AUSLANDER-REITEN COMPONENTS FOR SOME KNÖRR LATTICES

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1. Introduction

Let G be a finite group, and let R be the valuation ring of an algebraic extension K of $\mathbb{Q}_p$, the p-adic completion of the rational numbers. We shall be interested in RG-lattices, namely finitely generated RG-modules which are free as R-modules.

Among the most important RG-lattices are the Knörr lattices. These were introduced by Knörr in [13], where they are referred to as "virtually irreducible lattices." A striking connection is drawn there to a well known conjecture of Brauer, the height zero conjecture. Readers unfamiliar with this connection are referred to [13]. One purpose here is to give a sufficient condition for the almost split sequence terminating in a Knörr lattice to have an indecomposable middle term.

Before recalling the definition of a Knörr lattice, we fix some notation. First, p^a will denote the exact power of p dividing $|G|$. If M is an indecomposable RG-lattice, then D will denote a defect group of the block of G containing M, and p^d will denote the size of D. Writing the exact power of p dividing $\text{rank}(M)$ as p^{a-d+h}, it is known that the integer h, called the height of M, is non-negative. Observe that if G is a p-group, then $G = D$, and then the height of M is simply the exponent in the power of p dividing $\text{rank}(M)$. We also assume that R is a discrete valuation ring with uniformizer π and ramification index e, so that $(\pi^e) = (p)$, and that the residue field, k, of R is algebraically closed.

Continuing with the RG-lattice M, let $A = \text{End}_R(M)$. Identify A^G, the G-fixed points of A, as $\text{End}_{RG}(M)$. Consider now the trace map

$$\text{trace} : A \to R.$$

Following Thévenaz [14], we say that M is a Knörr lattice if M is indecomposable and

$$\text{trace}(J(A^G)) \neq \text{trace}(A^G),$$

where $J(A^G)$ denotes the Jacobson radical. In words, the traces of the invertible endomorphisms of M must generate a strictly larger ideal of R than the traces of the non-invertible endomorphisms. It is easy to see that absolutely irreducible lattices as well as lattices of p' rank are Knörr lattices. A main result of [13], different proofs for which are given in [2] and [14], says that if M is a Knörr lattice, then every source of M is also a Knörr lattice.

Turning now to almost split sequences, it is shown in [9] that such a sequence terminating in a non-projective absolutely irreducible lattice which remains indecomposable (mod π) has a middle term whose projective-free part is indecomposable.

The first main result here, proved in Section 2, generalizes this to a larger family of Knörr lattices.

Theorem 1.1. *Let M be a Knörr RG-lattice of height h in a block of defect d. Assume that*

(i) *The module $M/\pi M$ is indecomposable, and*
(ii) $e(d-h) \geq 2$.

Then the middle term of the almost split sequence terminating in M is indecomposable.

Recall now that the isomorphism classes of indecomposable RG-lattices form the vertices of a directed graph known as the Auslander-Reiten quiver of RG-lattices. To each connected component of this quiver is associated a certain tree known as the tree class of the (stable part of the) connected component. If M is an indecomposable RG-lattice, we will refer to the tree of the connected component containing M as simply the tree of M.

Examples of positive height Knörr lattices for elementary abelian p-groups have been given in [7] and [8]. These examples are cyclic lattices. An RD-lattice is called cyclic if it is generated by a single element. Equivalently, an RD-lattice is cyclic if it is a quotient of the regular lattice RD. The second main result here is the determination of the trees of these Knörr lattices.

The starting point is a celebrated theorem of Webb [16, Theorem A] which places strong restrictions on the possibilities for the tree of an arbitrary indecomposable RG-lattice. Specifically, unless M lies in a block with cyclic defect groups, then the tree of M is either a Euclidean diagram or one of the infinite Dynkin diagrams A_∞^∞, A_∞, B_∞, C_∞, or D_∞. These trees are displayed in [16].

