SUPPORT VARIETIES FOR WEYL MODULES OVER BAD PRIMES

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1. Introduction

1.1. Support varieties were introduced in the pioneering work of Alperin [A] and Carlson [Ca1, Ca2] nearly 25 years ago as a method to study complexes and resolutions of modules over group algebras. Since that time these ideas have been extended to encompass restricted Lie algebras by Friedlander and Parshall [FP], finite-dimensional sub Hopf algebras of the Steenrod algebra by Nakano and Palmieri [NPal], infinitesimal group schemes by Suslin, Friedlander and Bendel [SFB1, SFB2], and arbitrary finite-dimensional cocommutative Hopf algebras by Friedlander and Pevtsova [FPe]. Further recent attempts to generalize the theory have been made to finite-dimensional algebras by Solberg and Snashall [SS] via Hochschild cohomology and to Hecke algebras by Erdmann and Holloway [EH].

For a finite-dimensional cocommutative Hopf algebra A over an algebraically closed field k the cohomology ring $\mathrm{H}^\bullet(A,k)$ is a graded commutative finitely generated algebra [FS]. For any finitely generated A-module M, one can assign a conical subvariety $V_A(M)$ inside the spectrum of the cohomology ring. These support varieties provide a method to introduce the geometry of the spectrum of $\mathrm{H}^\bullet(A,k)$ into the representation theory of A.

The geometric implications become evident when one considers a reductive algebraic group G over an algebraically closed field k of characteristic $p>0$ and the rth Frobenius kernels G_r, $r \geq 1$. It is well-known that representations for G_1 are equivalent to modules for the restricted enveloping algebra $A := u(\mathfrak{g})$ where $\mathfrak{g} = \mathrm{Lie}\ G$. In this situation the spectrum of the cohomology ring is homeomorphic to the restricted nullcone [SFB1, SFB2]

$$\mathcal{N}_1(\mathfrak{g}) = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}.$$

Moreover, $\mathcal{N}_1(\mathfrak{g})$ is a G-stable conical subvariety inside the cone of nilpotent elements (nullcone) $\mathcal{N}(\mathfrak{g})$. The nullcone $\mathcal{N}(\mathfrak{g})$ has been well studied (see [Car, CM, Hum2]) because of its beautiful geometric properties with deep connections to Lie and representation theory.

1.2. Although support varieties are easy to define once the finite generation of cohomology is established, they are often very difficult to compute. In 1987, Jantzen [Jan2] stated a conjecture for reductive groups that the support varieties of the Weyl modules $V(\lambda)$ over G_1 are closures of certain Richardson orbits (depending on λ) when the characteristic of the underlying field is good. Nakano, Parshall and Vella [NPV, Thm. 6.2.1] proved this conjecture. The verification of this conjecture provides a bridge linking the representation theory, cohomology theory and conjugacy class theory of $\mathfrak{g}$. As an immediate corollary to the "Jantzen conjecture" on support varieties, it is shown that the restricted nullcone is an

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irreducible variety when k is of good characteristic. In [CLNP], Carlson, Lin, Nakano and Parshall, using techniques in [NPV], calculated $\mathcal{N}_1(\mathfrak{g})$ for good primes. This computation was replicated by the authors in [UGA1] using more elementary methods. The authors in [UGA2] determined the restricted nullcone over fields of bad characteristic. It was shown that $\mathcal{N}_1(\mathfrak{g})$ is always irreducible but need not be the closure of a Richardson orbit when p is bad. Calculating the support varieties of Weyl modules is a fundamental result in the theory. The calculation for good primes has advanced our understanding for both restricted and non-restricted representations of $\mathfrak{g}$ (see [GP, BN, CLN, CLNP, NT]).

For the sake of consistency with [NPV], we will compute the support varieties of the induced modules $H^0(\lambda) := \text{ind}_B^G \lambda$. Since $V(\lambda)$ is the contragredient dual of $H^0(-w_0\lambda)$ it follows that the support variety of $V(\lambda)$ is the same as the support variety of $H^0(-w_0\lambda)$. In this paper we will compute the support varieties of induced/Weyl modules over fields of bad characteristic. Our results in conjunction with [NPV, Thm. 6.2.1] give a complete description of the support varieties of induced/Weyl modules over algebraically closed fields of arbitrary positive characteristic.

1.3. Notation. The notation and conventions of this paper will follow those given in [Jan3]. Let k be an algebraically closed field of characteristic $p > 0$, and G a simple algebraic group defined over k with T a maximal torus of G. The root system associated to the pair (G, T) is denoted by Φ and identified with a subset of the set of weights $X(T)$. Let Φ^+ be a set of positive roots and Φ^- be the corresponding set of negative roots. The set of simple roots determined by Φ^+ is $\Delta = \{\alpha_1, \dots, \alpha_l\}$. We will use throughout this paper the ordering of simple roots given in [Hum1] following Bourbaki. Let B be the Borel subgroup relative to (G, T) given by the set of negative roots and let U be the unipotent radical of B. More generally, if $J \subseteq \Delta$, let P_J be the parabolic subgroup relative to $-J$ and let U_J be the unipotent radical of P_J. Let Φ_J be the root subsystem in Φ generated by the simple roots in J. Set $\mathfrak{g} = \text{Lie } G$, $\mathfrak{b} = \text{Lie } B$, $\mathfrak{u} = \text{Lie } U$, $\mathfrak{p}_J = \text{Lie } P_J$, and $\mathfrak{u}_J = \text{Lie } U_J$.

Let $\mathbb{E}$ be the Euclidean space associated with Φ, and the inner product on $\mathbb{E}$ will be denoted by $\langle\ ,\ \rangle$. Let $\check{\alpha}$ be the coroot corresponding to $\alpha \in \Phi$. Set α_0 to be the highest short root. Moreover, let ρ be the half sum of positive roots. The Coxeter number associated to Φ is $h = \langle \rho, \check{\alpha}_0 \rangle + 1$.

Let $X(T)$ be the integral weight lattice spanned by the fundamental weights $\{\omega_1, \dots, \omega_l\}$. The set $X(T)$ has a partial ordering defined as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$. The set of dominant integral weights is denoted by $X(T)_+$ and the set of p^r-restricted weights is $X_r(T)$. Set $H^0(\lambda) = \text{ind}_B^G \lambda$ where λ is the one-dimensional B-module obtained from the character $\lambda \in X(T)_+$ by letting U act trivially. The Weyl group corresponding to Φ is W and acts on $X(T)$ via the dot action ($w \cdot \lambda = w(\lambda + \rho) - \rho$ where $w \in W$, $\lambda \in X(T)$).

In this paper we will consider the case when p is a bad prime for Φ. A list of bad primes is provided below.

- Φ of type A_l, no primes;
- Φ of type B_l, C_l, D_l, $p = 2$;
- Φ of type E_6, E_7, F_4, G_2, $p = 2,\ 3$;
- Φ of type E_8, $p = 2,\ 3,\ 5$.

If the prime p does not appear on this list then p is a *good prime* relative to Φ.

Let $\mathcal{N}(\mathfrak{g})$ be the irreducible variety (of dimension $|\Phi|$) of nilpotent elements of $\mathfrak{g}$, which is often called the nullcone. The group G acts on $\mathcal{N}(\mathfrak{g})$ via the adjoint representation, and $\mathcal{N}(\mathfrak{g})$ has finitely many G-orbits [Lu]. For good primes the nilpotent orbits are classified in exactly the same way as over the complex numbers. On the other hand for bad primes the nilpotent orbits are determined in [He, Sp1, Sp2, Sp3, HS]. The conventions in [Law1] will be employed for the exceptional groups. If X is the Bala-Carter label for an orbit in characteristic zero which "splits" in characteristic p, then $X^{(p)}$ will denote the new orbit that arises.

Now view G as an algebraic group scheme defined over $\mathbb{F}_p$ and let $F : G \to G$ be the Frobenius morphism. Set $G_r = \text{Ker } F^r$ where F^r is the Frobenius map iterated with itself r times. Let $F|_B : B \to B$ be the restriction of F to B, let $B_r = \text{Ker } (F|_B)^r$ and let B_rT be the inverse image of T under $(F|_B)^r$. Set

$$R = \begin{cases} H^{2\bullet}(G_r, k) & \text{if char } k \neq 2 \\ H^{\bullet}(G_r, k) & \text{if char } k = 2. \end{cases}$$

According to [FS], R is a finitely generated commutative k-algebra. Furthermore, if M is a finite-dimensional module for G_r then $\text{Ext}^{\bullet}_{G_r}(M, M)$ is a finitely generated module over R. Let $J_r(M)$ denote the set of elements in R which annihilate $\text{Ext}^{\bullet}_{G_r}(M, M)$. Define $V_{G_r}(M) = \text{Maxspec}(R/J_r(M))$. The affine homogeneous variety $V_{G_r}(M)$ is called the *support variety* of M. Similarly, if M is a B_r-module one can define $V_{B_r}(M)$. We will mainly be interested in the case when $r = 1$, but will state results for arbitrary r when appropriate.

1.4. Motivation. For $\lambda \in X(T)$, define

$$\Phi_\lambda = \{\alpha \in \Phi \mid \langle \lambda + \rho, \check{\alpha} \rangle \in p\mathbb{Z}\}.$$

Then Φ_λ is either empty or a root subsystem of Φ, and when p is a good prime, there exist $w \in W$ and $I \subseteq \Delta$ such that $w(\Phi_\lambda) = \Phi_I$. The following result was proved by Nakano, Parshall and Vella for good primes [NPV, Thm. 6.2.1, Cor. 6.2.2].

Theorem. *Let G be a reductive algebraic group and assume that p is good. Let $\lambda \in X(T)_+$. Choose $w \in W$ such that $w(\Phi_\lambda) = \Phi_I$ for some $I \subseteq \Delta$. Then*

(a) $V_{G_1}(H^0(\lambda)) = G \cdot \mathfrak{u}_I$.

(b) $\dim V_{G_1}(H^0(\lambda)) = |\Phi| - |\Phi_\lambda|$.

The main issue relevent for bad primes is the fact that Φ_λ need not be conjugate to a root subsystem Φ_I. This makes formulating a precise conjecture on the support varieties of induced modules in this case somewhat intractable. Nevertheless, one might conjecture that

(1.4.1) $$\dim V_{G_1}(H^0(\lambda)) = |\Phi| - |\Phi_\lambda|.$$

for all primes. Indeed this is exactly what we will prove. The first step will be to prove the lower bound

(1.4.2) $$\dim V_{G_1}(H^0(\lambda)) \geq |\Phi| - |\Phi_\lambda|$$

for all primes. The next step is to prove a general version of the upper bound [NPV, Cor. 4.5.1]. These results along with our information about the restricted nullcone for bad primes will allow us to compute the support varieties of the induced modules while simultaneously verifying the dimension equality (1.4.1).

Note that our definition of Φ_λ differs from the stabilizer given in [NPV, §3.4]. For good primes, these two definitions agree, but for bad primes this will lead us to adapt several graded dimension results in [NPV, §3] to our setting.

2. General results

2.1. Restricted Nullcones. In [UGA2] the authors computed the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ over fields of bad characteristic. This result will be used throughout this paper because the support variety of any G_1-module is contained in $\mathcal{N}_1(\mathfrak{g})$.

Theorem (A). *Let G be a simple classical connected algebraic group over k with $p = 2$ a bad prime.*

(i) *If Φ is of type B_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2^l_2, 1_1)}$.*
(ii) *If Φ is of type C_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2^l_1)}$.*
(iii) *If Φ is of type D_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2^l_2)}$.*

Theorem (B). *Let G be an exceptional algebraic group with p a bad prime.*

(i) *If Φ is of type E_6 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = 2A_2 + A_1$ ($p = 3$), $3A_1$ ($p = 2$).*
(ii) *If Φ is of type E_7 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = 2A_2 + A_1$ ($p = 3$), $4A_1$ ($p = 2$).*
(iii) *If Φ is of type E_8 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = A_4 + A_3$ ($p = 5$), $2A_2 + 2A_1$ ($p = 3$), $4A_1$ ($p = 2$).*
(iv) *If Φ is of type F_4 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = A_1 + \widetilde{A_2}$ ($p = 3$), $A_1 + \widetilde{A_1}$ ($p = 2$).*
(v) *If Φ is of type G_2 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = G_2(a_1)$ ($p = 3$), $\widetilde{A_1}$ ($p = 2$).*

The inclusion relations among orbit closures in $\mathcal{N}_1(\mathfrak{g})$ are provided in Section 6.

2.2. Poles and rates of growth. Our first objective is to prove the inequality in (1.4.2). We will adapt the methods used in [NPV, §3] in order to accomplish this. Several of the arguments are also provided here for the sake of completeness and to assist the reader. We should also mention that our argument was inspired by Ostrik [Ost] who proved the inequality in the quantum setting when $p > h$.

Recall that if $\{a_n\}_{n\geq 0}$ is a sequence of complex numbers then the *rate of growth* $r(a_n)$ of this sequence is the smallest non-negative integer d such that there exists a positive number C with the property that

$$|a_n| \leq C \cdot n^{d-1}$$

for all $n > 0$. By convention, if no such d exists, then $r(a_n) = \infty$.

If there is a polynomial $f(t)$ of degree $d-1$ such that $a_n = f(n)$ for sufficiently large n then $r(a_n) = d$. Furthermore, if $S_n = |a_0| + \cdots + |a_n|$, then $r(S_n) = d+1$. We state a useful proposition from [NPV] which relates the rate of growth of $\{a_n\}_{n\geq 0}$ with the poles of the corresponding Poincaré series.

