The Dade group of a fusion system

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Outline

1. The original Dade group
2. The new Dade group
3. Results
4. Examples
5. Open problems

Setting

- p a prime;
- $k = \bar{k}$ a field of characteristic p;
- P a finite p-group;
- kP-module = finitely generated left kP-module;
- $M, \; N \;\; kP$-modules:
  - $M^* = \mathrm{Hom}_k(M, k)$ with $(u \cdot \varphi)(m) = \varphi(u^{-1} \cdot m)$;
  - $M \otimes N = M \otimes_k N$ with $u \cdot (m \otimes n) = (u \cdot m) \otimes (u \cdot n)$.

Recall

$\mathrm{Hom}_k(M, N) \cong M^* \otimes N \Longrightarrow \mathrm{End}_k M \cong M^* \otimes M$

Dade group

Let M be a kP-module.

- M is a cep=capped endo-permutation module if $\operatorname{End}_k M \cong k \oplus$ (perm) .
- M is et=endotrivial if $\operatorname{End}_k M \cong k \oplus$ (proj) .
- The Dade group of P is $D(P) = \{$equivalence classes of cep$\}$.

$$[M] + [N] = [M \otimes N]\,, \quad 0 = [k]\,, \quad -[M] = [M^*].$$

- The group of endotrivial modules is
  $T(P) = \{[M] \in D(P) \mid M \sim N \text{ for some et module } N\}.$

Operations

Let $f: Q \to P$. Restriction along f induces $\mathrm{Res}_f : D(P) \to D(Q)$. (For $u \in Q,\ x \in M$, we set $u \cdot x = f(u) \cdot x$.)

Assume $Q, N \leq P$ with $N \trianglelefteq P$.

- Restriction $\mathrm{Res}^P_Q : D(P) \to D(Q)$ for $f: Q \hookrightarrow P$;
- Inflation $\mathrm{Inf}^P_{P/N} : D(P/N) \to D(P)$ for $f: P \twoheadrightarrow P/N$.

There are converses:

- Tensor induction $\mathrm{Ten}^P_Q : D(Q) \to D(P)$;
- Deflation $\mathrm{Def}^P_{P/N} : D(P) \to D(P/N)$.

Compositions

If $Q \triangleleft R \leq P$, set

$$\mathrm{Defres}^P_{R/Q} = \mathrm{Def}^R_{R/Q} \circ \mathrm{Res}^P_R \quad \text{and} \quad \mathrm{Teninf}^P_{R/Q} = \mathrm{Ten}^P_R \circ \mathrm{Inf}^R_{R/Q} \ .$$

The example(s)

If $Q \leq P$ then $[\Omega_{P/Q}] \in D(P)$ where

$$\Omega_{P/Q} = \mathrm{Ker}(\varepsilon) \qquad \begin{array}{rcl} \varepsilon : k[P/Q] & \to & k \\ uQ & \mapsto & 1,\ \forall\ uQ \in P/Q\ . \end{array}$$

In particular, $\Omega_P \in T(P)$.
More generally, for a finite P-set X, Ω_X is a cep.
We have, for $N \trianglelefteq R \leq P$:

- $\mathrm{Res}^P_R[\Omega_X] = [\Omega_{\mathrm{Res}^P_R X}] \in D(R)$ for a P-set X;
- $\mathrm{Inf}^R_{R/N}[\Omega_X] = [\Omega_{\mathrm{Inf}^R_{R/N} X}] \in D(R)$, for a R/N-set X;
- $\mathrm{Ten}^P_R[\Omega_X] = \ldots$ tedious $\mathbb{Z}$-lin. com. of $[\Omega_{P/X_i}] \in D(P)$ for a R-set X;
- $\mathrm{Def}^R_{R/N}[\Omega_X] = [\Omega_{X^N}] \in D(R/N)$ for a R-set X.

