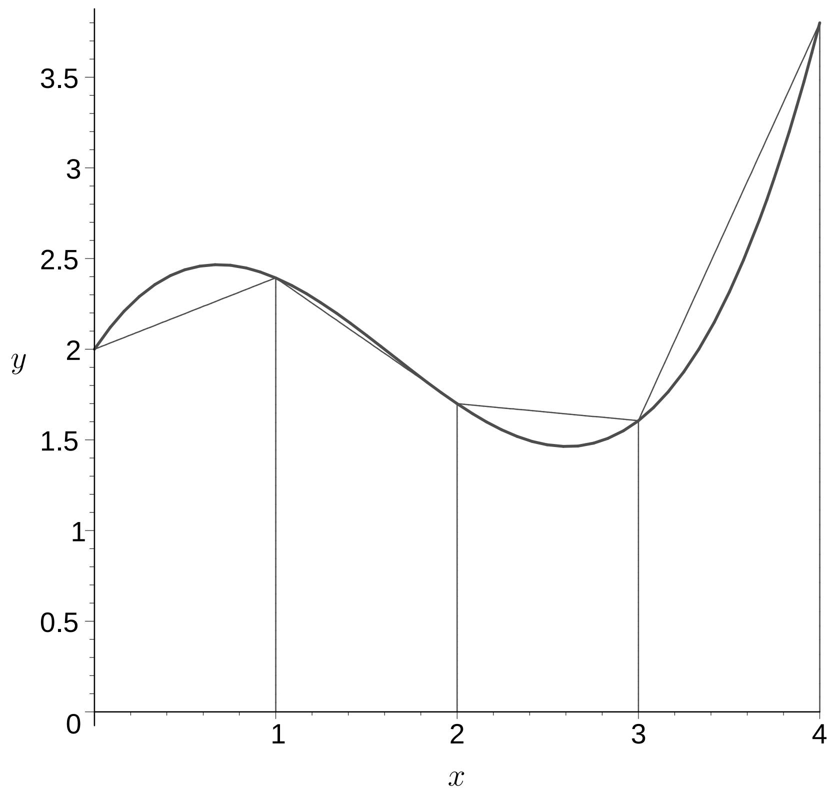

Figure 11: The trapezium rule

To find an approximate value for $\int_a^b f(x)\,\mathrm{d}x$, first subdivide the interval from a to b into n equal subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

with $x_i = x_{i-1} + h$ for $i = 1, \ldots, n$ and $h = (b - a)/n$. We approximate the area beneath the graph between x_{i-1} and x_i by the area A_i of a trapezium:

$$A_i = \tfrac{1}{2}\big(f(x_{i-1}) + f(x_i)\big)h.$$

Adding these, we see that

$$
\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \sum_{i=1}^{n} A_i \\
&= \frac{h}{2}\Big(\big(f(x_0)+f(x_1)\big)+\big(f(x_1)+f(x_2)\big)+\cdots+\big(f(x_{n-1})+f(x_n)\big)\Big) \\
&= \frac{h}{2}\big(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\big) \\
&= T(h).
\end{aligned}
$$

This is the *trapezium-rule approximation* to $\int_a^b f(x)\,\mathrm{d}x$ using n strips of width h.

Example

Use the trapezium rule with 4 strips to find an approximation for $I = \int_1^5 e^{-\sqrt{x}}\,\mathrm{d}x$. Work to 3 s.f.

As $h = (5-1)/4 = 1$, the strips have end points $1 < 2 < 3 < 4 < 5$. Letting $f(x) = e^{-\sqrt{x}}$, we see that

$$I \approx T(1) = \frac{1}{2}\big(f(1)+2f(2)+2f(3)+2f(4)+f(5)\big) = 0{\cdot}793 \qquad \text{(3 s.f.)}.$$

This is reasonably close to the actual value of I, which is 0·780 to 3 s.f.

Successive bisection

Often we do not know how many strips to use at first, so we repeat the process, increasing the number of strips until two successive answers agree to the required accuracy. We can save a great deal of work if we successively halve the width (i.e., double the number of strips by splitting each strip in half) since if n is even then

$$
\begin{aligned}
T(h) &= \frac{h}{2}\big(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\big) \\
&= \frac{h}{2}\big(2f(x_1)+2f(x_3)+\cdots+2f(x_{n-1})\big) \\
&\qquad + \frac{h}{2}\big(f(x_0)+2f(x_2)+2f(x_4)+\cdots+2f(x_{n-2})+f(x_n)\big) \\
&= h\big(f(x_1)+f(x_3)+\cdots+f(x_{n-1})\big)+\frac{1}{2}T(2h).
\end{aligned}
$$

To understand this formula, let us see it in action.

Example

Use the trapezium rule to estimate $I = \displaystyle\int_4^{20} \frac{1}{x^2-1}\,\mathrm{d}x$ accurate to 1 d.p.

If $f(x) = \dfrac{1}{x^2-1}$ then, recording our working accurate to 6 decimal places (for later),

1 strip with end points $4 < 20$ gives

$$\begin{aligned} T(16) = \frac{h}{2}\big(f(x_0)+f(x_1)\big) &= \frac{16}{2}\big(f(4)+f(20)\big) \\ &= 8(0{\cdot}066667 + 0{\cdot}002506) = 0{\cdot}553383. \end{aligned}$$

2 strips with end points $4 < 12 < 20$ give

$$T(8) = 8f(12) + \frac{1}{2}T(16) = 8(0{\cdot}006993) + \frac{1}{2}0{\cdot}553383 = 0{\cdot}332636.$$

(To get this figure, we add $h = 8$ times the new ordinate, $f(12)$, to half the previous answer, $T(16)$.)