Proposition. *Let $p(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}[[t]]$.*

(a) *If $p(t) = \frac{f(t)}{(1-t)^d}$ for some positive integer d and $f(t) \in \mathbb{C}[t]$ with $f(1) \neq 0$, then a_n is a polynomial in n of degree $d-1$. Hence, $r(a_n) = d$.*

(b) *Assume the poles of $p(t)$ are roots of unity. If $e^{i\theta}$ is a pole of order γ, then*

$$\gamma \leq r(a_n).$$

(c) *Assume $p(t) = \frac{f(t)}{(1-t^r)^b}$ for positive integers b, r and $f(t) \in \mathbb{C}[t]$, with $f(1) \neq 0$. Fix i, $0 \leq i < r$. For j sufficiently large, a_{i+rj} is a polynomial in j (say of degree $d_i - 1$).*

(d) *In (c), let $d = \max d_i$. Then $d = r(a_n)$. If $S_n = |a_0| + \cdots + |a_n|$, then $r(S_n) = d + 1$.*

2.3. Fix a weight Λ and suppose that $M \in \operatorname{mod}(B_rT)$ (the set of finite-dimensional B_rT-modules) and all weights of M satisfy $\mu \leq \Lambda$. That is, all weights of M lie in the cone defined by Λ. If $\mu \leq \Lambda$ then $0 \leq \Lambda - \mu = \sum_{\alpha\in\Delta} n_\alpha \alpha$ where $n_\alpha \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \Delta$. Set $\operatorname{ht}(\Lambda - \mu) = \sum_{\alpha\in\Delta} n_\alpha$. We define the graded dimension of M as

$$\dim_t M = \sum_{\mu\leq\Lambda} (\dim M_\mu) t^{\operatorname{ht}(\Lambda-\mu)} \tag{2.3.1}$$

where M_μ is the μ weight space with respect to T. This definition coincides with the principal grading given in [K, §10.8]. According to [K, Prop. 10.10],

$$\dim_t H^0(\Lambda) = \prod_{\alpha\in\Phi^+} \frac{1 - t^{\langle \Lambda+\rho, \check{\alpha}\rangle}}{1 - t^{\langle \rho, \check{\alpha}\rangle}}. \tag{2.3.2}$$

Set

$$h_r(t) = \prod_{\alpha\in\Phi^+} \frac{1 - t^{p^r\langle \rho, \check{\alpha}\rangle}}{1 - t^{\langle \rho, \check{\alpha}\rangle}}. \tag{2.3.3}$$

Let $P_r(\mu)$ be the projective cover of the simple B_rT-module $L_r(\mu)$ of highest weight $\mu \in X(T)$. Let $M \in \operatorname{mod}(B_rT)$ and let $P_\bullet$ be a minimal projective resolution of M in $\operatorname{mod}(B_rT)$. Upon restriction to B_r, this still provides a minimal projective resolution of M as a B_r-module. Since $P_\bullet$ is a minimal resolution we have

$$P_n \cong \bigoplus_{\mu\in X_r(T)} \operatorname{Ext}^n_{B_r}(M, \mu) \otimes P_r(\mu) \tag{2.3.4}$$

as T-modules. Note that $\dim_t P_r(\Lambda) = h_r(t)$ and

$$\dim_t P_r(\mu) = t^{\operatorname{ht}(\Lambda-\mu)} h_r(t) \tag{2.3.5}$$

for $\mu \leq \Lambda$, because $P_r(\mu) \cong \operatorname{coind}_T^{B_rT} \mu$ (cf. [NPV, (3.1.6)]).

Let $P_r(\sigma + p^r\nu)$ be a B_rT-direct summand of P_n and set $\tau_n = \sigma + p^r\nu$ (with $\sigma \in X_r(T)$ and $\nu \in X(T)$). Since $P_\bullet$ is a minimal resolution it follows that τ_n is a weight in the radical of P_{n-1}. Consequently, $\tau_{n-1} := \tau_n + \beta$ appears in the head of P_{n-1}, where β is some non-trivial sum of positive roots. Therefore, $\Lambda - \tau_n = \Lambda - \tau_{n-1} + \beta$. Applying heights we see that $\operatorname{ht}(\Lambda - \tau_n) \geq \operatorname{ht}(\Lambda - \tau_{n-1}) + 1$ and iterating this process n times yields

$$\operatorname{ht}(\Lambda - \tau_n) \geq \operatorname{ht}(\Lambda - \tau) + n \tag{2.3.6}$$

where τ is a weight in the head of M.

Now assuming that all weights μ of M satisfy $\mu \leq \Lambda$, then the smallest possible height $\operatorname{ht}(\Lambda - \tau)$, where τ is a weight of M, is zero. Hence, in this case $\operatorname{ht}(\Lambda - \tau_n) \geq n$.

2.4. For a positive integer d let $\Psi_d(t) \in \mathbb{Z}[t]$ be the dth cyclotomic polynomial in $\mathbb{Q}[t]$. The polynomials $\Psi_d(t)$ are irreducible over $\mathbb{Q}$. The following theorem extends [NPV, Thm. 3.3.1] which show that $\dim V_{B_r}(M)$ is related to the order of the poles of the rational function $(\dim_t M)/h_r(t)$.

Theorem. *Let M be a finite-dimensional B_rT module with all weights of M being less than or equal to some $\Lambda \in X(T)_+$. Let*

$$q(t) = \frac{\dim_t M}{h_r(t)} = \frac{f(t)}{(1-t^{p^r})^{\gamma}\, g(t)}$$

where $f(t), g(t) \in \mathbb{Q}[t]$ and $\Psi_{p^r}(t) \nmid f(t)g(t)$. Then $\gamma \leq \dim V_{B_r}(M)$.

Proof. Under our assumptions one can assume $\gamma \geq 0$ and γ is the order of the pole of any primitive p^rth root of unity in $q(t)$. Let $P_\bullet$ be a minimal projective resolution of M in the category of B_rT modules. Now express

$$\frac{\dim_t P_n}{h_r(t)} = \sum_m b(m,n)t^m,$$

where each $b(m,n)$ is a non-negative integer.

Since $\dim V_{B_r}(M)$ is the rate of growth of $\{\operatorname{Ext}^n_{B_r}(M,N)\}$ where $N = \bigoplus_{\mu \in X_r(T)} \mu$ is the direct sum of simple B_r-modules (cf. [Ben]), it follows from (2.3.4) and (2.3.5) that for some $C > 0$,

$$\sum_m b(m,n) = \dim \operatorname{Ext}^n_{B_r}(M,N) \leq Cn^{\dim V_{B_r}(M)-1}. \tag{2.4.1}$$

Suppose that $b(m,n) \neq 0$. Then for some $\sigma \in X_r(T)$ and $\nu \in X(T)$, $P_r(\sigma + p^r\nu)$ appears as a direct summand of P_n with $m = \operatorname{ht}(\Lambda - (\sigma + p^r\nu))$. From (2.3.6) one can conclude that if $b(m,n) \neq 0$ then $m \geq n$. Hence,

$$q(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\dim_t P_n}{h_r(t)}.$$

Now we are in the position to follow the line of reasoning given in [NPV, Thm. 3.3.1] directly. We include the argument for the convenience of the reader.

Expand $q(t) = \sum_{i=0}^{\infty} a_i t^i$ into a power series. From (2.4.1), it follows that for $i > 0$,

$$\begin{aligned} S_i = |a_0| + \cdots + |a_i| &\leq \sum_{m=0}^{i}\sum_{n=0}^{\infty} b(m,n) = \sum_{m=0}^{i}\sum_{n=0}^{m} b(m,n) \\ &\leq \sum_{n=0}^{i} Cn^{\dim V_{B_r}(M)-1} \leq Di^{\dim V_{B_r}(M)} \end{aligned} \tag{2.4.2}$$

for some $D > 0$.

From the definition of $h_r(t)$ one can deduce that the poles of $q(t)$ are roots of unity. Let b be the least common multiple of the orders of these poles. Then $q(t) = \frac{m(t)}{(1-t^b)^c}$ for some positive integer c and $m(t) \in \mathbb{C}[t]$. According to Proposition 2.2(c), $q(t) = q_0(t) + \cdots + q_{a-1}(t)$, with $q_i(t) = \sum_j b_{ij} t^{i+jb}$ with the property that b_{ij} is a polynomial in j of degree $d_i - 1$. Set $d = \max d_i$. Then by Proposition 2.2(d), $r(S_i) = d+1$ and $r(a_i) = d$. Thus, by (2.4.2), $r(a_i) \leq \dim V_{B_r}(M)$. The statement of the theorem now follows by Proposition 2.2(b). □

2.5. The following corollary establishes the inequality described in (1.4.2), which improves the lower bound given in [NPV, Cor. 3.4.3].

Corollary. *Let* $\lambda \in X(T)_+$. *Then* $\dim V_{G_1}(H^0(\lambda)) \geq |\Phi| - |\Phi_\lambda|$.

Proof. Recall that for any positive integer n, $t^n - 1 = \prod_{d|n} \Psi_d(t)$. Set $\Lambda = \lambda$ and $M := H^0(\lambda)$. Consider

$$q(t) = \frac{\dim_t M}{h_1(t)} = \prod_{\alpha \in \Phi^+} \frac{t^{\langle \lambda+\rho, \check{\alpha} \rangle} - 1}{t^{p\langle \rho, \check{\alpha}\rangle} - 1} \tag{2.5.1}$$

Observe that $\Psi_p(t)$ divides the numerator exactly $|\Phi_\lambda^+|$ times and $\Psi_p(t)$ divides the denominator exactly $|\Phi^+|$ times. Therefore, by Theorem 2.4 we have

$$\dim V_{B_1}(H^0(\lambda)) \geq |\Phi^+| - |\Phi_\lambda^+|. \tag{2.5.2}$$

According to [LN], if M is a rational G-module then $\dim V_{G_1}(M) = 2 \dim V_{B_1}(M)$. Hence,

$$\begin{aligned} \dim V_{G_1}(H^0(\lambda)) &= 2 \dim V_{B_1}(H^0(\lambda)) \\ &\geq 2\left(|\Phi^+| - |\Phi_\lambda^+|\right) \\ &= |\Phi| - |\Phi_\lambda|. \end{aligned}$$

□

2.6. An upper bound. In this section we provide an effective method for forcing the inclusion of the support variety of an induced module into the closure of certain nilpotent orbits; that is, an upper bound on the support variety. The technique involves checking a combinatorial condition which allows one to deduce that the module is projective over regular elements for Levi subalgebras.

For $J \subseteq \Delta$, let $x_J = \sum_{\alpha \in J} x_\alpha$, where x_α is a root vector in the root space $\mathfrak{g}_\alpha$. The theorem below was proved under the assumption that p is a good prime (see [NPV, Thm. 4.3.1]). The statement of this theorem still remains valid over arbitrary primes because if the regular element x_Δ is contained in $\mathcal{N}_1(\mathfrak{g})$ then $p \geq h$ by [CLNP, UGA2] (the determination of the restricted nullcone). This implies that the B-orbit of x_Δ is dense in $\mathcal{N}_1(\mathfrak{b})$. The arguments provided in [NPV, 4.4, 4.5] can then be used to prove the following statement.

Theorem. *Let* $J \subseteq \Delta$ *and* $\lambda \in X(T)_+$. *If* $w(\Phi_\lambda) \cap \Phi_J \neq \varnothing$ *for all* $w \in W$, *then* $x_J \notin V_{G_1}(H^0(\lambda))$.

2.7. Constrictors. Let $\mathcal{O}$ be an orbit in $\mathcal{N}_1(\mathfrak{g})$. The *constrictors* of $\mathcal{O}$ are the orbits contained in $\mathcal{N}_1(\mathfrak{g}) - \overline{\mathcal{O}}$ which are minimal with respect to the closure ordering of orbits in $\mathcal{N}(\mathfrak{g})$. The following result will be used in the next two sections to compute the support varieties of the induced modules. We will utilize Theorem 2.6 in addition to information about the constrictors of orbits.

Theorem. *Let* $\mathcal{O}$ *be an orbit in* $\mathcal{N}_1(\mathfrak{g})$ *and* $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_s\}$ *be the set of constrictors of* $\mathcal{O}$. *Let* $\lambda \in X(T)_+$ *and assume that the following conditions are satisfied:*

(i) $|\Phi| - |\Phi_\lambda| \geq \dim \mathcal{O}$;

(ii) *for* $i = 1, 2, \ldots, s$, $\mathcal{O}_i = G \cdot x_{J_i}$ *for some* $J_i \subseteq \Delta$;

(iii) *for* $i = 1, 2, \ldots, s$, $w(\Phi_\lambda) \cap \Phi_{J_i} \neq \varnothing$ *for all* $w \in W$.

Then $V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}}$.

Proof. Conditions (ii) and (iii) along with Theorem 2.6 show that $x_{J_i} \notin V_{G_1}(H^0(\lambda))$ for $i = 1, 2, \ldots, s$. In particular $V_{G_1}(H^0(\lambda)) \cap \mathcal{O}_j = \varnothing$ for $j = 1, 2, \ldots, s$ (by G-invariance). If $\mathcal{O}'$ is an orbit in $\mathcal{N}_1(\mathfrak{g}) - \overline{\mathcal{O}}$ and $V_{G_1}(H^0(\lambda)) \cap \mathcal{O}' \neq \varnothing$ then $\overline{\mathcal{O}'} \subseteq V_{G_1}(H^0(\lambda))$. But, this is impossible since every orbit closure in $\mathcal{N}_1(\mathfrak{g}) - \overline{\mathcal{O}}$ must contain a constrictor, hence $V_{G_1}(H^0(\lambda)) \cap \mathcal{O}' = \varnothing$. This shows that $V_{G_1}(H^0(\lambda)) \subseteq \overline{\mathcal{O}}$.