Observe that if Theorem 1.1 applies to M, then M lies on the edge of the Auslander-Reiten quiver, and in particular the tree of M cannot be A_∞^∞. Most of the work that goes into the following amounts to eliminating the D_∞ possibility.

Theorem 1.2. *Let D be a noncyclic elementary abelian p-group, and assume that R contains the pth roots of unity (so $e \geq p-1$). Assume also that $|D| \geq 8$ if $p = 2$ and $e = 1$.*

Let M be a cyclic Knörr RD-lattice. Then the tree of M is A_∞, and M lies on the edge of its connected component of the Auslander-Reiten quiver. Also, this component does not contain the regular RD-lattice.

Section 3 develops a technique which proves the above in all but four small cases. In three of these cases, arguments involving characters show that the Knörr lattices under investigation do not actually exist. Unfortunately, one of these three cases (that of a height 1 cyclic lattice for an elementary abelian group of size 3^3) requires a reference to [8]. The other two cases are eliminated in Section 3.

The fourth case is that of a height 1 cyclic lattice for an elementary abelian group of size 2^4. Such lattices do exist, and it is shown in Section 4 that they comply with Theorem 1.2.

Theorem 1.2 does not apply if D is a cyclic group. However, the quiver components for cyclic p-groups have long been known. The Auslander-Reiten quiver is either a single connected component whose tree is a finite Dynkin diagram, or a collection of tubes (each with A_∞ tree). See [17]. The cases in which finite Dynkin diagrams occur are treated in [5].

Finally, Theorem 1.2 is also inapplicable if D is a Klein four group and $e = 1$. The Auslander-Reiten quiver is given in [5] in this case as well. The only Knörr lattices are those of odd rank, and all such lattices lie in the same connected component as the regular lattice. The tree turns out to be the Euclidean diagram $\tilde{D}_4$.

2. Knörr lattices and almost split sequences

To set up the proof of Theorem 1.1, we will use a description of the almost split sequence terminating in a Knörr lattice given by Thévenaz in [14]. A concept introduced by Carlson and Jones in [2], known as the exponent of an RG-lattice, will also be needed.

2.1. **Duality and exponents.**

To begin, let M denote an arbitrary indecomposable RG-lattice, and continue to denote the G-algebra $\mathrm{End}_R(M)$ by A. The trace map extends to a function from $K \otimes_R A$ to K in a natural way. Now let L be an R-lattice inside the vector space $K \otimes_R A^G$. Following [14], we define the dual of L to be

$$L^* = \{\theta \in K \otimes_R A^G \mid \frac{1}{|G|}\,\mathrm{trace}(\theta L) \subseteq R\}.$$

It is shown in [14] that $(A^G)^* = A_1^G$, the image of the transfer map from A to A^G. Now because M is indecomposable, $A^G/J(A^G)$ is isomorphic to k. Letting $L(A^G)$ denote $J(A^G)^*$, basic properties of the duality construction imply that $L(A^G)/A_1^G$ is isomorphic to k as well. Therefore, there exists an element $\alpha \in A^G$ such that

$$L(A^G) = R\alpha + A_1^G.$$

Thévenaz calls such α an almost projective endomorphism of M. He proves [14, Proposition 6.3] that the top row of the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & E_M & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & \Omega M & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \end{array} \tag{1}$$

is the almost split sequence terminating in M. Here, $P_M \to M$ is the projective cover of M.

Now Carlson and Jones [2] define the exponent of the lattice M to be π^n where n is the smallest integer such that $\pi^n \,\mathrm{Id}_M$ lies in A_1^G. The following appears in [11, Remark 1.5].

Lemma 2.1. *Let M be a non-projective indecomposable RG-lattice with exponent π^n, $n \geq 2$. Then the almost split sequence terminating in M splits when reduced* $\pmod{\pi^{n-1}}$.