4 strips with end points $4 < 8 < 12 < 16 < 20$ give

$$\begin{aligned} T(4) &= 4\big(f(8)+f(16)\big) + \frac{1}{2}T(8) \\ &= 4\big(0{\cdot}015873 + 0{\cdot}003922\big) + \frac{1}{2}0{\cdot}332636 = 0{\cdot}245496. \end{aligned}$$

(Again, we add $h = 4$ times the sum of the new ordinates, $f(8)$ and $f(16)$, to half the previous answer, $T(8)$.)

8 strips with end points $4 < 6 < 8 < 10 < 12 < 14 < 16 < 18 < 20$ give

$$\begin{aligned} T(2) &= 2\big(f(6)+f(10)+f(14)+f(18)\big) + \frac{1}{2}T(4) \\ &= 2\big(0{\cdot}028571 + 0{\cdot}010101 + 0{\cdot}005128 + 0{\cdot}003096\big) + \frac{1}{2}0{\cdot}245496 \\ &= 0{\cdot}216541. \end{aligned}$$

(Once more, we add $h = 2$ times the sum of the new ordinates to half the previous answer.) As $T(4)$ and $T(2)$ agree sufficiently, $I = 0{\cdot}2$ to 1 d.p.

In this case, we can find I exactly. Since $x^2 - 1 = (x-1)(x+1)$, it is an exercise using partial fractions to show that

$$\int_4^{20} \frac{1}{x^2-1}\,\mathrm{d}x = \frac{1}{2}\left[\ln\frac{x-1}{x+1}\right]_4^{20} = \frac{1}{2}\ln\frac{95}{63} = 0{\cdot}205371 \quad \text{(6 d.p.)}.$$

The error

If we tabulate the estimates $T(h)$ and the errors

$$\epsilon_T(h) = T(h) - \int_4^{20} \frac{\mathrm{d}x}{x^2-1} = T(h) - \frac{1}{2}\ln\frac{95}{63}$$

then, extending the working above to give two further estimates, we obtain Table 1.

h	$T(h)$	$\epsilon_T(h)$
16	0·553383	0·348012
8	0·332636	0·127265
4	0·245496	0·040125
2	0·216541	0·011170
1	0·208271	0·002900
0·5	0·206104	0·000733

Table 1: Error in the trapezium-rule approximation to $\displaystyle\int_4^{20} \frac{1}{x^2-1}\,\mathrm{d}x$

The error decreases approximately by a factor of 4 as the width is halved, and this behaviour is typical for the trapezium rule. The error is roughly a constant times h^2 for small values of h: this is written as

$$T(h) = \int_a^b f(x)\,\mathrm{d}x + O(h^2),$$

where the notation $O(h^2)$ represents a function which is 'about the same order of magnitude' as h^2 for small h. It is possible to do better by approximating the curve more closely; this brings us to Simpson's rule.

3.2 Simpson's rule

The trapezium rule approximates the area under a curve by replacing that curve with a number of straight-line segments, which are curves of the form $y = mx + c$. Simpson's rule improves on this by using quadratic curves, i.e., curves of the form $y = ax^2 + bx + c$. A quadratic curve is determined by three points: for Simpson's rule, the first curve is chosen to go through $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$, the second through $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$, and so on. In order for this to work, the area must be divided into an even number of strips.

The derivation of Simpson's rule is **not** required for this course. For the interested reader, we include an explanation in Appendix A.

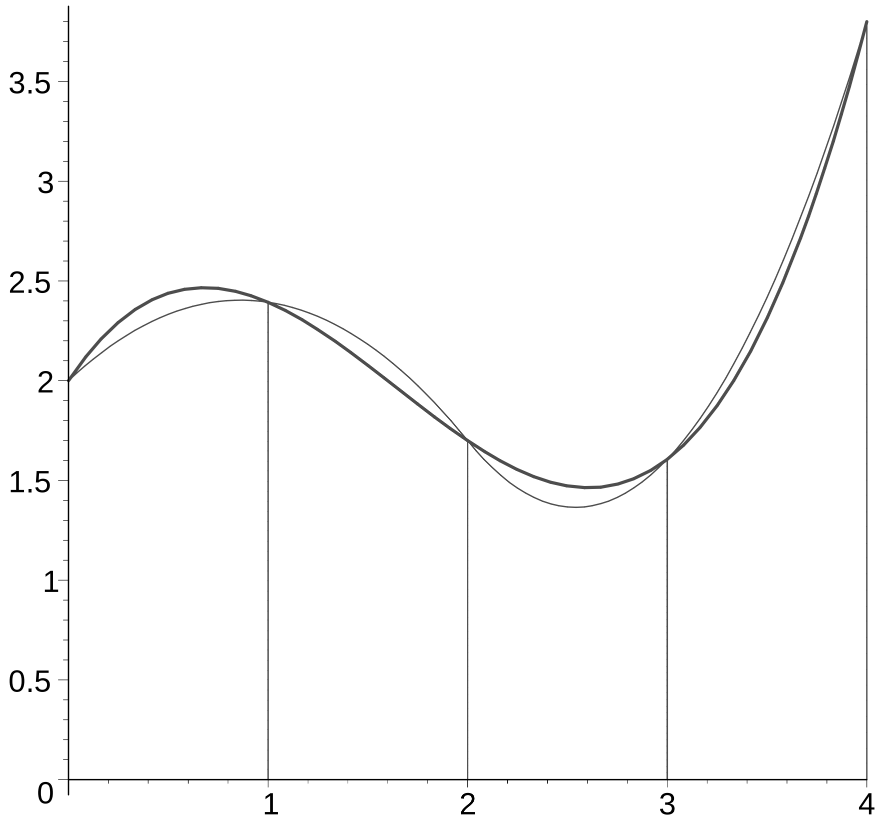

Figure 12: Simpson's rule

Simpson's rule states that

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big),$$

where n is an even number and h is the width of each strip. A shorter form is often used:

$$\int_a^b f(x)\,\mathrm{d}x \approx S(h) = \frac{h}{3}(F + L + 4O + 2E),$$

where F and L are the first and last ordinates, O is the sum of the odd-position ordinates, ($f(x_1)$, $f(x_3)$ et cetera) and E is the sum of the even-position ordinates ($f(x_2)$, $f(x_4)$, et cetera). For comparison, the trapezium rule has the form

$$\int_a^b f(x)\,\mathrm{d}x \approx T(h) = \frac{h}{2}(F + L + 2O + 2E).$$

Example

Use Simpson's rule with 2, 4 and 8 strips to estimate $\displaystyle\int_4^{20} \frac{1}{x^2-1}\,\mathrm{d}x$. Work to 6 d.p.

2 strips with $h = (20-4)/2 = 8$ and $f(x) = \dfrac{1}{x^2-1}$ give

$$S(8) = \frac{8}{3}\big(f(4)+f(20)+4f(12)\big) = \frac{8}{3}\big(0{\cdot}066667+0{\cdot}002506+4(0{\cdot}006993)\big) = 0{\cdot}259053,$$

4 strips with $h = (20-4)/4 = 4$ give

$$\begin{aligned} S(4) &= \frac{4}{3}\big(f(4)+f(20)+4\big(f(8)+f(16)\big)+2f(12)\big) \\ &= \frac{4}{3}\big(0{\cdot}066667+0{\cdot}002506+4(0{\cdot}015873+0{\cdot}003922)+2(0{\cdot}006993)\big) = 0{\cdot}216450 \end{aligned}$$

and **8 strips** with $h = (20-4)/8 = 2$ give

$$\begin{aligned} &S(2) \\ &= \frac{2}{3}\Big(f(4)+f(20)+4\big(f(6)+f(10)+f(14)+f(18)\big)+2\big(f(8)+f(12)+f(16)\big)\Big) \\ &= \frac{2}{3}\big(0{\cdot}066667+0{\cdot}002506+4(0{\cdot}028571+0{\cdot}010101+0{\cdot}005128+0{\cdot}003096) \\ &\qquad\qquad +2(0{\cdot}015873+0{\cdot}006993+0{\cdot}003922)\big) \\ &= 0{\cdot}206890. \end{aligned}$$

Hence $I \approx 0{\cdot}2$ to 1 d.p. Adding these (and two further calculations) to our previous table yields Table 2, where the error $\epsilon_S(h) = S(h) - \displaystyle\int_4^{20} \frac{\mathrm{d}x}{x^2-1}$.

h	$T(h)$	$\epsilon_T(h)$	$S(h)$	$\epsilon_S(h)$
16	0·553383	0·348012		
8	0·332636	0·127265	0·259053	0·053682
4	0·245496	0·040125	0·216450	0·011079
2	0·216541	0·011170	0·206890	0·001519
1	0·208271	0·002900	0·205514	0·000143
0·5	0·206104	0·000733	0·205382	0·000010

Table 2: Approximation errors for the trapezium and Simpson's rules

As the strip widths are halved, the errors are (approximately) divided by 16, i.e., the error term is $O(h^4)$. This is how the error behaves for Simpson's rule:

$$\int_a^b f(x)\,\mathrm{d}x = S(h) + O(h^4).$$

It becomes accurate much more rapidly than is the case with the trapezium rule.

Example

Find $I = \displaystyle\int_2^4 \frac{1}{\sqrt{1+x^3}}\,\mathrm{d}x$ using Simpson's rule with 4 and 8 strips. Work to 5 d.p. and state your answer with the appropriate degree of accuracy.

x	$f(x)$	4 strips	8 strips
2·00	0·33333	F	F
2·25	0·28409	–	O
2·50	0·24526	O	E
2·75	0·21419	–	O
3·00	0·18898	E	E
3·25	0·16824	–	O
3·50	0·15097	O	E
3·75	0·13642	–	O
4·00	0·12403	L	L

Table 3: Values of $f(x) = (1+x^3)^{-1/2}$ to 5 s.f.

