

## GLAUBERMAN'S AND THOMPSON'S THEOREMS FOR FUSION SYSTEMS

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ABSTRACT. We prove analogues of results of Glauberman and Thompson for fusion systems. Namely, given a (saturated) fusion system  $\mathcal{F}$  on a finite  $p$ -group  $S$ , and in the cases where  $p$  is odd or  $\mathcal{F}$  is  $S_4$ -free, we show that  $Z(N_{\mathcal{F}}(J(S))) = Z(\mathcal{F})$  (Glauberman) and that if  $C_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(J(S)) = \mathcal{F}_S(S)$ , then  $\mathcal{F} = \mathcal{F}_S(S)$  (Thompson). As a corollary, we obtain a stronger form of Frobenius' theorem for fusion systems, applicable under the above assumptions and generalizing another result of Thompson.

### 1. INTRODUCTION

Fusion systems were developed in the early 1990's as a way to unify fusion phenomena occurring both in finite groups and in the  $p$ -blocks of finite groups. This axiomatic approach introduced by Puig has proved successful in that many of the important theorems in local group analysis are now placed into this categorical framework (see [1], [2], [10]). For instance, there are fusion system analogues of Frobenius'  $p$ -nilpotency criterion (see [8, Theorem 3.11]), Alperin's fusion theorem (see [8, Theorem 5.2]), Glauberman and Thompson's  $p$ -nilpotency criterion and Glauberman's ZJ-theorem (see [7]). Following in the spirit of recent works of Kessar and Linckelmann, this paper generalizes some classical results in group theory of Glauberman and Thompson to the more abstract world of fusion systems.

In §2, we will state the definition of a *fusion system*, set up some notation and recall some of the standard tools. Additionally, we examine the properties of central elements in a fusion system and use this to determine the center of a fusion subsystem (Lemma 2.8). We also generalize Glauberman's definition of a *conjugation family* to fusion systems and prove that a set of representatives for the isomorphism classes of a conjugation family is again a conjugation family (Proposition 2.10). This will show that the  $(\mathcal{F}, W)$ -*well-placed* subgroups form a conjugation family, where  $\mathcal{F}$  is a fusion system and  $W$  is a *positive characteristic  $p$ -functor* (see Definition 2.11).

In §3, we will recall two theorems of Glauberman and Thompson (see Theorem G and Theorem T) as presented in Glauberman's article [6]. In §4, we will first prove

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the fusion system version of Theorem G in Theorem 4.1, adapting Glauberman's original proof (cf. [5, Theorem 6]), by examining a minimal counterexample and showing that it is *constrained* (see Remark 2.17). This enables us to conclude the proof by applying Theorem G. Thereafter, we prove a fusion system version of Theorem T in Theorem 4.5, which arises as an easy corollary of Theorem 4.1. Finally, we show how Theorem 4.5 implies a strengthening of Frobenius' theorem for fusion systems when  $p$  is odd or the fusion system is  $S_4$ -free (see Corollary 4.6).

## 2. GENERALITIES

Throughout, let  $p$  be a prime number and let  $S$  be a finite  $p$ -group.

**Definition 2.1.** A *fusion system* on  $S$  is a category  $\mathcal{F}$  having as objects the subgroups of  $S$  and for any two subgroups  $Q, R$  of  $S$ , the morphism set  $\text{Hom}_{\mathcal{F}}(Q, R)$  is a set of injective group homomorphisms  $Q \rightarrow R$  with the following properties:

- (1) composition of morphisms in  $\mathcal{F}$  is the usual composition of group homomorphisms;
- (2) if  $\varphi : Q \rightarrow R$  is a morphism in  $\mathcal{F}$ , then so is the induced group isomorphism  $Q \cong \varphi(Q)$  as well as its inverse;
- (3)  $\text{Hom}_{\mathcal{F}}(Q, R)$  contains the set  $\text{Hom}_S(Q, R)$  of group homomorphisms  $Q \rightarrow R$  given by conjugation with elements in  $S$ ;
- (4)  $\text{Aut}_S(S)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(S)$ ;
- (5) if  $\varphi : Q \rightarrow S$  is a morphism in  $\mathcal{F}$  such that  $|\text{N}_S(\varphi(Q))| \geq |\text{N}_S(\tau(Q))|$  for any  $\tau \in \text{Hom}_{\mathcal{F}}(Q, S)$ , then  $\varphi$  extends to a morphism  $\psi : N_{\varphi} \rightarrow S$  in  $\mathcal{F}$  where  $N_{\varphi}$  is the subgroup of  $\text{N}_S(Q)$  consisting of all  $y \in \text{N}_S(Q)$  for which there exists  $z \in \text{N}_S(\varphi(Q))$  with the property that  $\varphi(yuy^{-1}) = z\varphi(u)z^{-1}$  for all  $u \in Q$ .

This definition coincides with the one used by Linckelmann. Following Stancu [11], axioms (4) and (5) imply the *a priori* stronger axioms used in the work of Broto-Levi-Oliver [4], where fusion systems are called *saturated fusion systems*. We will refer to axiom (5) as the *extension axiom*. It is easy to see from its definition that  $N_{\varphi}$  is the largest subgroup of  $\text{N}_S(Q)$  to which  $\varphi$  extends and that  $QC_S(Q) \leq N_{\varphi}$ .

*Remark 2.2.* Let  $G$  be a finite group and let  $S$  be a Sylow  $p$ -subgroup of  $G$ . For any two subgroups  $Q, R$  of  $G$  denote by  $\text{Hom}_G(Q, R)$  the set of group homomorphisms  $Q \rightarrow R$  given by conjugation with elements of  $G$ . The fusion system of  $G$  on  $S$  is the category denoted by  $\mathcal{F}_S(G)$  having the subgroups of  $S$  as objects and the sets  $\text{Hom}_G(Q, R)$  as morphism sets, for any two subgroups  $Q, R$  of  $S$ . The category  $\mathcal{F}_S(G)$  is a fusion system on  $S$  (see [8, Theorem 2.11]). Axiom (3) in Definition 2.1 implies that any fusion system  $\mathcal{F}$  on  $S$  contains the fusion system  $\mathcal{F}_S(S)$  of  $S$  on itself. This latter is called the *trivial* fusion system on  $S$ .

Let us recall that  $G$  is  $p$ -nilpotent if and only if  $G$  has a normal  $p$ -complement, that is, there exists a normal subgroup  $H$  of  $G$  of order not divisible by  $p$  and such that the factor group  $G/H$  is a  $p$ -group. A theorem of Frobenius implies that  $G$  is  $p$ -nilpotent if and only if the factor groups  $\text{N}_G(Q)/C_G(Q)$  are  $p$ -groups for any nontrivial  $p$ -subgroup  $Q$  of  $G$  or, equivalently,  $\mathcal{F}_S(G) = \mathcal{F}_S(S)$ . This latter condition yields the corresponding notion of  $p$ -nilpotency for a fusion system  $\mathcal{F}$  on  $S$ . That is,  $\mathcal{F} = \mathcal{F}_S(S)$  or, equivalently,  $\text{Aut}_{\mathcal{F}}(Q)$  is a  $p$ -group for all nontrivial subgroups  $Q$  of  $S$  (cf. [8, Theorem 3.11]).