Since $\overline{\mathcal{O}}$ is an irreducible variety and $\dim V_{G_1}(H^0(\lambda)) \geq \dim \mathcal{O}$ by (i) and Corollary 2.5 one can deduce that $V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}}$. □

2.8. Strategy. Suppose that $\lambda \in X(T)_+$. We first observe that $\Phi_\lambda = \Phi_{\lambda + p\nu}$ for any $\nu \in X(T)$ and $\Phi_{w\cdot\lambda} = w^{-1}(\Phi_\lambda)$. This implies that if the conditions of Theorem 2.7 hold for λ then they also hold for any $w \cdot \lambda + p\nu \in X(T)_+$ where $w \in W$ and $\nu \in X(T)$. Therefore, it suffices to show that the conditions of Theorem 2.7 hold for representatives in $X(T)_+$ under the dot action of the extended affine Weyl group. These representatives can be chosen in $X_1(T)$.

Given such a representative the first step in our computation is to calculate $|\Phi| - |\Phi_\lambda|$ and find all orbits in $\mathcal{N}_1(\mathfrak{g})$ of that dimension. Many times there will be only one such orbit. Certainly the orbit(s) that we consider will satisfy (i) of Theorem 2.7. For each orbit we look at the constrictors (as defined in Section 2.7). It turns out that all such constrictors are of the form $G \cdot x_J$ for some $J \subseteq \Delta$ (i.e., an orbit representative is given by a regular Levi element), so condition (ii) is satisfied. For the set of constrictors of a given orbit of dimension $|\Phi| - |\Phi_\lambda|$ we check (using computer calculations in the exceptional cases) whether condition (iii) holds in Theorem 2.7. It turns out that all three conditions of Theorem 2.7 hold for a unique orbit $\mathcal{O}$ and this allows us to deduce that $V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}}$. In particular, $V_{G_1}(H^0(\lambda))$ is uniquely determined by Φ_λ. Once this is verified for all such representatives, we have by the above remarks,

$$V_{G_1}(H^0(\mu)) = V_{G_1}(H^0(w \cdot \mu + p\nu))$$

where $\mu \in X(T)_+$, $w \in W$, $\nu \in X(T)$ and $w \cdot \mu + p\nu \in X(T)_+$.

3. Classical Lie Algebras

In this section we compute the support varieties of the induced modules $H^0(\lambda)$ for the classical groups in bad characteristic (i.e., types B, C, D with $p = 2$). We follow the strategy of Section 2.8. We first determine orbit representatives x_J for the constrictor orbits which will arise later. We next show that every orbit of weights under the extended affine Weyl group has a fundamental dominant weight ω_k (or 0) as a representative. This significantly simplifies the remaining computations. We use this information to compute the lower bound $|\Phi| - |\Phi_\lambda|$ of Corollary 2.5, and identify a candidate orbit of that dimension, together with its associated constrictor orbits. Finally we verify condition (iii) of Theorem 2.7.

Throughout this section we let $N = 2l + 1$ (resp. $2l$, $2l$) in type B_l (resp. C_l, D_l).

3.1. Nilpotent orbit representatives. The nilpotent orbits in $\mathfrak{g}$ have been classified by Hesselink [He]. They are parametrized by pairs consisting of a partition μ of N and an *index function* $\chi\colon I \to \mathbb{Z}$, where I is the set of (non-zero) parts of μ. The partition gives, as usual, the sizes of the Jordan blocks for any representative of the orbit.

There is a one-to-one correspondence between nilpotent orbits and pairs μ, χ satisfying the following conditions. Write $n(m)$ for the multiplicity of $m \in I$ as a part in μ.

(3.1.1) (1) For $m > k$ in I, $\chi(m) \geq \chi(k)$ and $m - \chi(m) \geq k - \chi(k)$.
(2) For $G = \mathrm{Sp}(N)$ and $m \in I$:
  (a) $0 \leq \chi(m) \leq m/2$;
  (b) $\chi(m) = m/2$ if $n(m)$ is odd.
(3) For $G = \mathrm{O}(N)$ and $m \in I$:
  (a) $m/2 \leq \chi(m) \leq m$;
  (b) $\chi(m) = m$ if $n(m)$ is odd;
  (c) $\{\, m \in I \mid n(m) \text{ is odd }\} = \{i, i-1\} \cap \mathbb{N}$ for some $i \in \mathbb{Z}$.

If $I = \{m_1 > m_2 > \dots\}$, Hesselink displays the pair μ, χ as a *symbol*

$$\mu_\chi = \left({m_1}^{n(m_1)}_{\chi(m_1)}, {m_2}^{n(m_2)}_{\chi(m_2)}, \dots \right).$$

Write $\mathcal{O}(\mu_\chi)$ for the corresponding nilpotent orbit.

Proposition. *Let $G = O(N)$ or $Sp(N)$ over a field k of characteristic 2, and let $\mathfrak{g} = Lie(G)$ have rank l. The table lists certain regular nilpotent elements for Levi subalgebras of $\mathfrak{g}$ having type $A_1 \times \cdots \times A_1$, and the nilpotent orbits to which they belong.*

Type	*Representative*	*Orbit*
B_l	$x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{2r-1}}$ $(2r-1 < l)$	$\mathcal{O}(2^{2r}_1, 1^{N-4r}_1)$
B_l	$x_{\alpha_l} + x_{\alpha_{l-2}} + \cdots + x_{\alpha_{l-2r}}$ $(r \geq 0)$	$\mathcal{O}(2^{2r+1}_2, 1^{N-4r-2}_1)$
C_l	$x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{2r-1}}$ $(2r-1 < l)$	$\mathcal{O}(2^{2r}_0, 1^{N-4r}_0)$
C_l	$x_{\alpha_l} + x_{\alpha_{l-2}} + \cdots + x_{\alpha_{l-2r}}$ $(r \geq 0)$	$\mathcal{O}(2^{2r+1}_1, 1^{N-4r-2}_0)$
D_l	$x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{2r-1}}$ $(2r-1 < l)$	$\mathcal{O}(2^{2r}_1, 1^{N-4r}_1)$
D_l	$x_{\alpha_{l-1}} + x_{\alpha_l}$	$\mathcal{O}(2^2_2, 1^{N-4}_1)$

In type D_l when l is even, $x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{l-3}} + x_{\alpha_{l-1}}$ and $x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{l-3}} + x_{\alpha_l}$ are both representatives for the $O(2l)$-orbit with symbol (2^l_1), which lie in distinct $SO(2l)$-orbits. (Here $SO(2l)$ is by definition the component of the identity in $O(2l)$.)

Proof. We first treat the case $G = \mathrm{Sp}(N)$ with $N = 2l$. Following Hesselink, let V be an N-dimensional vector space over k equipped with a non-degenerate bilinear form β satisfying $\beta(v, v) = 0$ for all $v \in V$. Then G is the group of automorphisms of V leaving β invariant. Fix $x \in \mathfrak{g}$ with x nilpotent. For each $i \in \mathbb{N} \cup \{0\}$ define a quadratic form $\alpha_i\colon V \to k$ by $\alpha_i(v) = \beta(x^{i+1}v, x^i v)$. An extension to $\mathbb{N}$ of the index function given at the beginning of Section 3.1 is defined by

$$\chi(m) = \min\{\, i \geq 0 : \alpha_i|_{\mathrm{Ker}(x^m)} = 0 \,\}. \tag{3.1.2}$$

In particular, $\chi(1) = 0$, and if $x^2 = 0$ then $\alpha_1 = 0$, and

$$\chi(2) = \begin{cases} 0 & \text{if } \alpha_0 = 0; \\ 1 & \text{otherwise.} \end{cases} \tag{3.1.3}$$

We take the matrix of β to be

$$K = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

so that $x \in \mathfrak{g} \iff x^t K = Kx \iff Kx^t K = x$. The matrix $x' := Kx^t K$ is obtained from x by transposing across the antidiagonal, so $x \in \mathfrak{g}$ if and only if x is symmetric about the antidiagonal.

We can take a Cartan subalgebra to be the diagonal matrices in $\mathfrak{g}$. Then the simple root vectors can be chosen to be

$$x_{\alpha_i} = \begin{cases} E_{i,i+1} + E_{N-i,N+1-i} & \text{if } i < l; \\ E_{l,l+1} & \text{if } i = l; \end{cases}$$

(where E_{ij} are the matrix units). Consider

$$x = x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{2r-1}} \quad \text{where } 2r - 1 < l.$$

Then x is already in Jordan canonical form, with block sizes given by the partition $(2^{2r}, 1^{N-4r})$. It is elementary to check that $\beta(xv, v) = 0$ for all $v \in V$, so that $\alpha_0 \equiv 0$ and hence by (3.1.3), $\chi(2) = 0$. Thus x belongs to the orbit $\mathcal{O}(2_0^{2r}, 1_0^{N-4r})$. If

$$x = x_{\alpha_l} + x_{\alpha_{l-2}} + \cdots + x_{\alpha_{l-2r}}$$

then again x is in Jordan form, with block sizes $(2^{2r+1}, 1^{N-4r-2})$, but now $\alpha_0(v) = \beta(xv, v) = v_{n+1}^2 \neq 0$ so $\chi(2) = 1$. Thus x belongs to the orbit $\mathcal{O}(2_1^{2r+1}, 1_0^{N-4r-2})$.

For the orthogonal case, let V be an N-dimensional vector space over k, equipped with a quadratic form α and a bilinear form β satisfying $\beta(v, w) = \alpha(v + w) - \alpha(v) - \alpha(w)$ for all $v, w \in V$. We assume that V is non-degenerate in the sense of [He]. Then $G = \mathrm{O}(N)$ is the algebraic group of automorphisms of V leaving α invariant. Fix $x \in \mathfrak{g}$ with x nilpotent. For each $i \in \mathbb{N} \cup \{0\}$ define a quadratic form $\alpha_i \colon V \to k$ by $\alpha_i(v) = \alpha(x^i v)$. The index function χ is defined as in (3.1.2). Then $\chi(1) = 1$, and when $x^2 = 0$, $\chi(2)$ is given by

$$\chi(2) = \begin{cases} 1 & \text{if } \alpha_1 = 0; \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.4}$$

We may take $\alpha(v) = v_1 v_N + v_2 v_{N-1} + \cdots$, so that the matrix of β is K when $N = 2l$ is even, and is $K_0 = K - E_{n+1,n+1}$ when $N = 2l + 1$ is odd. A matrix x belongs to $\mathfrak{g}$ if and only if $\operatorname{tr} x = 0$ and $\beta(xv, v) = 0$ for all $v \in V$. When N is even the second condition is equivalent to x being symmetric about the antidiagonal, with antidiagonal elements all zero. If N is odd, say $N = 2l+1$, then x has the block form

$$x = \begin{pmatrix} A & 0 & C \\ w & 0 & u \\ B & 0 & A' \end{pmatrix} \tag{3.1.5}$$

where A, B, C are $l \times l$, w and u are $1 \times l$, and B and C are symmetric about the antidiagonal with antidiagonal elements all zero. In either case we may take a Cartan subalgbra consisting of the diagonal matrices in $\mathfrak{g}$, and simple root vectors

$$x_{\alpha_i} = \begin{cases} E_{i,i+1} + E_{N-i,N+1-i} & \text{if } i < l; \\ E_{l+1,l+2} & \text{if } i = l \text{ in type } B_l; \\ E_{l-1,l+1} + E_{l,l+2} & \text{if } i = l \text{ in type } D_l. \end{cases}$$

Assume $N = 2l+1$. Arguing similarly to the symplectic case, one finds that $x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{2r-1}}$ with $2r-1 < l$ lies in the orbit $\mathcal{O}(2_1^{2r}, 1_1^{N-4r})$, while $x_{\alpha_l} + x_{\alpha_{l-2}} + \cdots + x_{\alpha_{l-2r}}$ with $r \geq 0$ has $\alpha_1(v) = \alpha(xv) = v_{l+2}^2 \neq 0$ so it is in the orbit $\mathcal{O}(2_2^{2r+1}, 1_1^{N-4r-2})$.

Assume $N = 2l$. Using the same techniques as in types B_l and C_l, it is straightforward to check that $x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{2r-1}}$ with $2r-1 < l$ lies in the orbit $\mathcal{O}(2_1^{2r}, 1_1^{N-4r})$. Similarly when l is even, $x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{l-3}} + x_{\alpha_l}$ is in the orbit $\mathcal{O}(2_1^l)$. Consider $x = x_{\alpha_{l-1}} + x_{\alpha_l} = E_{l-1,l} + E_{l-1,l+1} + E_{l,l+2} + E_{l+1,l+2}$. Note that $x\colon e_{l+2} \mapsto e_l + e_{l+1} \mapsto 0,\ e_l \mapsto e_{l-1} \mapsto 0$ (where $\{e_i\colon 1 \leq i \leq N\}$ is the standard basis of V); this shows that x has Jordan block sizes $(2^2, 1^{N-4})$. And $\alpha_1(v) = \alpha(xv) = v_{l+2}^2 \neq 0$ so x is in the orbit $\mathcal{O}(2_2^2, 1_1^{N-4})$.