4 strips with $h = (4-2)/4 = 0{\cdot}5$ give

$$\begin{aligned} S(0{\cdot}5) &= \frac{0{\cdot}5}{3}\big(f(2) + f(4) + 4\big(f(2{\cdot}5) + f(3{\cdot}5)\big) + 2f(3)\big) \\ &= \frac{0{\cdot}5}{3}\big(0{\cdot}33333 + 0{\cdot}12403 + 4(0{\cdot}24526 + 0{\cdot}15097) + 2(0{\cdot}18898)\big) = 0{\cdot}40337 \end{aligned}$$

to 5 d.p., whereas **8 strips** with $h = (4-2)/8 = 0{\cdot}25$ give

$$\begin{aligned} S(0{\cdot}25) &= \frac{0{\cdot}25}{3}\big(f(2) + f(4) + 4\big(f(2{\cdot}25) + f(2{\cdot}75) + f(3{\cdot}25) + f(3{\cdot}75)\big) \\ &\qquad + 2\big(f(2{\cdot}5) + f(3) + f(3{\cdot}5)\big)\big) \\ &= \frac{0{\cdot}25}{3}\big(0{\cdot}33333 + 0{\cdot}12403 + 4(0{\cdot}28409 + 0{\cdot}21419 + 0{\cdot}16824 + 0{\cdot}13642) \\ &\qquad + 2(0{\cdot}24526 + 0{\cdot}18898 + 0{\cdot}15097)\big) \\ &= 0{\cdot}40330 \quad \text{to 5 d.p.} \end{aligned}$$

The two answers agree to 3 significant figures, so $I = 0{\cdot}403$ (3 s.f.).

3.3 The Newton-Raphson method

This is an iterative method of finding a root of the equation $f(x) = 0$. 'Iterative' means that we start with an initial approximation x_0, then use x_0 to find our next approximation x_1, which is hopefully more accurate. We then use x_1 to find x_2 and so on.

Consider the graph of $y = f(x)$ shown in Figure 3. The x value at the point A where the graph crosses the x-axis gives the solution to the equation $f(x) = 0$

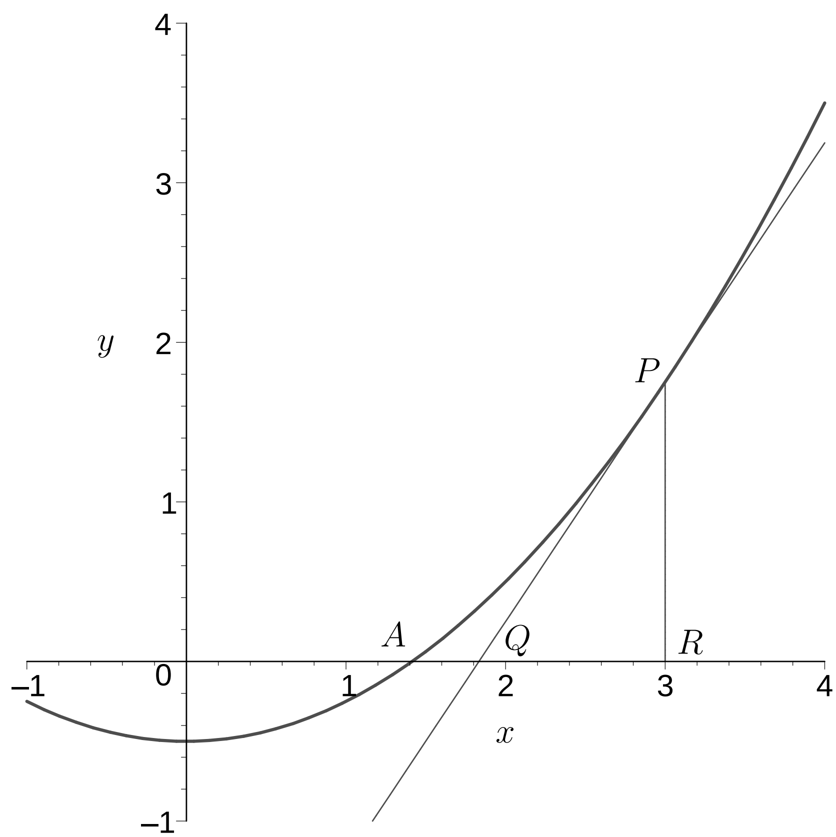

Figure 13: The Newton-Raphson method

If $P = \big(x_0, f(x_0)\big)$ is a point on the curve $y = f(x)$ near to A, then $x = x_0$ is an approximate value for the root of $f(x) = 0$. The error in the approximation is given by the distance AR, where $R = (x_0, 0)$.

Now if PQ is the tangent to the curve at P, crossing the x-axis at $Q = (x_1, 0)$, then $x = x_1$ is the next approximation to the root required.

The gradient of the tangent is $f'(x_0) = \dfrac{PR}{QR}$ and $f(x_0) = PR$, so

$$QR = \frac{PR}{f'(x_0)} = \frac{f(x_0)}{f'(x_0)} \qquad \text{and} \qquad x_1 = x_0 - QR = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};$$

this is the *Newton-Raphson method*.

As with the trapezium rule and Simpson's rule, we proceed with the iteration until two consecutive values agree to our desired level of accuracy.

Examples

1. Find, accurate to 4 decimal places, the root of $f(x) = 2x^3 - 7x + 2$ close to 0, starting with $x_0 = 0$.

   As $f'(x) = 6x^2 - 7$, recording the values to 5 decimal places gives the following.

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{2}{-7} &&= 0{\cdot}28571 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0{\cdot}28571 - \frac{0{\cdot}04665}{-6{\cdot}51020} &&= 0{\cdot}29288 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 0{\cdot}29288 - \frac{0{\cdot}00009}{-6{\cdot}48533} &&= 0{\cdot}29289 \end{aligned}$$

   The last two values agree to 4 decimal places., so the root required is $x = 0{\cdot}2929$ to 4 d.p. (If one more iteration is completed, x_3 and x_4 agree to 9 d.p.)

   Note that the value of $f'(x)$ is very large compared to the value of $f(x)$ for each iteration. Even for the first iteration, $f'(x)$ is more than 3 times the value of $f(x)$. This makes the 'correction term' $f(x_n)/f'(x_n)$ become small very quickly, giving rapid convergence to the root.