Proof. The map $\pi^{n-1}\operatorname{Id}_M$ does not lie in A_1^G. It follows that $(\pi^{n-1}A^G + A_1^G)/A_1^G$ must contain the simple socle of A^G/A_1^G, so that M has an almost projective endomorphism lying in $\pi^{n-1}A$. □

Letting M now denote a Knörr lattice, we must express the exponent of M in terms of the defect of the block containing M as well as the height of M. The following is [2, Proposition 4.3].

Lemma 2.2. *If M is a Knörr lattice of height h in a block of defect d, then the exponent of M is $\pi^{e(d-h)}$.*

Proof. Because M is a Knörr lattice, the dual of A^G can be determined by looking at Id_M. The exponent of M is then π^l with l satisfying

$$\frac{1}{|G|}\operatorname{trace}(\pi^l \operatorname{Id}_M) \in R^\times.$$

The power of p dividing $\operatorname{trace}(\operatorname{Id}_M)$ is p^{a-d+h}, and the result follows. □

Finally, [14, Proposition 8.1] shows that M is a Knörr lattice if and only if the almost projective endomorphism α can be chosen of the form $\pi^l \operatorname{Id}_M$ for some natural number l. Of course, we then have $l = e(d-h)-1$. The construction of the almost split sequence terminating in M as a pullback now yields the following:

Corollary 2.3. *An almost projective endomorphism of the Knörr lattice M above is given by $\pi^{e(d-h)-1}\operatorname{Id}_M$. In this case, the lattice E_M in* (1) *is given by*

$$E_M = \Omega M + \pi^{e(d-h)-1}P_M.$$

2.2. **Proof of Theorem 1.1.**

Assume now that we are in the situation of Theorem 1.1.

As has been discussed, we can take $\alpha = \pi^{e(d-h)-1}\operatorname{Id}_M$ to be an almost projective endomorphism of M. Because $e(d-h) \geq 2$, Lemma 2.1 implies that the almost split sequence terminating in M splits (mod π).

Take this sequence to be the top row of (1) and assume that $E_M \cong X \oplus Y$ for nonzero sublattices X and Y. Because $M/\pi M$ is indecomposable and $E_M/\pi E_M \cong M/\pi M \oplus \Omega M/\pi\Omega M$, we must have (relabeling if needed) $X/\pi X \cong M/\pi M$. In particular, $\operatorname{rank}(M) = \operatorname{rank}(X)$.

Now there results an endomorphism of P_M given by the composite

$$P_M \to \pi^{e(d-h)-1}P_M \to E_M \to X \to P_M$$

where the first map is $\pi^{e(d-h)-1}\operatorname{Id}_{P_M}$, the second map is the inclusion resulting from the identification $E_M = \Omega M + \pi^{e(d-h)-1}P$, and the third and fourth maps are projection and inclusion respectively.

The trace of this composite is $\pi^{e(d-h)-1}\operatorname{rank}(X)$. However, because P_M is projective, all endomorphisms of P_M have trace divisible by p^a. If the π-part of $\operatorname{rank}(X)$ is π^x, the result is $ea \leq e(d-h)-1+x$, or

$$e(a-d+h)+1 \leq x.$$

It follows that $\operatorname{rank}(X)$ is divisible by more p than $\operatorname{rank}(M)$, a contradiction.

2.3. **Applicability.**

The second hypothesis of Theorem 1.1 holds quite often. To see this, let us recall a lemma of Knörr, [13, Proposition 2.1]. Here, ν is an exponential valuation of R.

Lemma 2.4. *Let M be a Knörr RG-lattice, and let H be a subgroup of G. Let U be a summand of the restriction M_H, and let α be an RH-endomorphism of U. Then*

(i) $\nu(\mathrm{rank}(M)) \leq \nu(|G : H|) + \nu(\mathrm{trace}(\alpha))$
(ii) *Equality holds for some α if and only if M is a direct summand of the induced lattice U^G.*

As Knörr points out in [13], taking $H = 1$ shows that the largest power of p which can divide the rank of a Knörr lattice is p^a. Equivalently, the height of a Knörr lattice is bounded by the defect of the block containing the lattice. Further, Lemma 2.4 shows that the height of M is equal to this defect if and only if M is projective. In this case, Knörr shows [13, Lemma 1.9] that M belongs to a block of defect zero.