Finally, assume $N = 2l$ with l even. We have shown that $x = x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{l-3}} + x_{\alpha_{l-1}}$ and $y = x_{\alpha_1} + x_{\alpha_3} + \cdots + x_{\alpha_{l-3}} + x_{\alpha_l}$ both lie in the $\mathrm{O}(N)$-orbit $\mathcal{O}(2_1^l)$. By [He], this orbit splits into two distinct $\mathrm{SO}(N)$-orbits. Also by [He], the group $\mathrm{O}(N)$ is generated by the reflections r_w, for $w \in V$ with $\alpha(w) \neq 0$, given by

$$r_w(v) = v - \beta(v,w)\alpha(w)^{-1}w,$$

while $\mathrm{SO}(N)$ consists of the products of an even number of such reflections. Let $w = e_l + e_{l+1}$, with $\alpha(w) = 1$. Then r_w interchanges e_l and e_{l+1} while fixing all other e_i. Thus, conjugation by r_w interchanges $x_{\alpha_{l-1}}$ and x_{α_l}, and fixes all other x_{α_i}. In particular, r_w is an element of $\mathrm{O}(N) - \mathrm{SO}(N)$ which takes x to y. Since $\mathrm{SO}(N)$ has index 2 in $\mathrm{O}(N)$, it follows that x and y must belong to the two different $\mathrm{SO}(N)$ orbits in the $\mathrm{O}(N)$ orbit $\mathcal{O}(2_1^l)$. □

3.2. Generic W-orbits in $X_1(T)$. Let Φ be of type X_l, where $X = B, C$, or D, and let $p = 2$. In view of the strategy in Section 2.8, we consider the action of W on $X(T)/pX(T)$, which we identify with $X_1(T)$ by taking p-restricted parts of weights. In this subsection and the next, we will classify the W-orbits in $X_1(T)$. For convenience in stating the results, we work here with the ordinary (not the dot) action of W; the ρ shifts will be inserted later. And for simplicity of notation, we write equalities when we really mean congruences modulo $pX(T)$.

Set $\omega_0 = 0$ and $S_k = W(\omega_k)$ for $0 \leq k \leq l$. We plan to show that $X_1(T) = \cup_{k=0}^{l} S_k$. To do this, we first identify S_1 explicitly as a set of l elements in each type. We then show that for $k \leq m$ (a certain integer depending on the type, defined below), S_k consists of sums of k distinct elements of S_1. We determine when an element of S_k can be written in more than one way as such a sum, and also when $S_k = S_j$. In the next subsection we will identify $\cup_{k=m+1}^{l} S_k$. These data allow us to count $|\cup_{k=0}^{l} S_k| = 2^l = |X_1(T)|$.

We fix the integer m as follows:

$$m = \begin{cases} l-1, & X = B, \\ l, & X = C, \\ l-2, & X = D. \end{cases}$$

We will repeatedly use formulas (modulo $pX(T)$) for $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$. The following formulas are easily checked by taking the inner product of each side with $\check{\alpha}_k$ for all k:

$$s_i(\omega_i) = \begin{cases} \omega_{l-2} + \omega_{l-1}, & i = l-1, \text{ type } B_l; \\ \omega_l, & i = l, \text{ type } C_l; \\ \omega_{l-3} + \omega_{l-2} + \omega_{l-1} + \omega_l, & i = l-2, \text{ type } D_l; \\ \omega_{l-2} + \omega_{l-1}, & i = l-1, \text{ type } D_l; \\ \omega_{l-2} + \omega_l, & i = l, \text{ type } D_l; \\ \omega_{i-1} + \omega_i + \omega_{i+1}, & \text{otherwise} \end{cases} \tag{3.2.1}$$

with the convention that $\omega_0 = \omega_{l+1} = 0$. The following formula follows immediately:

$$s_{k-1}s_{k-2}\dots s_1(\omega_1) = \omega_{k-1} + \omega_k \text{ for } 1 \le k \le m. \tag{3.2.2}$$

The remaining elements of S_1 are determined case-by-case, again using (3.2.1). If $X = C$ then $s_l s_{l-1} \dots s_1(\omega_1) = \omega_{l-1} + \omega_l$, which is already accounted for in S_1, so we get no new elements. If $X = B$ then $s_{l-1}s_{l-2}\dots s_1(\omega_1) = \omega_{l-1}$, from which we can produce no additional new elements. Finally if $X = D$ then $s_{l-2}s_{l-3}\dots s_1(\omega_1) = \omega_{l-2} + \omega_{l-1} + \omega_l$, and $s_{l-1}s_{l-2}\dots s_1(\omega_1) = s_l s_{l-2} \dots s_1(\omega_1) = \omega_{l-1} + \omega_l$, from which we can produce no additional new elements. We summarize these results.

Proposition. *Let Φ be of type X_l where $X = B, C$, or D, and let $p = 2$. The W-orbit of ω_1 in $X_1(T)$ consists of l distinct elements, as follows:*

$$\begin{aligned} S_1 &= \{\omega_1, s_1(\omega_1), s_2s_1(\omega_1), \dots, s_{l-1}s_{l-2}\dots s_1(\omega_1)\} \\ &= \begin{cases} \{\omega_1, \omega_1+\omega_2, \dots, \omega_{l-2}+\omega_{l-1}, \omega_{l-1}\}, & X = B; \\ \{\omega_1, \omega_1+\omega_2, \dots, \omega_{l-2}+\omega_{l-1}, \omega_{l-1}+\omega_l\}, & X = C; \\ \{\omega_1, \omega_1+\omega_2, \dots, \omega_{l-2}+\omega_{l-1}+\omega_l, \omega_{l-1}+\omega_l\}, & X = D. \end{cases} \end{aligned}$$

Moreover, for $i \le l$, $s_{i-1}s_{i-2}\dots s_1(\omega_1)$ does not involve any ω_j with $j < i-1$.

Next we have a technical lemma which allows us to write sums of k elements of S_1 as elements of S_k.

Lemma. *Let $k \le m$ and $1 \le i_1 < i_2 < \dots < i_k \le l$ be positive integers. Then*

$$\begin{aligned} &s_{i_1-1}s_{i_1-2}\dots s_1(\omega_1) + s_{i_2-1}s_{i_2-2}\dots s_1(\omega_1) + \dots + s_{i_k-1}s_{i_k-2}\dots s_1(\omega_1) = \\ &\qquad (s_{i_1-1}s_{i_1-2}\dots s_1)(s_{i_2-1}s_{i_2-2}\dots s_2)\dots(s_{i_k-1}s_{i_k-2}\dots s_k)(\omega_k). \end{aligned}$$

Proof. We use induction on k. If $k = 1$ the result is trivial. In general, substitute $\omega_k = \omega_{k-1} + s_{k-1}s_{k-2}\dots s_1(\omega_1)$ from (3.2.1) (recall we are working mod 2) on the right hand side. Observe that $s_r s_{r-1}\dots s_j$ fixes $s_{i-1}s_{i-2}\dots s_1(\omega_1)$ for $r < i-1 < l$ (by the last statement of the Proposition), so the right hand side of the above equation is equal to

$$(s_{i_1-1}s_{i_1-2}\dots s_1)(s_{i_2-1}s_{i_2-2}\dots s_2)\dots(s_{i_k-1}s_{i_k-2}\dots s_k)(\omega_{k-1}) + s_{i_k-1}s_{i_k-2}\dots s_1(\omega_1).$$

Since s_r fixes ω_{k-1} for $r > k-1$, this is equal to

$$(s_{i_1-1}s_{i_1-2}\dots s_1)(s_{i_2-1}s_{i_2-2}\dots s_2)\dots(s_{i_{k-1}-1}s_{i_{k-1}-2}\dots s_{k-1})(\omega_{k-1}) + s_{i_k-1}s_{i_k-2}\dots s_1(\omega_1).$$

The result now follows by induction. □

Theorem. *For $1 \le k \le m$, S_k consists of all sums of k distinct elements of S_1. The expression for an element of S_k as such a sum is unique, except when $X = B$ or D and $k = l/2$, when every element of S_k can be written in precisely two ways as sum of k distinct elements of S_1. The orbits S_k are distinct, except that $S_k = S_{l-k}$ for $1 \le k \le l-1$ in type B, and for $2 \le k \le l-2$ in type D.*

Proof. Let $\widetilde{S}_k$ denote the set of sums of k distinct elements of S_1. According to the proposition, every $\mu \in \widetilde{S}_k$ can be expressed as on the left side of the equation in the lemma. Then $\mu \in S_k$, by the right side of that equation. To prove the reverse inclusion, notice that ω_k is the sum of the first k elements of S_1, in the order listed in the proposition. Hence any W-translate of ω_k is the sum of the W-translates of these k elements, which are in turn distinct elements of $S_1 = W\omega_1$.

A similar argument shows that the number of ways of writing ω_k as a sum of distinct elements of S_1 is the same as the number of such ways of writing every element of S_k. In type C it is clear from the explicit description of S_1 that $\omega_1 + (\omega_1+\omega_2) + \dots + (\omega_{k-1}+\omega_k)$ is the only way of so writing ω_k. In type B there is another way: $(\omega_k+\omega_{k+1}) + (\omega_{k+1}+\omega_{k+2}) + \dots + \omega_{l-1}$, a sum of $l-k$ elements. In type D there is also a second way of writing ω_k, namely $(\omega_k+\omega_{k+1}) + (\omega_{k+1}+\omega_{k+2}) + \dots + (\omega_{l-2}+\omega_{l-1}+\omega_l) + (\omega_{l-1}+\omega_l)$, also a sum of $l-k$ elements. This shows that $\widetilde{S}_k = \widetilde{S}_{l-k}$ in types B and D, and hence that $S_k = S_{l-k}$ (provided, in type D, that $k \ge 2$, since $m = l-2$). These second expressions for ω_k are in $\widetilde{S}_k$ (and hence every element of S_k has two realizations as a sum of k elements of S_1) if and only if $k = l/2$. □

3.3. Special W-orbits in $X_1(T)$. Here we use a different technique to identify $\cup_{k=m+1}^{l} S_k$. In type C_l there is nothing to do since $m = l$. In type B_l we need only identify S_l.

Proposition (A). *In type B_l, $\cup_{k=0}^{l-1} S_k = \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_l\rangle = 0\}$ and $S_l = \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_l\rangle = 1\}$.*

Proof. Since, by (3.2.1), the coefficients of ω_l in λ and $s_i(\lambda)$ are the same for every simple reflection s_i, it is clear that $L_0 := \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_l\rangle = 0\}$ and $L_1 := \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_l\rangle = 1\}$ are W-invariant. In particular, $S_l \subset L_1$ and $S_k \subset L_0$ for $0 \le k \le l-1$. In fact, by Theorem 3.2 we have

$$\left|\bigcup_{k=0}^{l-1} S_k\right| = 1 + \left|\bigcup_{k=1}^{l-1} S_k\right| = 1 + \frac{1}{2}\sum_{k=1}^{l-1}\binom{l}{k} = 1 + \frac{1}{2}(2^l - 2) = 2^{l-1} = |L_0|,$$

so that

$$L_0 = \bigcup_{k=0}^{l-1} S_k. \tag{3.3.1}$$

To prove that $L_1 \subset S_l$, we first claim that $\omega_{l-i} + \omega_l \in S_l$ for $1 \leq i \leq l-1$. We prove this by induction on i. Since $s_l(\omega_l) = \omega_{l-1} + \omega_l$ and $s_l s_{l-1} s_l(\omega_l) = \omega_{l-2} + \omega_l$ by (3.2.1), the claim is true for $i = 1$ and 2. Assume it is true for some $i \geq 2$. Use (3.2.1) to check that

$$s_{l-2}s_{l-3}\dots s_{l-i}(\omega_{l-i} + \omega_l) = \omega_{l-(i+1)} + \omega_{l-2} + \omega_{l-1} + \omega_l.$$

Apply s_{l-1} and finally s_l to obtain $\omega_{l-(i+1)} + \omega_l$, as required to complete the induction.

Now let $\lambda \in L_1$. Then $\lambda = \mu + \omega_l$ where $\mu \in L_0$. Then by (3.3.1), $\mu \in S_k$ for some $0 \leq k < l$. So we may write $\mu = w(\omega_k)$ for some $w \in W$. By Lemma 3.2, we may in fact assume that $w \in \langle s_1, \dots, s_{l-1}\rangle$. Then $w(\omega_l) = \omega_l$ so $\lambda = \mu + \omega_l = w(\omega_k) + \omega_l = w(\omega_k + \omega_l) \in S_l$, by the claim. □

Finally in type D_l we need to identify $S_{l-1} \cup S_l$.