   If the value of $f'(x)$ is not large in comparison with $f(x)$, convergence to the root can be much slower or, in certain circumstances, the successive values can diverge. There is a technique for dealing with this problem which will be described in Section 4.3, after we have introduced Taylor series.

2. Find the root of $\sinh x = x + 1$ to 4 s.f.

The equation has been given in the form $\sinh x = x+1$. The Newton-Raphson method requires the form $f(x) = 0$, so rearranging we have

$$f(x) = \sinh x - x - 1 \qquad \text{and} \qquad f'(x) = \cosh x - 1.$$

If we choose the starting point $x_0 = 0$ then $f(x_0) = -1$ and $f'(x_0) = 0$, so the Newton-Raphson formula makes no sense. When this happens, we start again with a better choice of x_0. From the sketch, $x_0 = 2$ is reasonable.

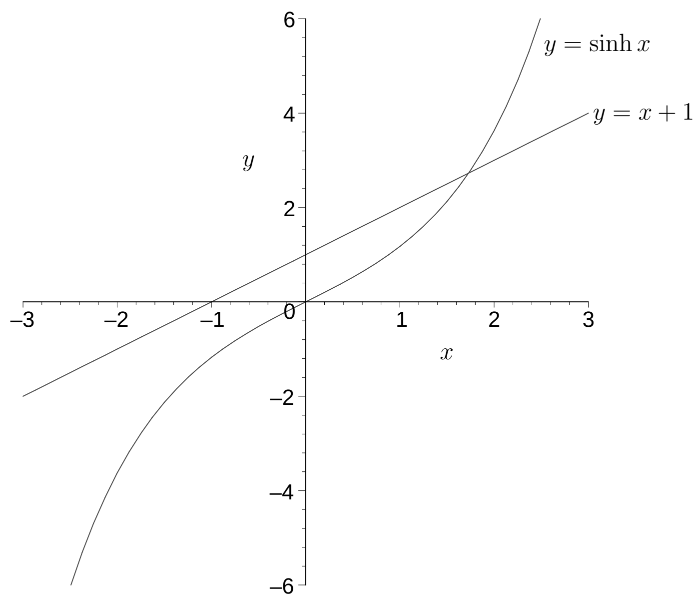

Figure 14: The graphs of $y = \sinh x$ and $y = x + 1$

With $x_0 = 2$ and working to 5 decimal places, we obtain the following.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{0{\cdot}62686}{2{\cdot}76220} = 1{\cdot}77306$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1{\cdot}77306 - \frac{0{\cdot}08646}{2{\cdot}02933} = 1{\cdot}73045$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1{\cdot}73045 - \frac{0{\cdot}00254}{1{\cdot}91020} = 1{\cdot}72913$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1{\cdot}72913 - \frac{0{\cdot}00002}{1{\cdot}90659} = 1{\cdot}72912$$

The last two values agree to at least 4 s.f., so the root is $x = 1{\cdot}729$ to 4 s.f.

4 Further topics

4.1 Taylor's theorem

If the function $f(x)$ is well behaved near a then, when h is small, the quantity $f(a+h)$ can be expressed as a power series, by writing

$$f(a+h) = A_0 + A_1 h + A_2 h^2 + \cdots = \sum_{n=0}^{\infty} A_n h^n,$$

where the coefficients A_0, A_1 et cetera are determined by the function $f(x)$.

Taking $h = 0$ on both sides shows that $A_0 = f(a)$.

To find the other coefficients we differentiate $f(a+h)$ with respect to h and set $h = 0$:

$$f'(a+h) = A_1 + 2A_2 h + 3A_3 h^2 + \cdots ,$$

so putting $h = 0$ gives $f'(a) = A_1$. Continuing to differentiate, we have

$$f''(a+h) = 2A_2 + (3 \times 2)A_3 h + (4 \times 3)A_4 h^2 + \cdots ,$$

so putting $h = 0$ gives $f''(a) = 2A_2$ and therefore $A_2 = \dfrac{f''(a)}{2}$.

In general, we have

$$A_n = \frac{f^{(n)}(a)}{n!},$$

where $f^{(n)}$ is the nth derivative of $f(x)$ and $n! = n(n-1)(n-2)\cdots(2)(1)$ is n factorial.

The series

$$f(a+x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}x^n$$

is called the *Taylor-series expansion* of $f(x)$ about $x = a$.

The corresponding series with $a = 0$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n,$$

and this is the *Maclaurin-series expansion* of $f(x)$.

These series expansions are usually valid wherever the series converges, but not always – we will not encounter any such difficulty in this course.

Examples

1. Find the Maclaurin series for $f(x) = e^x$.

   Since $f'(x) = e^x = f(x)$, it follows that $f^{(n)}(x) = e^x$ for $n = 1, 2, 3, \ldots$. Hence $f^{(n)}(0) = e^0 = 1$ for all n and the required series is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

2. Find the Maclaurin series for $f(x) = \sin x$.

   Here $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(iv)}(x) = \sin x$ et cetera. When $x = 0$, $\sin x = 0$ and $\cos x = 1$, so

$$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

   (The particularly alert reader will realise the previous two answers are based on circular reasoning, since we used the Maclaurin series of e^x and $\sin x$ to find these derivatives in the first place. Rest assured that it is possible to avoid this circularity, and there is nothing to worry about.)

   These two series are valid for any choice of x, but this is not true for all functions.