Structure

- Et are building bricks for cep if P is abelian (Dade, 1978).
- $D(P)$ is finitely generated (Puig, 1990).
- $2D^t(P) = 0$ if $p > 2$ or $4D^t(P) = 0$ (Carlson-Thévenaz, 2005).
- $T(P) = \langle$ generators $|$ relations $\rangle$ (Carlson-Thévenaz, 2004).[1]
- If $p \neq 2$ then $D(P) = \langle \Omega_{P/Q} \mid Q \leq P +$ relations$\rangle$ and if $p = 2$ then $D(P) = \langle$ gens $|$ rels$\rangle$ (Bouc, 2006).
- Detection: If $p \neq 2$ then there is a sectional characterization of $D(P)$ (Bouc-Thévenaz, 2008).
- Gluing:
  - abelian P: Puig, 1991;
  - $p \neq 2$, torsion cep: Bouc-Thévenaz, preprint;
  - $p \neq 2$, general case: Bouc, in preparation.

[1] publication 'bug'

Fusion systems

A fusion system on P is a category $\mathcal{F}$ with:

- objects: $Q \leq P$;
- morphisms: $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R) \subseteq \mathrm{Inj}(Q, R)$ + list of axioms.

The model

$G =$ finite group with P as Sylow p-subgroup.
$g \in G$ s.t. $gQg^{-1} \leq R \leq P$ induces $c_g \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ (conjugation).
We write $\mathcal{F} = \mathcal{F}_P(G)$.

If $\mathcal{F} \neq \mathcal{F}_P(G)$, we call $\mathcal{F}$ exotic.

$\mathcal{F} + D(P) = ?$

Definition

The *Dade group of* $\mathcal{F}$ is

$$D(\mathcal{F}) = \lim_{\overleftarrow{\mathcal{F}}} \mathbf{D} \qquad \begin{array}{rcl} \mathbf{D}: \quad \mathcal{F} & \rightarrow & \mathbb{Z}\text{-mod} \\ Q & \rightarrow & D(Q) \\ (\varphi: \ Q \rightarrow R) & \rightarrow & (\mathrm{Res}_{\varphi}: D(R) \rightarrow D(Q)) \end{array}$$

Note: $\mathbf{D}$ is a contravariant functor.

Down-to-earth

- $D(\mathcal{F}) \cong \{x \in D(P) \mid \ x$ is $\mathcal{F}$-stable $\} \subseteq D(P)$.
  ($x \in D(P)$ is $\mathcal{F}$-stable if $\mathrm{Res}_{\varphi} x = \mathrm{Res}^P_Q x, \ \forall \ \varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$)
- $D(P) \cong D(\mathcal{F}_P(P))$.

Motivation

- The cep are sources of simple modules in 'many' cases.
- $T(G)$ is defined for any finite group G, and $T(G) \subseteq (\mathbf{stmod}(kG))^{\times}$. Relationship $T(G) \leftrightarrow T(\mathcal{F}_P(G))$?
- Block theory and the gluing problem.
- Categorical approach is 'better'.

$\mathcal{F}$-stability

Assume M is a $\mathcal{F}$-stable cep, that is, $\mathrm{Res}_{\varphi} M \cong \mathrm{Res}_Q^P M$, for all morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and all $Q \leq P$. Then $[M] \in D(\mathcal{F})$. But...

Question:

For $x \in D(\mathcal{F})$ is there $M \in x$ which is $\mathcal{F}$-stable?

Answer:

Yes: take any $N \in x$ and set $M = kX \otimes_{kP} N$, where $X = \mathcal{F}$-stable BLO biset.

Detection

Objective

Determine a family $\mathcal{A}$ of p-groups such that $D(\mathcal{F})$ is detected by $\mathrm{Res}_{\mathcal{A}}$, i.e.

$$\mathrm{Res}_{\mathcal{A}} = \prod_{Q \in \mathcal{A}} \mathrm{Defres}^P_Q \; : \; D(\mathcal{F}) \to \prod_{Q \in \mathcal{A}} D(Q) \quad \text{is injective.}$$

The smaller $\mathcal{A}$ is, the better.