Proposition (B)**.** *In type D_l, $\cup_{k=0}^{l-2} S_k = \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_{l-1}\rangle = \langle \lambda, \check{\alpha}_l\rangle\}$ and $S_{l-1} \cup S_l = \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_{l-1}\rangle \neq \langle \lambda, \check{\alpha}_l\rangle\}$. When l is odd, $S_{l-1} = S_l$.*

Proof. Since, by (3.2.1), no simple reflection changes the coefficient of exactly one of ω_{l-1}, ω_l in a decomposition of $\lambda \in X_1(T)$ as a linear combination of fundamental weights, the sets $L_0 := \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_{l-1}\rangle = \langle \lambda, \check{\alpha}_l\rangle\}$ and $L_1 := \{\lambda \in X_1(T) \mid \langle \lambda, \check{\alpha}_{l-1}\rangle \neq \langle \lambda, \check{\alpha}_l\rangle\}$ are W-invariant. In particular, $S_{l-1} \cup S_l \subset L_1$ and $S_k \subset L_0$ for $0 \leq k \leq l-2$. A counting argument similar to the one in the previous proposition shows that

$$L_0 = \bigcup_{k=0}^{l-2} S_k \tag{3.3.2}$$

To prove that $L_1 \subset S_{l-1} \cup S_l$, we first claim that $\omega_{l-i} + \omega_l \in S_l$ for i even, $2 \leq i < l$, and $\omega_{l-i} + \omega_{l-1} \in S_l$ for i odd, $3 \leq i < l$. (The same statements, with ω_l and ω_{l-1} interchanged, hold for S_{l-1}.) Begin the induction on i with the computations $s_l(\omega_l) = \omega_{l-2} + \omega_l$ and $s_{l-1}s_{l-2}s_l(\omega_l) = \omega_{l-3} + \omega_{l-1}$. The inductive step follows from the computation $s_{l-1}s_{l-2}s_{l-3}\dots s_{l-i}(\omega_{l-i} + \omega_l) = \omega_{l-(i+1)} + \omega_{l-1}$ (and similarly interchanging ω_l and ω_{l-1}, and using s_l in place of s_{l-1}).

Now let $\lambda \in L_1$. Then without loss of generality $\lambda = \mu + \omega_l$ where $\mu \in L_0$. (The other possibility is $\lambda = \mu - \omega_l$, but these are congruent modulo $2\omega_l$ and we are working in $X(T)/2X(T)$.) Then by (3.3.2), $\mu \in S_k$ for some $0 \leq k \leq m$. So $\mu = w(\omega_k)$ for some $w \in W$, but in fact by Lemma 3.2, we may assume $w \in \langle s_1, \dots, s_{l-1}\rangle$. Then $w(\omega_l) = \omega_l$ so $\lambda = \mu + \omega_l = w(\omega_k) + \omega_l = w(\omega_k + \omega_l) \in S_l \cup S_{l-1}$, by the claim.

Finally we prove the last statement of the proposition. When l is odd, we have $l-1$ even, and so by the claim of the second paragraph with $i = l-1$, $\omega_1 + \omega_l \in S_l$. But $s_{l-1}s_{l-2}\dots s_1(\omega_1 + \omega_l) = \omega_{l-1}$, so $\omega_{l-1} \in S_l$ and thus $S_{l-1} \subset S_l$. The reverse inclusion follows by symmetry, so we have $S_{l-1} = S_l$ as desired. □

Theorem. *Let Φ be of type X_l where $X = B, C$, or D, and $p = 2$. Then $X_1(T) = \cup_{k=0}^{l} S_k$.*

Proof. First suppose $X = C$. Then by Theorem 3.2, the sets S_k are disjoint subsets of $X_1(T)$ for $0 \leq k \leq l$, and $|S_k| = \binom{l}{k}$. Thus $|\cup_{k=0}^{l} S_k| = 2^l = |X_1(T)|$. The theorem follows.

If $X = B$ or D the theorem follows from the two previous propositions, since $X_1(T) = L_0 \cup L_1$. □

3.4. Computation of $|\Phi| - |\Phi_\lambda|$. Define r as follows:

$$r = \begin{cases} l - 2k, & \Phi = B_l, C_l, D_l, \quad 1 \leq k \leq \frac{l}{2}, \\ -l + 2k, & \Phi = B_l, D_l, \quad \frac{l}{2} < k \leq l - 1, \\ -l + 2k - 1, & \Phi = C_l, \quad \frac{l}{2} < k \leq l. \end{cases}$$

When r is defined in this way and k is in the defined range, one has $0 \leq l - r \leq l$. This will make our labelling of partitions make sense later in Section 3.6. We now calculate $|\Phi| - |\Phi_\lambda|$ for classical Lie algebras. Recall the results of the previous two subsections for the values of k which index the W-orbits in $X_1(T)$.

Proposition. *Let Φ be of type X_l where $X = B$, C, or D. Let $\lambda = \omega_k - \rho$, and define r as above. Then*

$$|\Phi| - |\Phi_\lambda| = \begin{cases} l^2 - r^2, & X = B_l, \ 1 \leq k \leq \frac{l}{2}, \\ l^2 + l, & X = B_l, \ k = l, \\ l^2 + l - r^2 - r, & X = C_l, \ 1 \leq k \leq l, \\ l^2 - r^2, & X = D_l, \ 1 \leq k \leq \frac{l}{2}, \\ l^2 - l, & X = D_l, \ k = l - 1, l. \end{cases}$$

Proof. Set $\lambda = \omega_k - \rho$. Let us first assume that for types B_l and D_l, we have $1 \leq k \leq \frac{l}{2}$ and for type C_l, $1 \leq k \leq l$. Then $\Phi_\lambda = \{\alpha \mid \langle \omega_k, \check{\alpha} \rangle \in 2\mathbb{Z}\}$. Express $\check{\alpha} = n_1\check{\alpha}_1 + n_2\check{\alpha}_2 + \ldots n_l\check{\alpha}_l$. If $\alpha \in \Phi_\lambda$ then $n_k = 0, \pm 2$. If $n_k = 0$ then α is in the root subsystem $A_{k-1} \times X_{l-k}$. Moreover, the number of roots such that $n_k = \pm 2$ is $k(k+1)$ (resp. $k(k-1)$, $k(k-1)$) when $X = B$ (resp. C, D). This gives $|\Phi| - |\Phi_\lambda| = 4lk - 4k^2$ when $X = B$ or D and $4lk - 4k^2 + 2k$ when $X = C$. Substituting r as defined before the proposition yields the desired results.

On the other hand, for type B_l (resp. D_l) when $k = l$ (resp. $k = l-1, l$), Φ_λ is isomorphic to A_{l-1}. Therefore, $|\Phi| - |\Phi_\lambda| = 2l^2 - l(l-1) = l^2 + l$ (resp. $2(l^2 - l) - l(l-1) = l^2 - l$). □

3.5. Dimensions of Orbits. In [UGA2], the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ was calculated. The Hasse diagram for $\mathcal{N}_1(\mathfrak{g})$ was deduced from work of Großer [Gr] and Spaltenstein [Sp1] or can be obtained by using [UGA2, Thm. 2.3]. We provide this information in Section 6. The dimensions of the orbits in $\mathcal{N}_1(\mathfrak{g})$ can be computed from formulas given by Hesselink [He, Thm. 4.4]. This data will be essential for our computations so we have recorded it below.

Φ	Orbit	Dimension
B_l	$\mathcal{O}(2_2^{l-r}, 1_1^{2r+1})$	l^2+l-r^2-r
	$\mathcal{O}(2_1^{l-r}, 1_1^{2r+1})$	l^2-r^2
C_l	$\mathcal{O}(2_1^{l-r}, 1_0^{2r})$	l^2+l-r^2-r
	$\mathcal{O}(2_0^{l-r}, 1_0^{2r})$	l^2-r^2
D_l	$\mathcal{O}(2_2^{l-r}, 1_1^{2r})$	l^2-r^2
	$\mathcal{O}(2_1^{l-r}, 1_1^{2r})$	l^2-l-r^2+r

3.6. Supports of induced modules. The following theorem presents the computation of the supports of the induced modules $H^0(w\cdot\lambda)$, $w\cdot\lambda \in X(T)_+$ when G is $\mathrm{O}(N)$ or $\mathrm{Sp}(N)$ and char $k=2$. By Theorem 3.3 and the remarks in Section 2.8, it suffices to consider λ of the form $\omega_k-\rho$.

Theorem. *Let $G = O(2l+1)$, $Sp(2l)$, or $O(2l)$ be a classical group with root system Φ of type B_l, C_l or D_l and assume that* $\operatorname{char} k = 2$. *Let r be as defined in Section 3.4. Let $\lambda = \omega_k - \rho$. The support varieties of the induced modules $H^0(w\cdot\lambda)$ where $w\cdot\lambda \in X(T)_+$ are given in the following table.*

Φ	k	*Constrictors*	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w\cdot\lambda))$
B_l	$1 \le k \le \frac{l}{2}$	$\mathcal{O}(2_1^{l-r+2}, 1_1^{2r-3})$, $\mathcal{O}(2_2, 1_1^{2l-1})$	l^2-r^2	$\overline{\mathcal{O}(2_1^{l-r}, 1_1^{2r+1})}$
B_l	l		l^2+l	$\overline{\mathcal{O}(2_2^{l}, 1_1^{1})}$
C_l	$1 \le k \le \frac{l}{2}$	$\mathcal{O}(2_0^{l-(r-2)}, 1_0^{2(r-2)})$, $\mathcal{O}(2_1^{l-(r-1)}, 1_0^{2(r-1)})$	l^2+l-r^2-r	$\overline{\mathcal{O}(2_1^{l-r}, 1_0^{2r})}$
C_l	$\frac{l}{2} < k \le l$	$\mathcal{O}(2_0^{l-(r-1)}, 1_0^{2(r-1)})$	l^2+l-r^2-r	$\overline{\mathcal{O}(2_1^{l-r}, 1_0^{2r})}$
D_l	$1 \le k \le \frac{l}{2}$	$\mathcal{O}(2_1^{l-r+2}, 1_1^{2r-4})$	l^2-r^2	$\overline{\mathcal{O}(2_2^{l-r}, 1_1^{2r})}$
D_l *(l even)*	$l-1,\ l$	$\mathcal{O}(2_2^{2}, 1_1^{2l-4})$	l^2-l	$\overline{\mathcal{O}(2_2^{l})}$
D_l *(l odd)*	$l-1,\ l$	$\mathcal{O}(2_1^{2}, 1_1^{2l-4})$	l^2-l	$\overline{\mathcal{O}(2_1^{l-1}, 1_1^{2})}$

Proof. In Proposition 3.4, we calculated $\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$ for an element $\lambda = \omega_k - \rho$ in a typical W-orbit on $X_1(T)$. According to Section 3.5, there exists an orbit in $\mathcal{N}_1(\mathfrak{g})$ of dimension equal to $\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$. The orbits considered are given by $\mathcal{O} := \mathcal{O}(2_2^{l-r}, 1_1^{2r+1})$ (resp. $\mathcal{O}(2_1^{l-r}, 1_0^{2r})$, $\mathcal{O}(2_2^{l-r}, 1_1^{2r})$) when Φ is of type B_l (resp. C_l, D_l). From the Hasse diagrams given in Section 6, one can easily determine the constrictors for these orbits (which are given in the statement of the theorem).

From Proposition 3.1, it follows that the constrictors are all of the form $G\cdot x_J$ where $J \subseteq \Delta$. The results of this theorem will now follow from Theorem 2.7, if we verify condition (iii), which says that $w(\Phi_\lambda) \cap \Phi_J \neq \varnothing$ for all $w \in W$. Note that $w^{-1}(\Phi_\lambda) = \Phi_{w\cdot\lambda}$; indeed,

$$\langle w \cdot \lambda + \rho, \check{\alpha} \rangle = \langle w(\omega_k), \check{\alpha} \rangle.$$

Since $\Phi_J \cong A_1 \times A_1 \times \cdots \times A_1$, our task is equivalent to proving that for every $w \in W$, $2|\langle w(\omega_k), \check{\alpha} \rangle$ for some $\alpha \in J$.

We will proceed case by case. For type B_l, we have $N = 2l+1$. Assume $1 \leq k \leq l/2$, so $r = l - 2k$. Then $\mathcal{O}(2_1^{l-r+2}, 1_1^{2r-3}) = \mathcal{O}(2^{2(k+1)}, 1_1^{N-4(k+1)})$. Recall that $S_1 = \{\omega_1, \omega_1 + \omega_2, \ldots, \omega_{l-2} + \omega_{l-1}, \omega_{l-1}\}$. An element in the W orbit of ω_k must be the sum of k distinct elements of S_1. Consider such an element of the form

$$\begin{aligned} w(\omega_k) &= n_1\omega_1 + n_2(\omega_1+\omega_2) + \cdots + n_{l-1}(\omega_{l-2}+\omega_{l-1}) + n_l\omega_{l-1} \\ &= (n_1+n_2)\omega_1 + (n_2+n_3)\omega_2 + \ldots (n_{l-1}+n_l)\omega_{l-1} \end{aligned}$$

where $0 \leq n_j \leq 1$ for all j. Observe that $J = \{\alpha_1, \alpha_3, \ldots, \alpha_{2(k+1)-1}\}$ so $|J| = k+1$.

Suppose that $\langle w(\omega_k), \check{\alpha}_{2j-1} \rangle \equiv 1 \pmod 2$ for $j = 1, 2, \ldots, k+1$. Then the $k+1$ terms $n_1+n_2, n_3+n_4, n_5+n_6, n_{2(k+1)-1}+n_{2k}$ must all be equal to 1. But, this implies that there are $k+1$ distinct elements of S_1 in the expression of $w(\omega_k)$, which is a contradiction. Hence, $2|\langle w(\omega_k), \check{\alpha}_j \rangle$ for some $j = 1, 2, \ldots, 2(k+1)-1$. For the other constrictor $\mathcal{O}(2_2, 1_1^{2l-1})$, we have $J = \{\alpha_l\}$ according to the table in Proposition 3.1. Since $w(\omega_k)$ never contains ω_l as a summand (because of our description of S_k), it follows that $\langle w(\omega_k), \check{\alpha}_l \rangle = 0$ for all $w \in W$. The case when $k = l$ can be deduced immediately because the restricted nullcone is the closure of $\mathcal{O}(2_2^l, 1_1^1)$ (an irreducible variety). The variety has dimension $l^2 + l$ which is also a lower bound for the dimension of the support variety (i.e., $|\Phi| - |\Phi_\lambda|$).