3. If $f(x) = \dfrac{1}{1-x}$ then, by the linear-composite rule,

$$f'(x) = \frac{1}{(1-x)^2}, \qquad f''(x) = \frac{2}{(1-x)^3}$$

   and, in general, $f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$. The Maclaurin series is therefore

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

   and we have recovered the familiar geometric series. This expansion is valid for $|x| < 1$ but not for $|x| \geqslant 1$.

   In the examples above, finding $f^{(n)}(0)$ did not pose much of a problem. This may not always be so straightforward.

4. If $f(x) = \dfrac{1}{1+x^2}$ then $f'(x) = -\dfrac{2x}{(1+x^2)^2}$.

   To find the higher derivatives in this case becomes rather complicated very quickly – try it and see. Fortunately, there is an easier way: replace x by $-x^2$ in the series for $1/(1-x)$, to get that
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots ,$$
   which is valid whenever $|-x^2| < 1$ or, equivalently, $|x| < 1$.

5. Find the Maclaurin series for $\cos\bigl(x/(1-x^2)\bigr)$, up to and including the term in x^4.

   Again, to calculate this series by finding derivatives would be a bit complicated. Instead, recall that
$$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24} + \cdots ,$$
   where the dots represent terms of degree strictly greater than 4, so letting $y = x/(1-x^2) = x + x^3 + \cdots$ gives
$$\begin{aligned}\cos\bigl(x/(1-x^2)\bigr) &= 1 - \frac{1}{2}(x + x^3 + \cdots)^2 + \frac{1}{24}(x + x^3 + \cdots)^4 + \cdots \\ &= 1 - \frac{1}{2}(x^2 + 2x^4 + \cdots) + \frac{1}{24}(x^4 + \cdots) + \cdots \\ &= 1 - \frac{1}{2}x^2 - \frac{23}{24}x^4 + \cdots .\end{aligned}$$

Estimating integrals

The Taylor-series expansion gives us another way of finding approximate values for definite integrals. If $f(x)$ has the Taylor series
$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots$$
about a then $f(x)$ is approximately equal to the polynomial
$$p(x) = A_0 + A_1(x-a) + \cdots + A_n(x-a)^n$$
for all x close to a. Hence, if c and d are close to a,
$$\begin{aligned}\int_c^d f(x)\,\mathrm{d}x &\approx \int_c^d \bigl(A_0 + A_1(x-a) + \cdots + A_n(x-a)^n\bigr)\,\mathrm{d}x \\ &= \left[A_0(x-a) + \frac{A_1(x-a)^2}{2} + \cdots + \frac{A_n(x-a)^{n+1}}{n+1}\right]_c^d .\end{aligned}$$
Taking more terms of the Taylor series will give greater accuracy.

Examples

1. Find a numerical approximation to $\displaystyle\int_0^{0\cdot 5} e^{-x^2/2}\,\mathrm{d}x$.

   Replacing x by $-x^2/2$ in the Maclaurin series for e^x, we see that

$$e^{-x^2/2} = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots\,.$$

   Hence

$$\begin{aligned}\int_0^{0\cdot 5} e^{-x^2/2}\,\mathrm{d}x &\approx \int_0^{0\cdot 5}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)\mathrm{d}x\\ &= \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right]_0^{0\cdot 5}\\ &= 0\cdot 479925 \quad (6 \text{ d.p.}).\end{aligned}$$

   (The true value is 0·479925219 to 9 decimal places.)

2. Find an approximate value for $\displaystyle\int_3^4 \frac{1}{1+x^3}\,\mathrm{d}x$.

   A little care must be taken here; the expansion of the integrand must be valid. Although

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots\,,$$

   this holds only when $|x| < 1$. Instead, we re-write the integrand: note that

$$\begin{aligned}\int_3^4 \frac{1}{1+x^3}\,\mathrm{d}x = \int_3^4 \frac{x^{-3}}{x^{-3}+1}\,\mathrm{d}x &= \int_3^4 x^{-3}\bigl(1 - x^{-3} + x^{-6} + \cdots\bigr)\,\mathrm{d}x\\ &\approx \int_3^4 \bigl(x^{-3} - x^{-6} + x^{-9}\bigr)\,\mathrm{d}x\\ &= \left[-\frac{x^{-2}}{2} + \frac{x^{-5}}{5} - \frac{x^{-8}}{8}\right]_3^4\\ &= 0\cdot 02369 \quad (5 \text{ d.p.}).\end{aligned}$$

   (The correct value is 0·023694490 to 9 decimal places.)

4.2 L'Hôpital's rule

Let $f(x)$ and $g(x)$ be continuous functions. If $g(a) \neq 0$ then, by the rules for limits,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

If $g(a) = 0$ but $f(a) \neq 0$ then

$$\left|\frac{f(x)}{g(x)}\right| \to \infty \text{ as } x \to a,$$

since $|f(x)| \to |f(a)|$ but $|1/g(x)| = 1/|g(x)|$ is getting larger and larger as $x \to a$. This leaves the case where $f(a) = g(a) = 0$. In this situation, different possibilities arise. For example, as $x \to 0$,

$$\frac{\sin x}{x} \to 1, \qquad \frac{\sin^2 x}{x} \to 0 \qquad \text{and} \qquad \left|\frac{\sin x}{x^2}\right| \to \infty.$$

In general, *L'Hôpital's rule* lets us find such a limit.