**Definition 2.3.** Let  $\mathcal{F}$  be a fusion system on  $S$  and let  $Q$  be a subgroup of  $S$ .

- (1) We say that two subsets  $A, B \subseteq S$  are  $\mathcal{F}$ -conjugate, or  $\mathcal{F}$ -isomorphic, and write  $A =_{\mathcal{F}} B$ , if there exists an isomorphism  $\varphi$  in  $\mathcal{F}$  such that  $\varphi(A) = B$ . We write  $A^{\mathcal{F}}$  for the  $\mathcal{F}$ -conjugacy class of  $A$ .
- (2)  $Q$  is called *fully  $\mathcal{F}$ -normalized* if  $|N_S(Q)| \geq |N_S(R)|$  for any  $R \in Q^{\mathcal{F}}$ ; similarly,  $Q$  is called *fully  $\mathcal{F}$ -centralized* if  $|C_S(Q)| \geq |C_S(R)|$  for any  $R \in Q^{\mathcal{F}}$ .
- (3)  $Q$  is called  $\mathcal{F}$ -centric if  $C_S(R) = Z(R)$  for any  $R \in Q^{\mathcal{F}}$ , and  $Q$  is called  $\mathcal{F}$ -radical if  $\text{Aut}_Q(Q)$  is the largest normal  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$ .
- (4) Let us denote by  $\mathcal{F}^f, \mathcal{F}^z, \mathcal{F}^c$  and  $\mathcal{F}^r$  the set of all fully  $\mathcal{F}$ -normalized, fully  $\mathcal{F}$ -centralized,  $\mathcal{F}$ -centric and  $\mathcal{F}$ -radical subgroups of  $S$ , respectively. For  $X \subseteq \{f, z, c, r\}$ , set  $\mathcal{F}^X = \bigcap_{x \in X} \mathcal{F}^x$ .
- (5) Let  $R$  be a subgroup of  $S$  and let  $\varphi \in \text{Hom}_{\mathcal{F}}(R, S)$ . We say that  $\varphi$  *stably extends to  $Q$*  if  $QR$  is a subgroup of  $S$  and there exists a morphism  $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(QR, S)$  such that  $\tilde{\varphi}|_R = \varphi$  and  $\tilde{\varphi}(Q) = Q$ . If, in addition,  $\tilde{\varphi}|_Q = \text{Id}_Q$ , then we say that  $\varphi$  *centrally extends to  $Q$* .
- (6) The *normalizer of  $Q$  in  $\mathcal{F}$*  is the category  $N_{\mathcal{F}}(Q)$  with objects the subgroups of  $N_S(Q)$  and morphisms  $\text{Hom}_{N_{\mathcal{F}}(Q)}(U, V)$  all morphisms  $U \rightarrow V$  in  $\mathcal{F}$  that stably extend to  $Q$ . We say that  $Q$  is *normal in  $\mathcal{F}$*  (and write  $Q \triangleleft \mathcal{F}$ ) if  $\mathcal{F} = N_{\mathcal{F}}(Q)$ , i.e., if every morphism in  $\mathcal{F}$  stably extends to  $Q$ . In particular, if  $Q \triangleleft \mathcal{F}$ , then  $Q \leq S$ .
- (7) The *centralizer of  $Q$  in  $\mathcal{F}$*  is the category  $C_{\mathcal{F}}(Q)$  with objects the subgroups of  $C_S(Q)$  and morphisms  $\text{Hom}_{C_{\mathcal{F}}(Q)}(U, V)$  all morphisms  $U \rightarrow V$  in  $\mathcal{F}$  that centrally extend to  $Q$ . We say that  $Q$  is *central in  $\mathcal{F}$*  if  $\mathcal{F} = C_{\mathcal{F}}(Q)$ , i.e., if every morphism in  $\mathcal{F}$  centrally extends to  $Q$ . In particular, if  $Q$  is central in  $\mathcal{F}$ , then  $Q \leq Z(S)$ . Also, since the product of two central subgroups of  $\mathcal{F}$  is again central in  $\mathcal{F}$ , we call the largest central subgroup of  $\mathcal{F}$  the *center of  $\mathcal{F}$*  and denote it by  $Z(\mathcal{F})$ .
- (8) The  *$\text{Aut}_S(Q)$ -normalizer of  $Q$  in  $\mathcal{F}$*  is the category  $N_S(Q)C_{\mathcal{F}}(Q)$  with objects the subgroups of  $N_S(Q)$  and morphisms  $\text{Hom}_{N_S(Q)C_{\mathcal{F}}(Q)}(U, V)$  all morphisms  $\varphi : U \rightarrow V$  in  $\mathcal{F}$  for which there exists an extension  $\tilde{\varphi} : QU \rightarrow QV$  in  $\mathcal{F}$  such that  $\tilde{\varphi}|_Q \in \text{Aut}_S(Q)$ .
- (9) Assume that  $Q$  is normal in  $\mathcal{F}$ . The category  $\mathcal{F}/Q$  is the category whose objects are the subgroups of  $S/Q$  and for any two subgroups  $U$  and  $V$  of  $S$  containing  $Q$ , a group homomorphism  $\bar{\varphi} : U/Q \rightarrow V/Q$  is a morphism in  $\mathcal{F}/Q$  if there exists  $\varphi \in \text{Hom}_{\mathcal{F}}(U, V)$  such that  $\varphi(u)Q = \bar{\varphi}(uQ)$  for all  $u \in U$ .