Thus, if M is a Knörr lattice not belonging to a block of defect zero, then the height of M is at most $d - 1$. If $e \geq 2$, then indeed the second hypothesis of Theorem 1.1 holds. For the purpose of constructing RG-lattices affording absolutely irreducible characters, one expects R to contain the pth roots of unity. If this is the case, then $e \geq 2$ will hold as long as p is odd.

If $e = 1$, then the second hypothesis of the theorem will hold if G is a p-group with $d \geq 2$. This is seen by taking H in Lemma 2.4 to be a central subgroup of order p, and noting that the restriction M_H must have a summand of p' rank. Indeed if $e = 1$, the only indecomposable RH-lattice of rank divisible by p is the regular lattice [1], so if M_H had no summand of p' rank, then the character afforded by M would vanish on the generators of H. This quickly leads to a contradiction, as in [6].

The first hypothesis of Theorem 1.1 is perhaps a bit more mysterious. We mention two things. First, the main result of [9] also makes this assumption. Second, if χ is an absolutely irreducible character of G with values contained in R, then Thompson points out in [15] that there is always an RG-lattice affording χ which remains indecomposable (mod π). Indeed, this is crucial to Dade's analysis of blocks with cyclic defect groups [4].

Finally, because cyclic lattices are quotients of the regular lattice, if D is a p-group then cyclic RD-lattices remain indecomposable (mod π). This is needed in the next section.

3. Tree classes for cyclic Knörr lattices for abelian p-groups

Theorem 1.2 applies to the Knörr lattices constructed in [7] and [8]. In this section we prove this theorem in all cases but one. The remaining case is handled in Section 4. Recall that D denotes a group of size p^d, and e denotes the ramification index of $K/\mathbb{Q}_p$.

3.1. **Cyclic Knörr lattices.**

One advantage of cyclic lattices for abelian p-groups is that they are determined up to isomorphism by the characters they afford. Thus, the Knörr property can be reformulated using characters. This is essential in the argument eliminating the D_∞ case of Theorem 1.2.

Lemma 3.1. *Assume D is abelian, and let M be a cyclic RD-lattice of height h affording the character χ. Then M is a Knörr lattice if and only if*

$$\chi(g) \equiv \chi(1) \pmod{\pi p^h}$$

for all $g \in D$.

Proof. If M is a Knörr lattice, and $g \in D$, then multiplication by $1-g$ is a non-invertible endomorphism of M. It follows that its trace must be divisible by more π than the trace of Id_M. The power of p dividing the trace of Id_M is p^h, and the congruence follows.

Assume now that the displayed congruence holds for all $g \in D$. Let α be an endomorphism of M. Because M is cyclic, the natural map $RD \to \mathrm{End}_{RD}(M)$ is surjective. So α can be regarded as an element of RD. Then using the congruence we have

$$\mathrm{trace}(\alpha) \equiv \chi(1)\mathrm{aug}(\alpha) \pmod{\pi p^h}$$

where $\mathrm{aug} : RD \to R$ denotes the augmentation map. Now α is invertible if and only if its augmentation lies in $R^\times$, so the Knörr property is easily verified. □

3.2. **Further preliminaries.**

We shall also need some facts, due to Kawata, about the connected component Δ of the Auslander-Reiten quiver containing the regular RD-lattice, P. Let J denote the unique maximal sublattice of P. If J is indecomposable, then the inclusion $J \to P$ is an irreducible map, and there is an almost split sequence

$$0 \to J \to P \oplus M' \to \Omega^{-1}J \to 0 \tag{2}$$

for some RD-lattice M'.