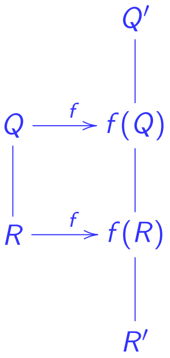
More categories

- $\mathcal{E}_{\geq 2}(\mathcal{F}) =$ full subcategory of $\mathcal{F}$.
  - Objects: the elementary abelian subgroups of P of order $\geq p^2$.
- $\mathcal{F}_{\mathcal{X}} =$ category with
  - Objects: sections (Q, R) of P with $|Q/R| \leq p^3$ and exponent $\leq p$.
  - A morphism $f : (Q, R) \to (Q', R')$ is

$$f \in \mathrm{Hom}_{\mathcal{F}}(Q, Q') \quad \text{s.t.} \quad R' \leq f(R) \ .$$

Write $\mathcal{E}_{\geq 2}(P)$ and $P_{\mathcal{X}}$ if $\mathcal{F} = \mathcal{F}_P(P)$.

Picture of a morphism in $\mathcal{F}_{\mathcal{X}}$:



$R \lhd Q$, $R' \lhd Q'$ and

Q/R , Q'/R' '*small*'.

Yet another Dade functor

Definition

Set $\underleftarrow{\mathbf{D}}(\mathcal{F}) = \varprojlim_{\mathcal{F}_{\mathcal{X}}} \mathbf{D}$, where $\mathbf{D}(Q,R) = D(Q/R)$, for all $(Q,R) \in \mathcal{F}_{\mathcal{X}}$.

Write $\underleftarrow{\mathbf{D}}(P)$ if $\mathcal{F} = \mathcal{F}_P(P)$.

Theorem (Bouc-Thévenaz, 2008)

If p is odd, then the deflation-restriction map induces an isomorphism

$$\mathrm{Defres}_{P_{\mathcal{X}}} \; : \; D(P) \xrightarrow{\cong} \underleftarrow{\mathbf{D}}(P) \; .$$

What for an arbitrary $\mathcal{F}$?

Proposition (L-M, 2008)

The diagram

$$\begin{array}{ccc}
D(\mathcal{F}) & \xrightarrow{\mathrm{Defres}_{\mathcal{F}\mathcal{X}}} & \underset{\leftarrow}{\mathbf{D}}(\mathcal{F}) \\
\big\downarrow & & \big\downarrow \\
D(P) & \xrightarrow{\mathrm{Defres}_{P\mathcal{X}}} & \underset{\leftarrow}{\mathbf{D}}(P)
\end{array}$$

is a pull-back diagram.

Theorem (L-M, 2008)

If p is odd then the deflation-restriction map induces an isomorphism

$$\mathrm{Defres}_{\mathcal{F}\mathcal{X}} \ : \ D(\mathcal{F}) \xrightarrow{\cong} \underset{\leftarrow}{\mathbf{D}}(\mathcal{F}) \ .$$

The gluing problem

- Given:
  - for each $1 < Q \leq P$ an element $x_Q \in D(N_P(Q)/Q)$, such that:
  - $(x_Q)_Q \in \varprojlim_{1<Q\leq P} \mathbf{D}$.
- Determine:
  - is there $x \in D(P)$ such that $x_Q = \mathrm{Defres}^P_{N_P(Q)/Q}\, x$, for all $1 < Q \leq P$?
  - is there a general formula for x?
  - if x exists, is x unique?
- What for?
  - local analysis of the structure of blocks of group algebras.

Remark

This problem generalizes to $\mathcal{F}$ instead of $\mathcal{F}_P(P)$.

Partial answers

- Puig, 1991: P abelian – explicit formula for x.
- Bouc-Thévenaz: p odd and all the $x_Q \in D^t(P)$.