Puig proved (see [8, Theorem 3.2]) that if  $Q \in \mathcal{F}^f$ , then  $N_{\mathcal{F}}(Q)$  is a fusion system on  $N_S(Q)$ . Similarly, if  $Q \in \mathcal{F}^z$ , then  $C_{\mathcal{F}}(Q)$  is a fusion system on  $C_S(Q)$  and  $N_S(Q)C_{\mathcal{F}}(Q)$  is a fusion system on  $N_S(Q)$ . Finally, if  $Q$  is normal in  $\mathcal{F}$ , then  $\mathcal{F}/Q$  is a fusion system on  $S/Q$  (see [8, Theorem 6.2]). Let us also recall a useful result of Stancu (see [9, Proposition 1.6]) which says

$$(2.4) \quad \text{if } Q \in \mathcal{F}^f, \text{ then } Q \in \mathcal{F}^z \text{ and } \text{Aut}_S(Q) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(Q)).$$

Notice that the extension axiom says that any morphism  $\varphi$  in  $\mathcal{F}$  whose image is fully  $\mathcal{F}$ -normalized extends to a morphism in  $\mathcal{F}$  whose domain is  $N_{\varphi}$ . Obvious examples of fully  $\mathcal{F}$ -normalized (resp., fully  $\mathcal{F}$ -centralized) subgroups of  $S$  are the normal (resp., central) subgroups of  $S$ .

The following lemma gives a criterion for when the  $\text{Aut}_S(Q)$ -normalizer is the entire fusion system. This result will be used in the proof of Corollary 4.6.

**Lemma 2.5.** *Let  $\mathcal{F}$  be a fusion system on  $S$  and let  $Q$  be a subgroup of  $S$  such that  $Q \triangleleft \mathcal{F}$ . If  $\text{Aut}_{\mathcal{F}}(Q)$  is a  $p$ -group, then  $\mathcal{F} = SC_{\mathcal{F}}(Q)$ .*

*Proof.* By (2.4),  $Q \triangleleft \mathcal{F}$  implies that  $\text{Aut}_S(Q) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(Q))$  and so, by assumption,  $\text{Aut}_S(Q) = \text{Aut}_{\mathcal{F}}(Q)$ . Thus, every morphism in  $\mathcal{F}$  stably extends to  $Q$  and the restriction to  $Q$  of each extension is induced by  $S$ -conjugation.  $\square$

**Definition 2.6.** Let  $\mathcal{F}$  be a fusion system on  $S$ . Motivated by Glauberman (cf. [6, §3]), we define a *conjugation family* for  $\mathcal{F}$  to be a set  $\mathcal{C}$  of subgroups of  $S$  such that any isomorphism in  $\mathcal{F}$  can be written as a composition of  $\mathcal{F}$ -isomorphisms  $\phi : Q \rightarrow Q'$  for which there exists  $R \in \mathcal{C}$  containing  $Q$  and  $Q'$  and an automorphism  $\alpha \in \text{Aut}_{\mathcal{F}}(R)$  such that  $\alpha|_Q = \phi$ .

Examples of conjugation families include the set of fully  $\mathcal{F}$ -normalized radical centric subgroups of  $S$  (see [4, A.10]) and the fully  $\mathcal{F}$ -normalized essential subgroups of  $S$  (this result is due to Puig; see [8, Theorem 5.2]). Though it varies in the literature, the fact that these are conjugation families is widely known as *Alperin's fusion theorem for fusion systems*. Let us point out that the definition does not require a conjugation family to be closed under  $\mathcal{F}$ -isomorphisms (cf. Proposition 2.10); however any conjugation family for  $\mathcal{F}$  contains  $S$ .

