Geometric Constraint System Stories:
Evariste Galois, CAD and Crystallography. [1]

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Thank you for coming to a talk by a pure mathematician. The dean of our faculty has occasionally remarked that the pure mathematicians at Lancaster are not very visible. Hiding under rocks. Well here I am, emerged from the subterranean, a little pallid, but very content to tell you about current research in the world of constraint systems. This is a mathematical world with many significant connections with the real one.

I would like to start by showing you a video clip of computer aided design in action. It shows the CAD imagery one might see during component design and adjustment. I'll come back to some mathematics underlying this later.

Pure mathematics divides into two great disciplines:

ALGEBRA and ANALYSIS

ALGEBRA concerns discrete structures, such as symmetry groups, combinatorics and counting, finite matrix algebra, algebraic surfaces, modular arithmetic, cryptography, and so on. ANALYSIS, on the other hand, embraces infinities, the infinitesimally small, the infinitely large, the probabilistic, and the time evolving solutions of equations.

However there are no absolute divides. Indeed, some of the most extreme forms of excitement amongst mathematicians - if you can imagine such a thing - occur when a breakthrough resonates across the divide, involving a mix of mathematical structures and methods.

Let me briefly mention one famous example: Knot theory is a branch of algebra. Imagine two wildly complicated knots. To determine if they are different - and to develop a general theory - we need mathematical fingerprints. These are called invariants, and they usually come from algebra, from combinatorics and counting. In 1990 the mathematician Vaughan Jones at the University of California, Berkeley, won the Fields medal (Mathematics' Nobel prize) for finding new knot invariants. His route was via Analysis. In fact, to be more specific, it was via the higher level subject of Operator Algebras.

[1] These notes lay behind the presentation for the Faculty Christmas Conference at Lancaster University, Tuesday 15th December 2009. Not all that is written here may have been said, and certainly vice versa. My brief: a 25 minute talk about research to a scientific audience including comments on "impact".

The subject of Geometric Constraint Systems is a hybrid one, both mathematically and in the scope of its practitioners. These include engineers, chemists, CAD software developers, material scientists, and mathematicians. I am going to outline three related mathematical stories - sketches really - surrounding some theorems and results in this area.

Story 1: The Rigid Structure Problem

The mathematical structure that appears throughout my talk is that of a **bar-joint framework.** These are also called linkages or linkworks, especially when they are flexible with one degree of freedom. Here is a famous example in two dimensions with one degree of freedom, the Peaucellier mechanism.

In 1784 James Watt designed a bar-joint linkage which transformed circular motion into approximate linear motion. This had rather a profound impact on steam engine transmission. The mechanism was approximate and was superseded by an exact linear motion mechanism eighty years later. This is the Peaucellier mechanism which you now behold.

The subject of bar-joint frameworks was all the rage in the 19th century. James Clerk Maxwell took an interest and observed the following simplifying principle: if a 3D bar-joint framework is rigid but only "just rigid" (meaning not over-constrained) then

$$3V - E = 6.$$

where V is the number of joints, E the number of edges. (The number 6 is the number of spatial degrees of freedom of any rigid object.) The equation is a combinatorial/counting fact about the underlying graph (or shape or topology) of the framework. However, it was only in 1970 that an engineer, G. Laman[2], obtained a necessary *and sufficient* condition in 2D. This was the true beginning of the rigidity theory side of Geometric Constraint Frameworks.

Laman's Theorem: A *generic* framework in 2D is "just rigid" if and only if it satisfies the Maxwell count $2V - E = 3$ and the counting inequalities $2V' - E' \geq 3$ for all sub-frameworks.

Remarkably, the 3D version of this problem is open:

Open Problem: Find counting/combinatorial/graph conditions for "just rigidity" in 3D.

[2] G. Laman, On graphs and the rigidity of plane skeletal structures, J. Engineering Mathematics, 4 (1970), 331-340.

This problem area is significant because many general 3D equation systems are reducible to a 3D bar-joint system. Computer Aided Design, for example, with its structures of lines/planes/spheres/cylinders/pipes/ yields just such equation systems.

Current research: Generic frameworks, frameworks with symmetries, line-point frameworks, body-pin frameworks, polytope frameworks.

Current techniques: matrix analysis, matroid analysis, multi-linear algebra, combinatorics, manifolds, geometry, rigidity matrices.

Story 2: Galois' Obstacle

I began this lecture with a glimpse at some sophisticated CAD software, courtesy of my collaborator John Owen. At the core of the software operation, groups of equations are being solved to determine exactly the new shape as the designer changes various measurements and dependencies. The problem of recomputing shapes and structures can usually be reduced to problems about bar-joint frameworks. These can be in 2D or 3D. Schematically we have

CAD designs $\Rightarrow$ geometrical structures $\Rightarrow$ framework equation systems

Let's formulate the essential mathematical problem in 2D: Given a graph (= my rough design shape) together with edge lengths (= my chosen design dimensions) **solve** for the vertex positions (determine the exact shape).