Lemma 3.2. *With the above notation, the following hold:*

(i) *The lattice J is indecomposable unless $d = 1$ and $e = 1$.*
(ii) *The lattice M' is indecomposable unless $d = 1$ and $e \leq 2$, or $d = 2$ and $e = 1$.*
(iii) *If J and M' are indecomposable, the tree class of Δ is not D_∞.*

Proof. (i) This is [12, Lemma 1.1].
(ii) This is [12, Proposition 3.6].
(iii) This is [12, Lemma 6.1].

□

Next is a fact about the exponent of the middle term of an almost split sequence terminating in a Knörr lattice. This is pointed out in [2, Remark 2.6]:

Lemma 3.3. *Let M be a Knörr lattice for RG with exponent π^l. Then the exponent of the middle term of the almost split sequence terminating in M is π^{l-1}.*

Proof. Taking the top row of (1) as the sequence, we have $E_M = \Omega M + \pi^{l-1} P_M$. Let π^n denote the exponent of E_M. There is a factorization of $\pi^{l-1}\,\mathrm{Id}_{E_M}$ as

$$E_M \to P_M \to \pi^{l-1} P_M \to E_M$$

showing that $n \le l-1$. On the other hand, the map $\pi\,\mathrm{Id}_M$ must factor through E_M. It follows that $\pi^{n+1}\,\mathrm{Id}_M$ factors through a projective. So $l \le n+1$. □

Finally, the following theorem of Maranda will also be needed:

Theorem 3.4. *Let M_1 and M_2 be RG-lattices. Assume that*

$$M_1/\pi p^a M_1 \cong M_2/\pi p^a M_2$$

where $p^a = |G|_p$. Then $M_1 \cong M_2$.

Proof. See [3, Theorem 30.14]. □

3.3. **Eliminating D_∞.**
The next two propositions show that many cyclic Knörr lattices for abelian p-groups do not have tree D_∞.

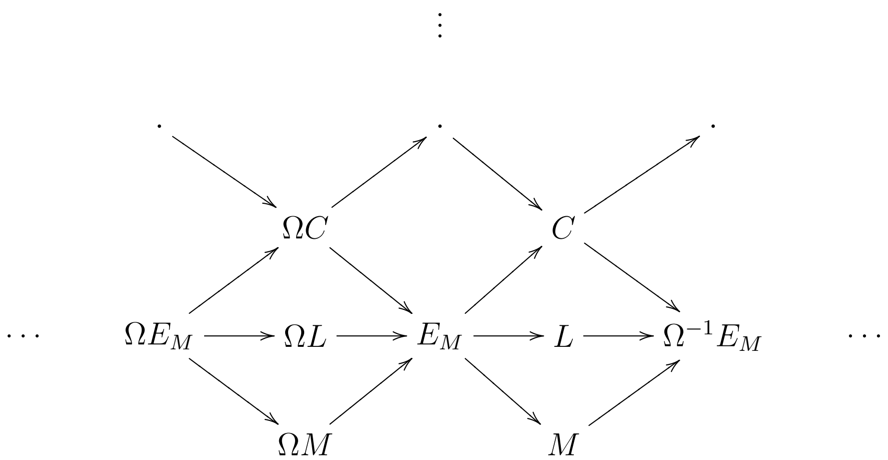
Proposition 3.5. *Let D be an abelian p-group of size p^d, and let M be a cyclic Knörr RD-lattice of height h. Denote the exponent of D by p^m and assume*

$$e(d-h-m) \ge 3.$$

Then the tree of M is not D_∞.

Proof. Theorem 1.1 applies to M. Therefore, if the tree of M is D_∞, then the component of the Auslander-Reiten quiver containing M must look like this:

(3)



By Lemma 3.2, the regular RD-lattice is not connected to the diagram above. Thus, there are two almost split sequences:

$$0 \to \Omega M \to E_M \to M \to 0 \tag{4}$$

$$0 \to \Omega L \to E_M \to L \to 0. \tag{5}$$

Let P_M and P_L denote the projective covers of M and L respectively. The sequences imply that

$$\begin{aligned}\operatorname{rank}(P_M) &= \operatorname{rank}(\Omega M) + \operatorname{rank}(M) = \operatorname{rank}(E_M)\\ &= \operatorname{rank}(\Omega L) + \operatorname{rank}(L) = \operatorname{rank}(P_L).\end{aligned}$$

The fact that M is cyclic now implies that L is cyclic also. In particular, $L/\pi L$ is indecomposable.