**Definition 2.7.** A subset  $A$  of  $S$  is *weakly closed in  $S$  with respect to  $\mathcal{F}$*  if for any morphism  $\varphi : \langle A \rangle \rightarrow S$ , we have  $\varphi(A) = A$ .

The next lemma shows that  $Z(\mathcal{F})$  is the set of weakly closed elements in  $S$  with respect to  $\mathcal{F}$ , justifying the convention of calling such elements *central in  $\mathcal{F}$* .

**Lemma 2.8.** *Let  $\mathcal{F}$  be a fusion system on  $S$ . The center  $Z(\mathcal{F})$  of  $\mathcal{F}$  is the set of weakly closed elements in  $S$  with respect to  $\mathcal{F}$ . In particular, if  $\mathcal{G}$  is a fusion subsystem of  $\mathcal{F}$  on a subgroup  $T$  of  $S$ , then  $T \cap Z(\mathcal{F}) \leq Z(\mathcal{G})$ .*

*Proof.* As  $Z(\mathcal{F})$  is central in  $\mathcal{F}$ , every element of  $Z(\mathcal{F})$  is weakly closed in  $S$  with respect to  $\mathcal{F}$ . Conversely, suppose that  $x \in S$  is weakly closed in  $S$  with respect to  $\mathcal{F}$  and let  $Q \in \mathcal{F}^{frc}$ . Then,  $x \in Q$  since  $x \in Z(S)$  and  $Q$  is  $\mathcal{F}$ -centric. By assumption, any  $\mathcal{F}$ -automorphism of  $Q$  is the identity on  $\langle x \rangle$  and so by Alperin's fusion theorem,  $\langle x \rangle \leq Z(\mathcal{F})$ .  $\square$

**Lemma 2.9.** *Let  $\mathcal{F}$  be a fusion system on  $S$ , and let  $Q$  and  $R$  be subgroups of  $S$ . If  $Q$  is a characteristic subgroup of  $R$ , then  $N_{\mathcal{F}}(R)$  is a subcategory of  $N_{\mathcal{F}}(Q)$ ; i.e., any morphism in  $\mathcal{F}$  that stably extends to  $R$  also stably extends to  $Q$ .*

*Proof.* Since  $Q$  is a characteristic subgroup of  $R$ , any morphism in  $N_{\mathcal{F}}(R)$  induces an automorphism of  $R$  and hence of  $Q$ .  $\square$

The following proposition on conjugation families is well-known (and is used implicitly in [7]), but does not seem to occur formally in the literature. We thank Professor Linckelmann for bringing it to our attention.

**Proposition 2.10.** *Let  $\mathcal{F}$  be a fusion system on  $S$  and let  $\mathcal{C}$  be a conjugation family for  $\mathcal{F}$ . Any set of representatives for the  $\mathcal{F}$ -isomorphism classes of subgroups in  $\mathcal{C}$  is a conjugation family for  $\mathcal{F}$ .*