Theorem (Bouc-Thévenaz, 2008)

If p is an odd prime and P a non cyclic finite p-group[a]*, there is a short exact sequence of $\mathbb{F}_2$-vector spaces*

$$0 \longrightarrow D^t(P) \xrightarrow{\mathrm{Defres}_{P^1_{\mathcal{X}}}} \varprojlim_{P^1_{\mathcal{X}}} \mathbf{D}^t \xrightarrow{\Phi_P} (\mathbb{F}_2)^n \longrightarrow 0 \quad , \text{ where}$$

- $P^1_{\mathcal{X}}$ *is the full subcategory of* $P_{\mathcal{X}}$ *with objects the sections* (Q, R) *with* $1 \neq R$ *and* $\mathbf{D}^t(Q, R) = D^t(Q/R)$;
- $n + 1 = \sharp$ *connected components of* $\mathcal{E}_{\geq 2}(P)$;
- Φ_P *explicitly constructed.*

[a] If P is cyclic, then cf. Puig

Partial answers cont'd

- L-M: fusion-version of B-T as follows

Theorem (L-M: 2008)

Assume p is odd and and P is a non cyclic finite p-group. Let $\mathcal{F}^1_{\mathcal{X}}$ be the full subcategory of $\mathcal{F}_{\mathcal{X}}$ with objects the sections (Q, R) with $1 \neq R$ and let $\mathbf{D}^t(Q, R) = D^t(Q/R)$. Then there is a short exact sequence of $\mathbb{F}_2$-vector spaces

$$0 \longrightarrow D^t(\mathcal{F}) \xrightarrow{\mathrm{Defres}_{\mathcal{F}^1_{\mathcal{X}}}} \varprojlim_{\mathcal{F}^1_{\mathcal{X}}} \mathbf{D}^t \longrightarrow (\mathbb{Z}/2)^{n_{\mathcal{F}}} \longrightarrow 0 \quad ,$$

where $n_{\mathcal{F}} + 1 = \sharp$ connected components of $\mathcal{E}_{\geq 2}(\mathcal{F})$.

Trivial situations

Throughout: $\mathcal{F}$ is a fusion system on a finite p-group P.

Theorem (L.-M., 2008)

If $p = 2$ and the only $\mathcal{F}$-essential subgroups of P are Klein four groups or quaternion groups of order 8, then $D(\mathcal{F}) = D(P)$.

Proof.

AFT & all the elements of $D(P)$ are $\mathcal{F}$-stable. □

Examples of such 2-groups

Cyclic, dihedral, semi-dihedral, quasi-dihedral, generalized quaternion 2-groups.

Classical and exotic examples

Classical example

p odd, $G = \mathrm{PSL}_3(p)$ and $\mathcal{F} = \mathcal{F}_P(G)$, with $P \cong p_+^{1+2}$. Then,

$$D(P) \cong \mathbb{Z}^{p+2} \oplus (\mathbb{Z}/2)^{p+2}\ ,\ D(\mathcal{F}) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2\ ,\ \text{and}\ T(\mathcal{F}) \cong \mathbb{Z}^3.$$

More precisely, $D(\mathcal{F}) = \langle \text{gens \& rels} \ldots \rangle$ and $T(\mathcal{F}) = \langle \text{gens \& rel} \ldots \rangle$.

Exotic example

Let $p = 7$, $P \cong 7_+^{1+2}$ and $\mathcal{F}$ be the exotic fusion system on P (Ruiz-Viruel). Then,

$$D(\mathcal{F}) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3.$$

More precisely, $D(\mathcal{F}) = \langle \text{gens \& rels} \ldots \rangle$ and $T(\mathcal{F}) = \langle \text{gens \& rel} \ldots \rangle$.

Conjectures and dreams

- Gluing problem: puzzling for about 20 years!
- Higher limits of the Dade functor: only conjectures. . . w.i.p.
- Endotrivial modules and cohomology ring: link $T(G) \leftrightarrow T(\mathcal{F}_P(G))$?.
- Still many more to come. . .

Thanks!