Next, the exponent of M is $\pi^{e(d-h)}$ by Lemma 2.2. It follows from Lemma 2.1 that the sequence (4) splits when reduced (mod $\pi^{e(d-h)-1}$).

Now consider the almost split sequence

$$0 \to E_M \to M \oplus L \oplus C \to \Omega^{-1}E_M \to 0. \tag{6}$$

The exponent of E_M is $\pi^{e(d-h)-1}$ by Lemma 3.3. It follows from Lemma 2.1 that (6) splits when reduced (mod $\pi^{e(d-h)-2}$).

Combining the split sequences that result from reducing (4) and (6) (mod $\pi^{e(d-h)-2}$) and using the Krull-Schmidt theorem (which applies to the Artinian ring $\overline{R}D$) to cancel an $\overline{M}$ yields

$$\Omega\overline{M} \oplus \overline{M} \oplus \Omega^{-1}\overline{M} \cong \overline{L} \oplus \overline{C},$$

where "bar" denotes reduction (mod $\pi^{e(d-h)-2}$).

All three terms on the left are indecomposable. The module $\overline{L}$ is also indecomposable, so the Krull-Schmidt theorem implies that $\overline{L}$ is isomorphic to one of the terms on the left, say $\overline{X}$. In particular, L and M have the same height.

If H is a subgroup of D, Maranda's theorem now implies that the restrictions L_H and X_H are isomorphic if $|H|$ divides $\frac{p^{d-h}}{\pi^3}$. The assumption $e(d-h-m) \geq 3$ implies that this holds whenever H is a cyclic subgroup of D. We deduce that L affords the same character as X, a Heller translate of M. In particular, L affords the character of a Knörr lattice.

Because L is cyclic, it follows from Lemma 3.1 that L is a Knörr lattice. Because L and M have the same height, we can then write

$$E_M = \Omega M + \frac{p^{d-h}}{\pi}RD = \Omega L + \frac{p^{d-h}}{\pi}RD.$$

An argument using Maranda's theorem, similar to the one above, now shows that ΩM and ΩL afford the same character. However, ΩM and ΩL are pure sublattices of the regular lattice RD, and are hence determined by the characters they afford. The contradiction $\Omega M \cong \Omega L$ finally follows. □

An argument simpler than that above can be used if M has height zero. Several cases where the inequality of Proposition 3.5 fails are covered this way. For example, the following applies to height zero cyclic lattices if D is elementary abelian of order 8 and $e = 1$, or if D is elementary abelian of order 9 and $e = 2$.

Proposition 3.6. *Let D be an abelian p-group of size p^d, and let M be a cyclic Knörr RD-lattice of height zero. Let p^m denote the exponent of D, and assume that*

$$e(d-m) \geq 2.$$

Then the tree of M is not D_∞.

Proof. As before, we have the diagram (3) and the almost split sequences (4), (5), and (6). By reducing (mod π) and arguing as above, it follows that L has height zero. We conclude that L is a Knörr lattice immediately. We also know that L is cyclic by the same argument as above. It follows that

$$\Omega M + \frac{p^d}{\pi} RD = \Omega L + \frac{p^d}{\pi} RD.$$

Maranda's theorem now says that if H is a subgroup of D, then ΩM becomes isomorphic to ΩL when restricted to H, provided $|H|$ divides $\frac{p^d}{\pi^2}$. The assumption $e(d-m) \geq 2$ ensures that all cyclic subgroups of D satisfy this condition, so that ΩL and ΩM afford the same character. The contradiction $\Omega M \cong \Omega L$ follows as before. □

3.4. Getting A_∞.

Most cases of Theorem 1.2 can now be deduced.