Put $x = a + h$, so that $x \to a$ as $h \to 0$. Using Taylor series,

$$\frac{f(x)}{g(x)} = \frac{f(a+h)}{g(a+h)} = \frac{f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \cdots}{g(a) + g'(a)h + \frac{g''(a)}{2!}h^2 + \cdots}.$$

If $f(a) = g(a) = 0$ then dividing the numerator and denominator by h shows that

$$\frac{f(x)}{g(x)} = \frac{f'(a) + \frac{f''(a)}{2!}h + \cdots}{g'(a) + \frac{g''(a)}{2!}h + \cdots} \to \frac{f'(a)}{g'(a)}$$

as $h \to 0$, as long as $g'(a) \neq 0$. Hence

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \qquad \text{if } f(a) = g(a) = 0 \text{ and } g'(a) \neq 0.$$

What if $g(a) = 0$ and $g'(a) = 0$? The working above shows that this situation is controlled by the second derivatives: the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)} \qquad \text{if } f(a) = g(a) = 0,\ f'(a) = g'(a) = 0 \text{ and } g''(a) \neq 0.$$

Similarly, if $f''(a) = g''(a) = 0$ but $g'''(a) \neq 0$ then the method can be repeated, and so on. Summarising, we have the following general rule.

L'Hôpital's rule

If $f(x)$ and $g(x)$ are functions such that

$$f(a) = g(a) = \cdots = f^{(n-1)}(a) = g^{(n-1)}(a) = 0$$

but $g^{(n)}(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \cdots = \lim_{x \to a} \frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

Examples

1. Find $\lim_{x \to 0} \frac{\sin x}{x}$.

   If $f(x) = \sin x$ and $g(x) = x$ then $f(0) = \sin 0 = 0$ and $g(0) = 0$. However, $f'(x) = \cos x$ and $g'(x) = 1$, so $f'(0) = \cos 1$ and $g'(0) = 1$. Hence L'Hôpital's rule gives that

   $$\lim_{x \to 0} \frac{\sin x}{x} = \frac{f'(0)}{g'(0)} = \frac{1}{1} = 1.$$

   (L'Hôpital's rule is not really needed here: note that

   $$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1,$$

   by the definition of $f'(0)$.)

2. Find $\lim_{x \to 0} \frac{3x - \sin x}{x}$.

   If $f(x) = 3x - \sin x$ and $g(x) = x$ then $f(0) = 0$ and $g(0) = 0$, but $f'(x) = 3 - \cos x$ and $g'(x) = 1$. Hence L'Hôpital's rule applies and

   $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{f'(0)}{g'(0)} = \frac{3 - 1}{1} = 2.$$

   (Again, L'Hôpital's rule isn't really needed here: by the rules for limits,

   $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3x}{x} - \lim_{x \to 0} \frac{\sin x}{x} = 3 - 1 = 2,$$

   as expected.)

3. Find $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1-\frac{1}{2}x}{x^2}$.

Let $f(x) = \sqrt{1+x}-1-\frac{1}{2}x$ and $g(x) = x^2$. Then $f(0) = g(0) = 0$, but

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} - \frac{1}{2} \quad \text{and} \quad g'(x) = 2x,$$

so $f'(0) = g'(0) = 0$ as well.

Differentiating again gives $f''(x) = -\dfrac{1}{4}(1+x)^{-3/2}$ and $g''(x) = 2$, so

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f''(0)}{g''(0)} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}.$$

(We could also find this limit by using the binomial theorem to write down the first few terms of the Maclaurin series for $(1+x)^{1/2}$.)

4. Find $\lim_{x\to\pi/2} \dfrac{\cos x}{\cos 3x}$.

Let $f(x) = \cos x$ and $g(x) = \cos 3x$; note that $f(0) = g(0) = 0$. However, $f'(x) = -\sin x$ and $g'(x) = -3\sin 3x$, so

$$\lim_{x\to\pi/2} \frac{\cos x}{\cos 3x} = \frac{f'(0)}{g'(0)} = \frac{-1}{+3} = -\frac{1}{3}.$$

(Here, L'Hôpital's rule really is the best way of finding the desired limit.)

4.3 The modified Newton-Raphson method

The following material is worth noting for future reference and is included here **for interest only**. It is another application of Taylor-series expansion.

Consider the equation $f(x) = 0$. If $x = x_0$ is an approximate root and $x = x_0 + h$ is an exact root then, using a Taylor series, we have

$$0 = f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots.$$

Considering the first two terms gives $0 \approx f(x_0) + hf'(x_0)$, so $h \approx -\dfrac{f(x_0)}{f'(x_0)}$. Hence

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

should be a more accurate approximation to $x_0 + h$ than x_0 is; this is the standard Newton-Raphson method.

If $f'(x_0)$, the slope of the tangent to the curve $y = f(x)$ at $x = x_0$, is small then the approximate value of h may be large and the second approximation x_1 may be further away from the exact root than x_0. In other words, if $f'(x_0)$ is small then the Newton-Raphson method as it stands may fail to converge. (See Figure 15.)