For type C_l (resp. D_l) when $r = l - 2k$ (i.e., $0 \leq k \leq \frac{l}{2}$) one has $N = 2l$ and $\mathcal{O}(2_0^{l-r+2}, 1_0^{2r-4}) = \mathcal{O}(2_0^{2(k+1)}, 1_0^{N-4(k+1)})$, $\mathcal{O}(2_1^{l-r+1}, 1_0^{2r-2}) = \mathcal{O}(2_1^{2K+1}, 1_0^{N-4k-2})$, (resp. $\mathcal{O}(2_1^{l-r+2}, 1_1^{2r-4}) = \mathcal{O}(2_1^{2(k+1)}, 1_1^{N-4(k+1)})$). The same argument as for type B_l can be used to verify that for $w \in W$, $2|\langle w(\omega_k), \check{\alpha} \rangle$ for some $\alpha \in J$.

For type C_l when $r = -l + 2k - 1$ (i.e., $\frac{l}{2} < k \leq l$) we have $\mathcal{O}(2_0^{l-(r-1)}, 1_0^{2(r-1)}) = \mathcal{O}(2_0^{2(l-k+1)}, 1_0^{N-4(l-k+1)})$. From the tables in Proposition 3.1, we have $|J| = l - k + 1$. Since $\frac{l}{2} < k$, we have $|J| = l - k + 1 \leq k$ with equality holding if and only if $k = \frac{l+1}{2}$. If $|J| < k$ then we can use the same argument as in the type B_l case to deduce that condition (iii) of Theorem 2.7 holds. On the other hand, if $k = \frac{l+1}{2}$ then l must be odd and $l - (r-1) = 2(l-k+1) = l+1$. But in the case when l is odd the only allowable orbits $\mathcal{O}(2_0^{l-s}, 1_0^{2s})$ occur when $l - s < l$. That is, there are no constrictors in this case.

The remaining cases for D_l when $k = l-1,\ l$ can be easily verified using arguments similar to those given in the preceding paragraphs. □

4. Exceptional Lie algebras

In this section we compute the support varieties for the induced modules for exceptional Lie algebras. The strategy follows the one outlined in Section 2.8. Let us illustrate this with the following example.

Example. Let $\Phi = G_2$ and $p = 3$. The positive roots with their corresponding coroots are given in the following table. Note that $\Delta = \{\alpha_1, \alpha_2\}$ where α_1 is the short root.

root	coroot
α_1	$\check{\alpha}_1$
α_2	$\check{\alpha}_2$
$\alpha_1+\alpha_2$	$\check{\alpha}_1+3\check{\alpha}_2$
$2\alpha_1+\alpha_2$	$2\check{\alpha}_1+3\check{\alpha}_2$
$3\alpha_1+\alpha_2$	$\check{\alpha}_1+\check{\alpha}_2$
$3\alpha_1+2\alpha_2$	$\check{\alpha}_1+2\check{\alpha}_2$

Recall that $\Phi_\lambda = \{\alpha \in \Phi \mid \langle \lambda+\rho, \check{\alpha}\rangle \in p\mathbb{Z}\}$. The restricted region $X_1(T)$ contains nine weights and their stablizers are computed below.

λ	$\lambda+\rho$	Φ_λ
$(0,0)$	$(1,1)$	$\{\pm(3\alpha_1+2\alpha_2)\}$
$(1,0)$	$(2,1)$	$\{\pm(3\alpha_1+\alpha_2)\}$
$(2,0)$	$(3,1)$	$\{\pm\alpha_1, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$
$(0,1)$	$(1,2)$	$\{\pm(3\alpha_1+\alpha_2)\}$
$(1,1)$	$(2,2)$	$\{\pm(3\alpha_1+2\alpha_2)\}$
$(2,1)$	$(3,2)$	$\{\pm\alpha_1, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$
$(0,2)$	$(1,3)$	$\{\pm\alpha_2\}$
$(1,2)$	$(2,3)$	$\{\pm\alpha_2\}$
$(2,2)$	$(3,3)$	Φ

One can verify that there are 3 W-orbits in $X_1(T)$ under the dot action of the extended affine Weyl group, namely, $\{(0,0),(1,0),(0,1),(1,1),(0,2),(1,2)\}$, $\{(2,0),(2,1)\}$, and $\{(2,2)\}$. The classes have stabilizer Φ_λ isomorphic to A_1, A_2 and Φ respectively.

First assume that Φ_λ is isomorphic to A_1. Then $|\Phi|-|\Phi_\lambda| = 10$. The restricted nullcone $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(G_2(a_1))}$ by Theorem 2.1(B)(v), an irreducible 10-dimensional variety. Hence, $V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}(G_2(a_1))}$ by Corollary 2.5.

Next assume that Φ_λ is isomorphic to A_2. In this case $|\Phi|-|\Phi_\lambda| = 6$. There are two 6-dimensional G-orbits in $\mathcal{N}_1(\mathfrak{g})$. Consider $\mathcal{O} = \mathcal{O}(A_1)$ and its constrictor $\mathcal{O}_1 = \mathcal{O}(\widetilde{A}_1)$ as in Theorem 2.7. Condition (i) is satisfied because $\dim \mathcal{O} = 6$. Moreover, condition (ii) holds because $\mathcal{O}(\widetilde{A}_1) = G\cdot x_{\alpha_1}$. Finally, condition (iii) is satified by inspecting the table above for the weights $(2,0)$ and $(2,1)$. Hence, $V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}(A_1)}$ by Theorem 2.7.

When $\Phi_\lambda = \Phi$ then $\lambda = (2,2)$ and $H^0((2,2))$ is the Steinberg module for G_1. This module is projective so $V_{G_1}(H^0(\lambda)) = \{0\}$.

For the exceptional groups the computation of $V_{G_1}(H^0(\lambda))$ follows the paradigm given in the preceding example. The explicit calculation of $w(\Phi_\lambda)\cap\Phi_J$ was done with the help of GAP [Sch]. The following tables record our results. In the leftmost column, we give an orbit representative in $X_1(T)$ of $X(T)$ under the extended affine Weyl group. Our results show that $V_{G_1}(H^0(\lambda)) = V_{G_1}(H^0(w\cdot\lambda + p\nu))$ for all $\lambda, w\cdot\lambda + p\nu \in X(T)_+$ where $w \in W$ and $\nu \in X(T)$.

$E_6,\ \ p=2$

λ	$\|W\cdot\lambda\|$	Constrictors	$\|\Phi\| - \|\Phi_\lambda\|$	$V_{G_1}(H^0(w\cdot\lambda))$
$(1,0,1,1,1,1)$	36		40	$\overline{\mathcal{O}(3A_1)}$
$(0,1,1,1,1,1)$	27	$\mathcal{O}(3A_1)$	32	$\overline{\mathcal{O}(2A_1)}$
$(1,1,1,1,1,1)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$E_6,\ \ p=3$

λ	$\|W\cdot\lambda\|$	Constrictors	$\|\Phi\| - \|\Phi_\lambda\|$	$V_{G_1}(H^0(w\cdot\lambda))$
$(2,2,2,0,2,2)$	80		54	$\overline{\mathcal{O}(2A_2+A_1)}$
$(2,2,2,2,0,2)$	216	$\mathcal{O}(2A_2)$	50	$\overline{\mathcal{O}(A_2+2A_1)}$
$(2,2,0,2,2,2)$	216	$\mathcal{O}(2A_2)$	50	$\overline{\mathcal{O}(A_2+2A_1)}$
$(2,2,0,0,0,2)$	90	$\mathcal{O}(A_2+2A_1)$	48	$\overline{\mathcal{O}(2A_2)}$
$(2,0,2,2,2,2)$	72	$\mathcal{O}(A_2+A_1)$	42	$\overline{\mathcal{O}(A_2)}$
$(2,2,2,2,2,0)$	27	$\mathcal{O}(3A_1)$	32	$\overline{\mathcal{O}(2A_1)}$
$(0,2,2,2,2,2)$	27	$\mathcal{O}(3A_1)$	32	$\overline{\mathcal{O}(2A_1)}$
$(2,2,2,2,2,2)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$E_7,\ \ p=2$

λ	$\|W\cdot\lambda\|$	Constrictors	$\|\Phi\| - \|\Phi_\lambda\|$	$V_{G_1}(H^0(w\cdot\lambda))$
$(1,0,1,1,1,1,1)$	36		70	$\overline{\mathcal{O}(4A_1)}$
$(0,1,1,1,1,1,1)$	63	$\mathcal{O}((3A_1)')$	64	$\overline{\mathcal{O}((3A_1)'')}$
$(1,1,1,1,1,1,0)$	28	$\mathcal{O}((3A_1)'')$	54	$\overline{\mathcal{O}((3A_1)')}$
$(1,1,1,1,1,1,1)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$E_7,\ \ p=3$

λ	$\|W\cdot\lambda\|$	Constrictors	$\|\Phi\| - \|\Phi_\lambda\|$	$V_{G_1}(H^0(w\cdot\lambda))$
$(0,0,0,0,0,0,0)$	672		90	$\overline{\mathcal{O}(2A_2+A_1)}$
$(0,2,0,0,0,0,0)$	576	$\mathcal{O}(2A_2)$	84	$\overline{\mathcal{O}(A_2+3A_1)}$
$(0,0,0,1,0,1,0)$	756	$\mathcal{O}(A_2+3A_1)$	84	$\overline{\mathcal{O}(2A_2)}$
$(0,2,0,2,2,2,2)$	126	$\mathcal{O}((3A_1)'')$	66	$\overline{\mathcal{O}(A_2)}$
$(0,2,1,2,0,1,2)$	56	$\mathcal{O}((3A_1)')$	54	$\overline{\mathcal{O}((3A_1)'')}$
$(2,2,2,2,2,2,2)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$E_8, \;\; p = 2$

λ	$\lvert W \cdot \lambda \rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,1,1,1,1,1,1,1)$	135		128	$\overline{\mathcal{O}(4A_1)}$
$(1,1,0,1,1,1,1,1)$	120	$\mathcal{O}(4A_1)$	112	$\overline{\mathcal{O}(3A_1)}$
$(1,1,1,1,1,1,1,1)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$E_8, \;\; p = 3$

λ	$\lvert W \cdot \lambda \rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,0,0,0,0,0,0,0)$	1920		168	$\overline{\mathcal{O}(2A_2+2A_1)}$
$(0,0,1,0,0,0,0,0)$	2240	$\mathcal{O}(2A_2+2A_1)$	162	$\overline{\mathcal{O}(2A_2+A_1)}$
$(0,0,0,0,0,2,1,0)$	2160	$\mathcal{O}(2A_2+A_1)$	156	$\overline{\mathcal{O}(2A_2)}$
$(1,1,0,2,2,2,0,1)$	240	$\mathcal{O}(4A_1)$	114	$\overline{\mathcal{O}(A_2)}$
$(2,2,2,2,2,2,2,2)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$E_8, \;\; p = 5$

λ	$\lvert W \cdot \lambda \rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,0,0,0,0,0,0,0)$	48,384		200	$\overline{\mathcal{O}(A_4+A_3)}$
$(1,0,0,0,0,0,0,0)$	69,120	$\mathcal{O}(A_4+A_3)$	196	$\overline{\mathcal{O}(A_4+A_2+A_1)}$
$(3,1,1,1,1,1,1,1)$	69,120	$\mathcal{O}(A_4+A_3)$	196	$\overline{\mathcal{O}(A_4+A_2+A_1)}$
$(1,0,0,0,1,0,0,0)$	60,480	$\mathcal{O}(A_4+A_2+A_1)$	194	$\overline{\mathcal{O}(A_4+A_2)}$
$(3,1,1,1,3,1,1,1)$	60,480	$\mathcal{O}(A_4+A_2+A_1)$	194	$\overline{\mathcal{O}(A_4+A_2)}$
$(0,4,0,4,4,4,4,4)$	17,280	$\mathcal{O}(A_4)$, $\mathcal{O}(2A_3)$	184	$\overline{\mathcal{O}(D_4(a_1)+A_2)}$
$(1,4,1,4,4,4,4,4)$	17,280	$\mathcal{O}(A_4)$, $\mathcal{O}(2A_3)$	184	$\overline{\mathcal{O}(D_4(a_1)+A_2)}$
$(0,2,0,0,0,0,4,4)$	30,240	$\mathcal{O}(A_3+A_2+A_1)$	180	$\overline{\mathcal{O}(A_4)}$
$(0,4,0,0,1,2,1,4)$	6720	$\mathcal{O}(2A_2+2A_1)$	166	$\overline{\mathcal{O}(D_4(a_1))}$
$(1,4,1,1,3,0,3,4)$	6720	$\mathcal{O}(2A_2+2A_1)$	166	$\overline{\mathcal{O}(D_4(a_1))}$
$(0,4,1,1,0,1,4,4)$	2160	$\mathcal{O}(2A_2+A_1)$, $\mathcal{O}(A_3)$	156	$\overline{\mathcal{O}(2A_2)}$
$(1,4,3,3,1,3,4,4)$	2160	$\mathcal{O}(2A_2+A_1)$, $\mathcal{O}(A_3)$	156	$\overline{\mathcal{O}(2A_2)}$
$(0,4,0,3,4,4,4,4)$	240	$\mathcal{O}(4A_1)$	114	$\overline{\mathcal{O}(A_2)}$
$(1,4,1,2,4,4,4,4)$	240	$\mathcal{O}(4A_1)$	114	$\overline{\mathcal{O}(A_2)}$
$(4,4,4,4,4,4,4,4)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$F_4, \ \ p = 2$