Theorem 3.7. *The tree of M in Propositions 3.5 and 3.6 is A_∞. Also, the component Θ of the Auslander-Reiten quiver containing M does not contain the projective indecomposable RD-lattice.*

Proof. This requires input from many people, notably Inoue [10], Kawata [10] [12], Okuyama (credited in [10] and [12]), and Webb [16].

Let us first assume that the tree is A_∞ and argue that Θ cannot contain the regular lattice.

Recall the almost split sequence (2). Note that J and M' are indecomposable by Lemma 3.2 (ii). If J and M lie in the same $\mathbb{Z}A_\infty$ graph, then they both must lie on the edge. It follows that J is a Heller translate of M which is impossible because J cannot be a Knörr lattice. For example, the character it affords vanishes on the nonidentity elements of D.

We must now show that the tree class is in fact A_∞. The possibilities A_∞^∞ and D_∞ have been discussed. By Webb's theorem [16, Theorem A], it remains to eliminate B_∞, C_∞, and the Euclidean diagrams.

For the Euclidean case, arguments from both [10] and [12] must be combined. If Θ contains the trivial lattice, then [10, Lemma 4.2] says the tree is not Euclidean. If Θ does not contain the trivial lattice, then [12, Proposition 5.2] says that the tree is not Euclidean.

Finally, a standard argument (see [12, Remark 4.3]) eliminates the B_∞ and C_∞ cases. It suffices to show that the radical quotient of the endomorphism ring of M is k. This holds because M is cyclic. It also follows from the assumption that k is algebraically closed. □

To finish the proof of Theorem 1.2, we must isolate the cases of that theorem in which Propositions 3.5 and 3.6 do not apply. Let D be as in the theorem, and write $|D| = p^d$.

Let us begin with the case of a height zero lattice. Noting that $m = 1$, Proposition 3.6 is inapplicable when $e(d-1) \leq 1$. Because D is assumed noncyclic, we must have $d = 2$, and $e = 1$. Because R is assumed to contain the pth roots of unity, we then have $p = 2$. However, this case is excluded in the statement of the theorem.

To handle the case of a lattice with positive height h, we need a further relationship between d and h which comes from considering the character afforded by the lattice.

Lemma 3.8. *Let M be a cyclic Knörr lattice of height h and rank greater than 1 for the elementary abelian p-group D of size p^d. Then*

$$p^{2h} < \operatorname{rank}(M) < p^d.$$

Proof. Let χ be the character afforded by M. By Lemma 3.1, we know that $\nu(\chi(g)) = \nu(p^h)$ for every $g \in D$. It follows that

$$\prod_{g \in D} |\chi(g)|^2$$

is a positive integer divisible by $p^{2h|D|}$. Extracting a $|D|th$ root and applying the inequality relating geometric and arithmetic means yields

$$p^{2h} < [\chi, \chi] = \chi(1) = \operatorname{rank}(M) < p^d$$

because χ is multiplicity free. □

Now if $m = 1$ and Proposition 3.5 is inapplicable, then $e(d - h - 1) \leq 2$, so

$$d \leq \frac{2}{e} + 1 + h \leq 3 + h.$$

Note that $d = 3 + h$ can only hold if $e = 1$, hence $p = 2$ (recall that R contains the pth roots of unity). Similarly, $d = h + 2$ can only hold if $e \leq 2$, hence $p \leq 3$.

Combining this with the condition $d > 2h$ from Lemma 3.8, it is elementary to check that the following four cases are those which must be examined separately:

p	(h, d)
3	(1,3)
2	(1,3), (1,4), (2,5).

It is shown in [8] that no cyclic Knörr lattice exists in the $p = 3$ case above. Also, it is elementary to check that the $p = 2$, $h = 1$, $d = 3$ case cannot occur.