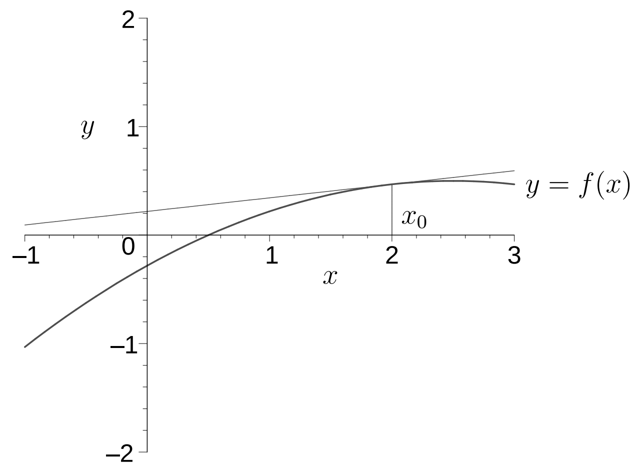

Figure 15: A function with a shallow gradient near a root

For a more accurate first step in this case, we consider the first three terms in the Taylor series, so that

$$0 \approx f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0),$$

which is a quadratic equation in h. However, as $f'(x_0)$ is very small (or we would not be doing this working) we may neglect the h term and

$$0 \approx 2f(x_0) + h^2 f''(x_0) \qquad \Longleftrightarrow \qquad h \approx \pm\sqrt{\frac{-2f(x_0)}{f''(x_0)}}.$$

Hence we should take

$$x_1 = x_0 \pm \sqrt{\frac{-2f(x_0)}{f''(x_0)}}$$

in this case, where either sign may be chosen. After this step we revert to the standard formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

A The derivation of Simpson's rule

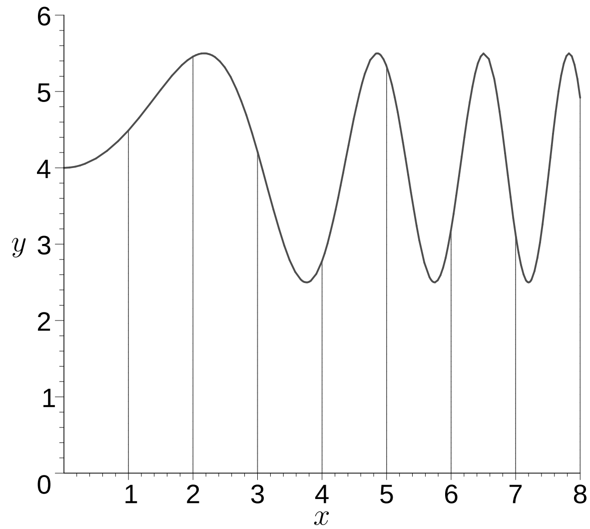

Figure 16: Estimation of the area under a graph

Consider the diagram above. If the region between the x-axis, the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is divided into an even number of strips, each of width h, then Simpson's rule can be used to estimate this area. (In Figure 16 above, the end point $a = 1$, $b = 8$ and the width $h = 1$.) The area of two consecutive strips is approximated by fitting a quadratic curve, i.e., a curve of the form

$$y = q(x) = rx^2 + sx + t,$$

through the three points on the original curve at the end points of those strips: the points with ordinates $y_- = f(x_i - h)$, $y_0 = f(x_i)$ and $y_+ = f(x_i + h)$. (In comparison, the trapezium rule fits a straight line between the end points of each strip.) Note that

$$y_- = r(x_i - h)^2 + s(x_i - h) + t, \qquad \text{(i)}$$

$$y_0 = rx_i^2 + sx_i + t \qquad \text{(ii)}$$

$$\text{and} \qquad y_+ = r(x_i + h)^2 + s(x_i + h) + t, \qquad \text{(iii)}$$

so taking (i) $-$ 2(ii) + (iii) shows that

$$y_- - 2y_0 + y_+ = r(x_i - h)^2 + s(x_i - h) + t \\ - 2rx_i^2 - 2sx_i - 2t + r(x_i + h)^2 + s(x_i + h) + t = 2rh^2.$$

If A_i is the area under the fitted quadratic between $x = x_i - h$ and $x = x_i + h$ then

$$\begin{aligned}
A_i &= \int_{x_i-h}^{x_i+h} q(x)\,\mathrm{d}x \\
&= \left[\frac{rx^3}{3} + \frac{sx^2}{2} + tx\right]_{x_i-h}^{x_i+h} \\
&= \frac{r(x_i+h)^3}{3} + \frac{s(x_i+h)^2}{2} + t(x_i+h) - \frac{r(x_i-h)^3}{3} - \frac{s(x_i-h)^2}{2} - t(x_i-h) \\
&= \frac{6rx_i^2h + 2rh^3}{3} + \frac{4sx_ih}{2} + 2th \\
&= \frac{2rh^3}{3} + 2(rx_i^2 + sx_i + t)h \\
&= \frac{h}{3}\big(y_- - 2y_0 + y_+\big) + 2y_0h \\
&= \frac{h}{3}\big(y_- + 4y_0 + y_+\big).
\end{aligned}$$

The area of the first pair of strips is, therefore,

$$A_1 = \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big);$$

the area under the quadratic curve fitted to the second pair of strips is A_3, the estimated area of the third pair of strips is A_5, and so on, where

$$A_3 = \frac{h}{3}\big(f(x_2) + 4f(x_3) + f(x_4)\big), \qquad A_5 = \frac{h}{3}\big(f(x_4) + 4f(x_5) + f(x_6)\big)$$

et cetera. Hence the estimated total area of the first three pairs of strips is

$$A_1 + A_3 + A_5 = \frac{h}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\big).$$

For a general number of pairs of strips, we have

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\big(F + L + 4O + 2E\big),$$

where F and L are the values of the first and last ordinates $f(x_0)$ and $f(x_n)$, O is the sum of all the odd-placed ordinates ($f(x_1)$, $f(x_3)$ et cetera) and E is the sum of all the even-placed ordinates ($f(x_2)$, $f(x_4)$ et cetera).

This is *Simpson's rule.*