λ	$\lvert W \cdot \lambda\rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,0,0,0)$	12		28	$\overline{\mathcal{O}(A_1 + \widetilde{A_1})}$
$(1,0,1,1)$	3	$\mathcal{O}(\widetilde{A}_1)$	16	$\overline{\mathcal{O}(A_1)}$
$(1,1,1,1)$	1	$\mathcal{O}(A_1)$, $\mathcal{O}(\widetilde{A}_1)$	0	$\{0\}$

$F_4, \ \ p = 3$

λ	$\lvert W \cdot \lambda\rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,0,0,0)$	32		36	$\overline{\mathcal{O}(A_1 + \widetilde{A}_2)}$
$(0,2,2,2)$	24	$\mathcal{O}(A_2)$	30	$\overline{\mathcal{O}(\widetilde{A}_2)}$
$(0,2,2,2)$	24	$\mathcal{O}(\widetilde{A}_2)$	30	$\overline{\mathcal{O}(A_2)}$
$(2,2,2,2)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$G_2, \ \ p = 2$

λ	$\lvert W \cdot \lambda\rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,0)$	3		8	$\overline{\mathcal{O}(\widetilde{A}_1)}$
$(1,1)$	1	$\mathcal{O}(A_1)$	0	$\{0\}$

$G_2, \ \ p = 3$

λ	$\lvert W \cdot \lambda\rvert$	Constrictors	$\lvert\Phi\rvert - \lvert\Phi_\lambda\rvert$	$V_{G_1}(H^0(w \cdot \lambda))$
$(0,0)$	6		10	$\overline{\mathcal{O}(G_2(a_1))}$
$(2,0)$	2	$\mathcal{O}(\widetilde{A}_1)$	6	$\overline{\mathcal{O}(A_1)}$
$(2,2)$	1	$\mathcal{O}(A_1)$, $\mathcal{O}(\widetilde{A}_1)$	0	$\{0\}$

5. Applications

In this section we prove several consequences of our support variety computations.

5.1. Dimension equality. Here we prove the dimension equality conjecture stated earlier as (1.4.1).

Theorem. *Let G be a simple algebraic group defined over an algebraically closed field of characteristic $p > 0$. Let $\lambda \in X(T)_+$. Then* $\dim V_{G_1}(H^0(\lambda)) = |\Phi| - |\Phi_\lambda|$.

Proof. For p good this is Theorem 1.4(b). For p bad it follows by observation from the tables in Sections 3 and 4. □

5.2. Irreducibility. For induced modules we can now show that their supports are always irreducible varieties.

Theorem. *Let G be a simple algebraic group defined over an algebraically closed field of characteristic $p > 0$. Let $\lambda \in X(T)_+$. Then $V_{G_1}(H^0(\lambda))$ is an irreducible variety.*

Proof. For p good this follows from Theorem 1.4(a). It is also true for p bad since, from the tables in Sections 3 and 4, $V_{G_1}(H^0(\lambda))$ is always the closure of a single G-orbit. □

5.3. Realization of orbit closures. Another open problem from [FP] is: For a simple algebraic group G, which G-stable, closed, conical subvarieties of $\mathcal{N}_1(\mathfrak{g})$ can be realized as $V_{G_1}(M)$ for some G-module M? Because the support of a direct sum is the union of the supports, it is enough to determine which G-orbit closures can be realized. In [Jan2], Jantzen showed that all orbit closures can be realized in type A and in type B_2 (when $p \neq 2$). It follows from [CLNP] that when p is good, the closure of every Richardson orbit in $\mathcal{N}_1(\mathfrak{g})$ can be realized as the support variety of an induced module. Recently, Nakano and Tanisaki [NT] have proved that when p is good, with the possible exception of a few non-Richardson orbits in type E, all other orbit closures can be realized.

We will employ the ideas used in [NT] with some modifications. Assume that $\widehat{\rho} : G \to \mathrm{GL}(V)$ is a representation such that the differential $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ satisfies the property that $\mathrm{Ker}\ \rho \cap \mathcal{N}(\mathfrak{g}) = \{0\}$ (here we are also viewing $\rho : G_1 \to \mathrm{GL}(V)$ via the equivalence between restricted representations of $\mathfrak{g}$ and representations of G_1). For any finite-dimensional $\rho(G_1)$-module M, we have

$$V_{\mathrm{GL}(V)_1}(M) \cap \rho(\mathfrak{g}) = V_{\rho(G_1)}(M).$$

Now M becomes a G_1-module by composing with ρ. Furthermore, ρ induces a natural map $\rho^* : V_{G_1}(M) \to V_{\rho(G_1)}(M)$ which is surjective. This map is injective because $\ker\ \rho \cap \mathcal{N}(\mathfrak{g}) = \{0\}$. Hence,

$$V_{\mathrm{GL}(V)_1}(M) \cap \rho(\mathfrak{g}) \cong V_{G_1}(M). \tag{5.3.1}$$

This isomorphism will be used to realize orbit closures as support varieties of G-modules.

Theorem. *Assume that p is a bad prime for a simple algebraic group G. Then the closure of every G-orbit in $\mathcal{N}_1(\mathfrak{g})$ is realized as $V_{G_1}(M)$ for some G-module M, with the following possible exceptions.*

1. *Type C_l: $\mathcal{O}(2_0^k 1_0^{2(n-k)})$ for $k \in \mathbb{N}$ even ($p = 2$).*
2. *Type F_4: $\mathcal{O}(\widetilde{A}_1)$ ($p = 2$).*
3. *Type G_2: $\mathcal{O}(\widetilde{A}_1)$ ($p = 3$).*

Proof. We use the following principle, [UGA1, Prop. 2.4]. Assume $\rho\colon G \to \mathrm{GL}(V)$ is a representation satisfying $\mathrm{Ker}\ \rho \cap \mathcal{N}(\mathfrak{g}) = \{0\}$. Given a G-orbit $\mathcal{O} \subset \mathcal{N}(\mathfrak{g})$, we identify $\mathcal{O}$ with $\rho(\mathcal{O})$ (and similarly for $\overline{\mathcal{O}}$), and let $\lambda(\mathcal{O})$ be the Jordan block partition associated with $\rho(\mathcal{O})$. Let $\mathcal{O}_{\lambda(\mathcal{O})}$ denote the corresponding $\mathrm{GL}(V)$ orbit in $\mathfrak{gl}(V)$. Then

$$\overline{\mathcal{O}_{\lambda(\mathcal{O})}} \cap \rho(\mathfrak{g}) = \bigcup \overline{\mathcal{O}'}, \tag{5.3.2}$$

where the union is over all G-orbits $\mathcal{O}' \subseteq \mathcal{N}_1(\mathfrak{g})$ which are maximal with respect to $\preceq$ (the inclusion of orbit closures) and satisfy $\lambda(\mathcal{O}') \trianglelefteq \lambda(\mathcal{O})$ (where $\trianglelefteq$ is the dominance ordering on partitions). In particular, if

$$\mathcal{O}' \not\preceq \mathcal{O} \implies \lambda(\mathcal{O}') \ntrianglelefteq \lambda(\mathcal{O}), \tag{5.3.3}$$

then $\overline{\mathcal{O}_{\lambda(\mathcal{O})}} \cap \rho(\mathfrak{g}) = \overline{\mathcal{O}}$. In this case, by [NT] and (5.3.1) there is a $\mathrm{GL}(V)$-module M such that

$$\overline{\mathcal{O}} = \overline{\mathcal{O}_{\lambda(\mathcal{O})}} \cap \rho(\mathfrak{g}) = V_{\mathrm{GL}(V)_1}(M) \cap \rho(\mathfrak{g}) = V_{G_1}(M|_{G_1}),$$

whence $\overline{\mathcal{O}}$ is realized as a support variety of a G-module. In each case there is precisely one orbit $\mathcal{O}'$ besides $\mathcal{O}$ on the right side of (5.3.2).

Begin with the exceptional groups and let ρ be the adjoint representation. Using the tables of adjoint partitions associated to the G-orbits in $\mathcal{N}_1(\mathfrak{g})$ from [UGA2, §6] along with the Hasse diagrams giving the order $\preceq$ (see Section 6), we find that there are four cases where (5.3.3) does not hold (and $\mathcal{O}$ is not already realized as the support of an induced module). These are tabulated below.

Type	p	$\mathcal{O}$	$\mathcal{O}'$	$\lambda(\mathcal{O})$	$\lambda(\mathcal{O}')$
F_4	2	$\mathcal{O}(\widetilde{A}_1)$	$\mathcal{O}(A_1)$	$(2^{16}, 1^{20})$	$(2^{16}, 1^{20})$
F_4	3	$\mathcal{O}(A_2 + \widetilde{A}_1)$	$\mathcal{O}(A_1 + \widetilde{A}_2)$	$(3^{16}, 2^2)$	$(3^{16}, 2^2)$
G_2	2	$\mathcal{O}(A_1)$	$\mathcal{O}(\widetilde{A}_1)$	$(2^6, 1^2)$	$(2^6, 1^2)$
G_2	3	$\mathcal{O}(\widetilde{A}_1)$	$\mathcal{O}(A_1)$	$(3^3, 1^5)$	$(3, 2^4, 1^3)$

But we can investigate these same orbits $\mathcal{O}$ under the minimal representation ρ, using the orbit representatives in [UGA2, §5] and MAGMA [BC, BCP]. The Jordan block partitions are given in the next table. (Note that in F_4, $p = 3$ we have replaced $\mathcal{O}' = A_1 + \widetilde{A}_2$ by the smaller orbit $\widetilde{A}_2$.)

Type	p	$\mathcal{O}$	$\mathcal{O}'$	$\lambda(\mathcal{O})$	$\lambda(\mathcal{O}')$
F_4	2	$\mathcal{O}(\widetilde{A}_1)$	$\mathcal{O}(A_1)$	$(2^{10}, 1^6)$	$(2^6, 1^{14})$
F_4	3	$\mathcal{O}(A_2 + \widetilde{A}_1)$	$\mathcal{O}(\widetilde{A}_2)$	$(3^7, 2^2, 1)$	$(3^8, 2)$
G_2	2	$\mathcal{O}(A_1)$	$\mathcal{O}(\widetilde{A}_1)$	$(2^2, 1^3)$	$(2^3, 1)$
G_2	3	$\mathcal{O}(\widetilde{A}_1)$	$\mathcal{O}(A_1)$	$(3, 2^2)$	$(2^2, 1^3)$

In the two middle rows of the table, $\lambda(\mathcal{O}') \ntrianglelefteq \lambda(\mathcal{O})$, so by the discussion at the beginning of the proof, $\overline{\mathcal{O}}$ is realized as a support variety in those two cases, leaving $\widetilde{A}_1$ in F_4, $p = 2$ and in G_2, $p = 3$ as the only exceptional orbits possibly not realized as support varieties of G-modules in bad characteristic.

For the classical groups, let ρ be the standard representation V with $\dim V = N$. Assume first that we are in type B_l, where $N = 2l + 1$. The orbit closures not realized as support varieties of induced modules are parametrized by symbols $\mu_\chi = (2_2^k, 1_1^{N-2k}),\ k \geq 1$. Recall that the associated Jordan block partition for the minimal representation is obtained from μ_χ by deleting the subscripts χ; i.e., it is simply μ. Observe from the Hasse diagram that condition (5.3.3) is satisfied for these orbits. Thus they are all realized as supports of restrictions to G of $\mathrm{GL}(N)$-modules.

Next consider type D_l, where $N = 2l$. The orbit closures not realized as support varieties of induced modules are parametrized by symbols $\mu_\chi = (2_1^k, 1_1^{N-2k}),\ 2 \le k \le l-2,\ k$ even. Here condition (5.3.3) is not satisfied, but we can use another trick. Assume for definiteness that l is even and $G = \mathrm{O}(2l)$. There are G-modules M and N whose support varieties are $\overline{\mathcal{O}(2_1^l)}$ and $\overline{\mathcal{O}(2_2^{l-2}, 1_1^4)}$, respectively. Now by a basic property of support varieties,

$$V_{G_1}(M \otimes N) = V_{G_1}(M) \cap V_{G_1}(N) = \overline{\mathcal{O}(2_1^{l-2}, 1_1^4)}.$$

Iterating this process, we realize all the remaining orbit closures in type D. The argument is identical, with a shift in indices, if l is odd. If $G = \mathrm{SO}(2l)$ and l is even then the $\mathrm{O}(2l)$-orbit $\mathcal{O}(2_1^l)$ splits into a union of two G-orbits, both of whose closures are support varieties of induced G-modules, and both of which contain $\mathcal{O}(2_1^{l-2}, 1_1^4)$, so the same argument works. □

Remark. In type C the argument used for type B does not apply because each "unrealized" orbit is dominated (in the closure ordering) by an orbit having the same partition. And the argument used for type D does not work because we do not know that $\overline{\mathcal{O}(2_0^{l-\varepsilon}, 1_0^{2\varepsilon})}$ is realized (where $\varepsilon = 0$ if l is even and $\varepsilon = 1$ if l is odd). However, if that orbit *were* realized, then all the remaining ones in type C would be, too, by the same tensor product construction used in type D.