Consider now a height 2 cyclic Knörr lattice M for an elementary abelian group of order 2^5. Replacing M by a cyclic lattice affording the same character as ΩM if necessary, we can assume that $\operatorname{rank}(M) < 16$. But now Lemma 3.8 says $2^4 < \operatorname{rank}(M) < 16$, a contradiction. This eliminates the $(2, 5)$ case above.

The remaining case needed to prove Theorem 1.2 is handled in the next section.

4. The rank 6 lattice for the elementary abelian group of order 16

Let D be an elementary abelian 2-group of size 2^4, and let the group of irreducible characters of D be $\hat{D} = \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle$. Let M be the quotient of RD affording the character

$$\chi = \chi_M = 1_D + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_4.$$

Then M is indeed a Knörr lattice of rank 6 (see [7]).

Theorem 3.7 does not apply to M, so we give a special argument to show that the tree for this lattice is A_∞ too. This uses the symmetry of the character χ. The argument in Theorem 3.7 shows that it suffices to eliminate the tree D_∞. Thus, we return to the diagram (3). As in Proposition 3.5, we know that L is cyclic. In particular $L/\pi L$ is indecomposable.

The group $\mathrm{Aut}(D)$ acts on the isomorphism classes of RD-lattices. Let Y denote the stabilizer in $\mathrm{Aut}(D)$ of M. Because M is cyclic, Y is the same as the stabilizer in $\mathrm{Aut}(D)$ of χ. A cyclic permutation of the nontrivial irreducible constituents of χ is achieved by an automorphism of D, revealing a 5-cycle in Y that together with the permutations of the λ_i show $Y \cong S_5$.

Now $\mathrm{Aut}(D)$ also acts on the set of almost split sequences of RD-lattices. So Y stabilizes the sequence terminating in M, namely (4).

But E_M, the middle term of that sequence, is indecomposable by Theorem 1.1. It follows that Y stabilizes the isomorphism class of E_M. If the tree is indeed D_∞, then Y must also stabilize the almost split sequence (5). We conclude that Y stabilizes the isomorphism class of L so in particular stabilizes the character χ_L it affords.

Now $\hat{D}$ splits into three Y-orbits, of sizes $1, 5$, and 10. The character χ_L must be comprised of some of these. However, the isomorphism class of L is determined by χ_L because L is cyclic. It follows that $\chi_L \neq \chi$.

Next, the exponent of E_M is 2^2, so the almost split sequence (6) splits modulo 2, giving the isomorphism

$$\Omega\overline{M} \oplus \overline{M} \oplus \Omega^{-1}\overline{M} \cong \overline{L} \oplus \overline{C},$$

where now "bar" denotes reduction (mod 2). In this situation, Maranda's theorem tells us nothing nontrivial, but we can still conclude from the above that L has the same height as M, namely 1.

This is enough for us to conclude that χ_L must be precisely the sum of the irreducibles in the orbit of Y on $\hat{D}$ of size 10. However, we now have $\chi_L = \chi_{\Omega M}$. In particular, L affords the character of a Knörr lattice. Because L is cyclic, it follows from Lemma 3.1 that L is a Knörr lattice.

Noting that the exponent of M is 2^3, we now have

$$E_M = \Omega M + 2^2 RD = \Omega L + 2^2 RD.$$

From this, the contradiction $\mathrm{rank}(M) = \mathrm{rank}(L)$ follows.

Finally, we must show that no other Knörr lattices of height 1 for D need be considered. If M is now an arbitrary cyclic Knörr RD-lattice of height 1, then the only choices for the rank of M are 6 and 10. If M has rank 6, then it is no loss to assume that M is as above by tensoring with a rank 1 lattice and applying a suitable automorphism of D.

If M has rank 10, then $(\Omega M)^*$ is a cyclic Knörr lattice of rank 6 in a component of the Auslander-Reiten quiver with the same tree as that of M.

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