Solving numerically won't do. For speed and compatibility one needs exact algebraic methods. It was precisely this that was developed commercially in the 1980's by the Cambridge company D-cubed (formed by John Owen, and now part of Siemens PLC). However, there are limits to what one can do algebraically and this is where the famous and poignant figure of the mathematician Evariste Galois (1811-1832) enters in.

Galois' obstacle: There is no formula (no matter how complicated) for the solution x of certain polynomial equations of high degree (such as $x^5 - 10x + 2 = 0$).

More precisely one might say that there is no formulae based on arithmetic and n^{th} roots, or that there is "no solution by radical extensions." Making use of this theory and more we have found when Galois' obstacle is unavoidable in a 2D CAD graph:

Theorem (Owen-Power, Trans. Amer. Math. Soc., 2007)

Let G be a Laman graph which is 3-*connected* and planar together with given generic edge distances. Then the vertex solution does not lie in a radical extension of the input distance data.

The proof is deep, being hybrid and long.[3]

[3]Here "planar" means in the usual sense for graphs (planar embeddable). It seems likely that this

Story 3: Infinite bar-joint frameworks

Mathematicians are usually not afraid of the infinite. I find it interesting therefore that, to my knowledge, there has been no extended mathematical development of infinite bar-joint frameworks. This is probably because the finite case is already hard enough. Perhaps there is a further reason: Analysing infinite framework rigidity and flexibility requires continuous methods from Analysis as well as the discrete methods from Algebra.

One can certainly start a theory of infinite Peaucellier type linkages and frameworks. This is what John Owen and I have done in recent work. Here is a whimsical esoteric structure that appeared in a recent paper of an infinitely bifurcating linkage. This particular infinite framework goes down forever but with diminishing scale and so occupies a finite region of space. It has one degree of flexibility and uncountably many downward paths.

One can easily imagine similar infinite strange structures in 3D that extend outward without bound beyond the horizon.

But this type of mathematical enquiry is not simply blue skies contemplation by subterranean mathematicians. Infinite bar-joint frameworks already appear in models in the physical and material sciences. I am thinking mainly of crystals.

Here is part of the infinite 2D crystal framework known as the kagome lattice. And here is an indication of its 3D variant, the kagome net. It resmebles the following picture of the crystal lattice atomic structure of silicon dioxide, quartz. You can see that it is a network of vertex connected tetrahedra - tough little pyramids centred on the silicon atoms. The mathematical model extends without bound, to infinity. Of course it is periodic and therefore essentially a finite structure. But when it flexes and deforms that periodicity and finiteness can be broken, so a broader theory than classical crystallography seems advisable.

On the other hand the low energy excitation modes of crystals, visible in diffraction experiments in the lab, really do correspond to periodic infinitesimal flexing as a bar-joint framework. And for decades material scientists have been studying and mapping these so called Rigid Unit Modes.

It is my view that a more sophisticated treatment of the mathematical model can be beneficial, both to physics and to mathematics. This is the kind of research that is being developed at Lancaster.

While finite frameworks require rigidity matrices, infinite frameworks require $r$igidity operators on infinite dimensional spaces, such as Hilbert spaces and Banach spaces.

The fascination for me of this area is, firstly, that a great multiplicity of concepts

condition can be removed

seem necessary to understand specific infinite structures. Secondly there seem to be interesting connections with established mathematics, such as operator theory.

Pure Mathematics and Impact

Let me make a few comments about the impact of pure mathematics in general and a few comments about Lancaster in this regard.

Pure mathematics is two things, at least. Its developments provide tools and a methodology for science and innovation. On the other hand it is a subject in its own right. Progress in the subject per se benefits science which in turn benefits society. By a mathematical tool I mean rather sophisticated things. One example would be the *Theory of Wavelets*. This is a mathematical topic that was not here thirty years ago. Now wavelets often provide a substitute for Fourier series and they appear in many scientific discourses and applications (fingerprint data compression for example).

For another tool I could mention *Mathematical Quasicrystals*. This is another subject that was not here thirty years ago. This theory seems to play a significant role in the classification of material quasicrystals. We might also recall that the representation theory of symmetry groups, another sophisticated tool, has long been wedded to molecular crystallography.

So certainly pure mathematics permeates the models in science and is essential for new paradigms. It benefits society profoundly, but in an indirect way. I would suggest that it is evident that the REF concept of impact is quite inappropriate to the discipline of pure mathematics which operates, so to speak, on a longer wavelength and with wider participation.

Now how about Lancaster and Impact ?

Well it is true to say that my student Sean Olphert and I have made what we think is a highly original contribution to the theory of multidimensional wavelets. But rather than mentioning this and more distant things I would like to look forward. It is the unknown which is really interesting and which is the driver of research:

There will be a conference here in July 2010: *Rigidity of Frameworks and Applications.* My expectation is that there will be key researchers from many universities: Lancaster, Cambridge, London, Sheffield, York (Toronto), Cornell, Florida, Arizona, Berlin, Budapest,....

The event should embrace a hybrid group - diverse mathematicians, chemists, physicists (probably), and academic engineers will exchange ideas and innovations. We expect presentations from the worlds leading rigidity theorists. It could get a bit technical but if anyone here is interested then I warmly welcome you.

Thankyou.