5.4. Richardson orbits. When p is a good prime, it follows from the solution to the Jantzen conjecture on support varieties [NPV] that $V_{G_1}(H^0(\lambda))$ is the closure of a Richardson orbit, and every Richardson orbit closure in $\mathcal{N}_1(\mathfrak{g})$ is realized in this fashion. We now consider to what extent these statements are true for bad primes.

Theorem. *Let G be a simple algebraic group defined over an algebraically closed field of bad characteristic p. Let $\mathcal{O}$ be a Richardson orbit in $\mathcal{N}_1(\mathfrak{g})$. Then there exists $\lambda \in X(T)_+$ such that $V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}}$, except when $\mathcal{O} = \mathcal{O}(2_2^2, 1_1^{2l-3})$ in type B_l.*

Proof. Recall that $\mathcal{O}$ is Richardson if and only if $\overline{\mathcal{O}} = G \cdot \mathfrak{u}_J$ for some subset $J \subset \Delta$. Also $\dim G \cdot \mathfrak{u}_J = 2 \dim \mathfrak{u}_J = |\Phi| - |\Phi_J|$. The zero orbit is trivially Richardson (corresponding to $J = \varnothing$), and is also the support variety of the Steinberg module, so we assume henceforth that $\mathcal{O}$ is not the zero orbit and $J \ne \varnothing$.

Assume first that G is classical, so $p = 2$. The formula $(x+y)^{[2]} = x^{[2]} + [x,y] + y^{[2]}$ (cf. [NPV, Section 6.3]) implies that $\mathfrak{u}_J \subset \mathcal{N}_1(\mathfrak{g})$ if and only if $\mathfrak{u}_J$ is abelian. One checks that $\mathfrak{u}_J$ is abelian (in characteristic 2) if and only if $\Delta - J$ is a single simple root at an "end" of the Dynkin diagram (i.e., a simple root α such that $\langle \alpha, \beta \rangle \ne 0$ for exactly one other simple root β). Comparing $|\Phi| - |\Phi_J|$ for these J with the dimensions of the orbits from Hesselink (see Section 3), there is always a unique orbit of the correct dimension. The orbit parameters for the nonzero Richardson orbits are given in the next table.

Type	Richardson Orbits
B_l	$\mathcal{O}(2_2^2, 1_1^{2l-3})$, $\mathcal{O}(2_2^l, 1_1^1)$
C_l	$\mathcal{O}(2_1^2, 1_0^{2l-4})$, $\mathcal{O}(2_1^l)$
D_l, l even	$\mathcal{O}(2_2^2, 1_1^{2l-4})$, $\mathcal{O}(2_1^l)$
D_l, l odd	$\mathcal{O}(2_2^2, 1_1^{2l-4})$, $\mathcal{O}(2_1^{l-1}, 1_1^2)$

(Recall that for l even, the $\mathrm{O}(2l)$ orbit $\mathcal{O}(2_1^l)$ splits into two $\mathrm{SO}(2l)$ orbits; these are both Richardson.) The theorem now follows in the classical cases by comparison with the circled orbits in Figures 1 and 2.

Now assume G is exceptional. The Richardson orbits in characteristic zero are given by Hirai in [Hi]. None of these orbits (in $\mathcal{N}_1(\mathfrak{g})$) "split" in bad characteristic; therefore the Richardson orbits in characteristic p are given by the same parameters. The list of nontrivial Richardson orbits in $\mathcal{N}_1(\mathfrak{g})$ is given in the following table.

Type	Richardson Orbits
E_6	$\mathcal{O}(2A_1)$, $\mathcal{O}(A_2)$, $\mathcal{O}(A_2+2A_1)$, $\mathcal{O}(2A_2)$
E_7	$\mathcal{O}((3A_1)'')$, $\mathcal{O}(A_2)$, $\mathcal{O}(A_2+3A_1)$, $\mathcal{O}(2A_2)$
E_8	$\mathcal{O}(A_2)$, $\mathcal{O}(2A_2)$, $\mathcal{O}(D_4(a_1))$, $\mathcal{O}(A_4)$, $\mathcal{O}(D_4(a_1)+A_2)$, $\mathcal{O}(A_4+A_2)$, $\mathcal{O}(A_4+A_2+A_1)$
F_4	$\mathcal{O}(A_2)$, $\mathcal{O}(\widetilde{A}_2)$
G_2	$\mathcal{O}(G_2(a_1))$

Comparison with Figures 3–7 shows that the closures of all these orbits arise as support varieties of induced modules. □

Remark. It is clear from the data that, in contrast to the situation when p is good, many non-Richardson orbit closures arise as support varieties of induced modules when p is bad.

6. Hasse diagrams

In this section, we provide the Hasse diagrams for $\mathcal{N}_1(\mathfrak{g})$ when k is a field of bad characteristic for Φ. The orbits whose closures are support varieties of induced/Weyl modules are circled (cf. Sections 3.6 and 4).

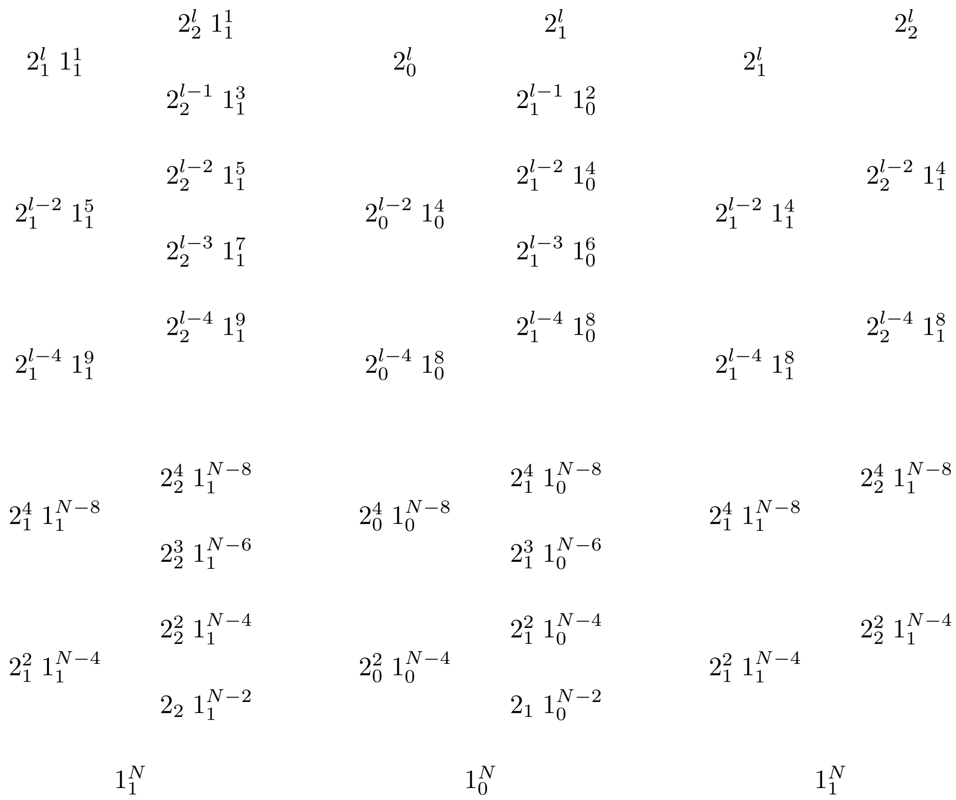

FIGURE 1. $\mathrm{O}(2l+1)$, $\mathrm{Sp}(2l)$, $\mathrm{O}(2l)$ for l even

	$2_2^l\ 1_1^1$		2_1^l		
	$2_2^{l-1}\ 1_1^3$		$2_1^{l-1}\ 1_0^2$		$2_2^{l-1}\ 1_1^2$
$2_1^{l-1}\ 1_1^3$		$2_0^{l-1}\ 1_0^2$		$2_1^{l-1}\ 1_1^2$	
	$2_2^{l-2}\ 1_1^5$		$2_1^{l-2}\ 1_0^4$		
	$2_2^{l-3}\ 1_1^7$		$2_1^{l-3}\ 1_0^6$		$2_2^{l-3}\ 1_1^6$
$2_1^{l-3}\ 1_1^7$		$2_0^{l-3}\ 1_0^6$		$2_1^{l-3}\ 1_1^6$	
	$2_2^{l-4}\ 1_1^9$		$2_1^{l-4}\ 1_0^8$		
	$2_2^{l-5}\ 1_1^{11}$		$2_1^{l-5}\ 1_0^{10}$		$2_2^{l-5}\ 1_1^{10}$
$2_1^{l-5}\ 1_1^{11}$		$2_0^{l-5}\ 1_0^{10}$		$2_1^{l-5}\ 1_1^{10}$	
	$2_2^4\ 1_1^{N-8}$		$2_1^4\ 1_0^{N-8}$		$2_2^4\ 1_1^{N-8}$
$2_1^4\ 1_1^{N-8}$		$2_0^4\ 1_0^{N-8}$		$2_1^4\ 1_1^{N-8}$	
	$2_2^3\ 1_1^{N-6}$		$2_1^3\ 1_0^{N-6}$		
	$2_2^2\ 1_1^{N-4}$		$2_1^2\ 1_0^{N-4}$		$2_2^2\ 1_1^{N-4}$
$2_1^2\ 1_1^{N-4}$		$2_0^2\ 1_0^{N-4}$		$2_1^2\ 1_1^{N-4}$	
	$2_2\ 1_1^{N-2}$		$2_1\ 1_0^{N-2}$		
1_1^N		1_0^N		1_1^N	

FIGURE 2. $\mathrm{O}(2l+1)$, $\mathrm{Sp}(2l)$, $\mathrm{O}(2l)$ for l odd

$2A_2 + A_1$

$A_2 + 2A_1$ $\quad$ $2A_2$

$A_2 + A_1$

A_2

$3A_1$ $\quad$ $3A_1$

$2A_1$ $\quad$ $2A_1$

A_1 $\quad$ A_1

$\varnothing$ $\quad$ $\varnothing$

FIGURE 3. E_6 for $p = 2$ (left) and $p = 3$ (right)

$2A_2 + A_1$

$A_2 + 3A_1$ $\quad$ $2A_2$

$A_2 + 2A_1$

$A_2 + A_1$

$4A_1$ $\quad$ $4A_1$ $\quad$ A_2

$(3A_1)''$ $\quad$ $(3A_1)'$ $\quad$ $(3A_1)''$ $\quad$ $(3A_1)'$

$2A_1$ $\quad$ $2A_1$

A_1 $\quad$ A_1

$\varnothing$ $\quad$ $\varnothing$

FIGURE 4. E_7 for $p = 2$ (left) and $p = 3$ (right)

		$A_4 + A_3$
		$A_4 + A_2 + A_1$
		$A_4 + A_2$
		$A_4 + 2A_1$
		$A_4 + A_1$, $2A_3$
		$D_4(a_1) + A_2$
		A_4
		$A_3 + A_2 + A_1$
		$A_3 + A_2$
		$D_4(a_1) + A_1$
		$D_4(a_1)$, $A_3 + 2A_1$
	$2A_2 + 2A_1$	$A_3 + A_1$, $2A_2 + 2A_1$
	$2A_2 + A_1$	$2A_2 + A_1$
	$2A_2$	A_3, $2A_2$
	$A_2 + 3A_1$	$A_2 + 3A_1$
	$A_2 + 2A_1$	$A_2 + 2A_1$
	$A_2 + A_1$	$A_2 + A_1$
$4A_1$	$4A_1$, A_2	$4A_1$, A_2
$3A_1$	$3A_1$	$3A_1$
$2A_1$	$2A_1$	$2A_1$
A_1	A_1	A_1
$\varnothing$	$\varnothing$	$\varnothing$

FIGURE 5. E_8 for $p = 2$ (left), $p = 3$, (center), and $p = 5$ (right)

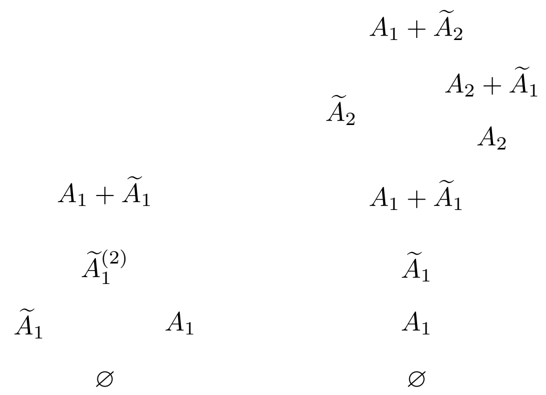

FIGURE 6. F_4 for $p = 2$ (left) and $p = 3$ (right)

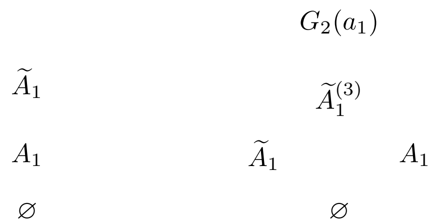

FIGURE 7. G_2 for $p = 2$ (left) and $p = 3$ (right)

7. VIGRE Algebra Group at the University of Georgia

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