*Proof.* Let  $\mathcal{C}'$  be a set of representatives for the  $\mathcal{F}$ -isomorphism classes of subgroups in  $\mathcal{C}$ . Let  $P \in \mathcal{C}$  and assume that  $P$  is not in  $\mathcal{C}'$ . In particular,  $P$  is a proper subgroup of  $S$ . We need to show that any automorphism in  $\text{Aut}_{\mathcal{F}}(P)$  is contained in  $\mathcal{F}(\mathcal{C}')$ , where  $\mathcal{F}(\mathcal{C}')$  is the subcategory of  $\mathcal{F}$  on  $S$  with morphism sets given by the composition of restrictions of  $\mathcal{F}$ -automorphisms of subgroups in  $\mathcal{C}'$ . By induction, we assume that  $\text{Aut}_{\mathcal{F}}(T) \subseteq \mathcal{F}(\mathcal{C}')$  for all subgroups  $T \in \mathcal{C}$  whose order is greater than  $|P|$ . Let  $P' \in \mathcal{C}'$  such that there exists an isomorphism  $\varphi \in \text{Hom}_{\mathcal{F}}(P, P')$ . As  $\mathcal{C}$  is a conjugation family,  $\varphi$  can be written as a composition of isomorphisms that extend to  $\mathcal{F}$ -automorphisms of subgroups  $Q \in \mathcal{C}$ . If any of these subgroups  $Q$  has order equal to  $|P|$ , remove the corresponding automorphism to obtain a (possibly different) isomorphism from  $P$  to  $P'$ . Abusing notation, we again call this new isomorphism  $\varphi$ . By induction, any automorphism of  $Q$ , where  $Q$  is larger than  $P$ , is in  $\mathcal{F}(\mathcal{C}')$ . Hence  $\varphi \in \mathcal{F}(\mathcal{C}')$ . Now take  $\psi \in \text{Aut}_{\mathcal{F}}(P)$  and set  $f = \varphi \circ \psi \circ \varphi^{-1} \in \text{Aut}_{\mathcal{F}}(P')$ . Conjugating by  $\varphi^{-1}$ , we get  $\psi = \varphi^{-1} \circ f \circ \varphi \in \mathcal{F}(\mathcal{C}')$ .  $\square$

As in [7, Definition 1.2, 5.1], we make the following definitions.

**Definition 2.11.**

- (1) A *positive characteristic  $p$ -functor*  $W$  is a map sending any finite  $p$ -group  $Q$  to a characteristic subgroup  $W(Q)$  of  $Q$  such that
  - (a)  $W(Q) \neq 1$  if  $Q \neq 1$ ;
  - (b) if  $\varphi : Q \rightarrow T$  is a group isomorphism, then  $\varphi(W(Q)) = W(T)$ .
- (2) Let  $\mathcal{F}$  be a fusion system on  $S$  and let  $W$  be a positive characteristic  $p$ -functor. For a subgroup  $Q$  of  $S$ , define  $W_1(Q) = Q$  and for any positive integer  $i$ , define  $W_{i+1}(Q) = W(N_S(W_i(Q)))$ . We say that  $Q$  is  $(\mathcal{F}, W)$ -*well-placed* if  $W_i(Q) \in \mathcal{F}^f$  for all positive integers  $i$ .

**Corollary 2.12.** *Let  $\mathcal{F}$  be a fusion system on  $S$  and let  $W$  be a positive characteristic  $p$ -functor. The  $(\mathcal{F}, W)$ -well-placed subgroups of  $S$  form a conjugation family for  $\mathcal{F}$ .*

*Proof.* Let  $\mathcal{C}$  be any conjugation family for  $\mathcal{F}$ . By [7, Proposition 5.2], every subgroup in  $\mathcal{C}$  is  $\mathcal{F}$ -isomorphic to an  $(\mathcal{F}, W)$ -well-placed subgroup of  $S$ . The result now follows from Proposition 2.10.  $\square$

In the next sections, we consider the positive characteristic  $p$ -functor  $J$  sending a finite  $p$ -group  $S$  to its *Thompson subgroup*. For the sake of self-containment and clarity (there are at least three nonequivalent definitions for the Thompson subgroup), we briefly recall its definition and state some of its useful properties. Let  $d(S)$  be the maximum of the orders of the abelian subgroups of  $S$ , and let  $\mathcal{A}(S)$  be the set of abelian subgroups of  $S$  of order  $d(S)$ . The Thompson subgroup,  $J(S)$ , of  $S$  is defined by

$$J(S) = \langle A \mid A \in \mathcal{A}(S) \rangle$$

and satisfies the following properties:

- (2.13) If  $A \in \mathcal{A}(S)$ , then  $C_S(A) = A$ ; in particular,  $Z(S) \leq J(S)$ .
- (2.14)  $d(J(S)) = d(S)$ ,  $\mathcal{A}(J(S)) = \mathcal{A}(S)$ ,  $J(J(S)) = J(S)$ .
- (2.15)  $J$  is a positive characteristic  $p$ -functor.

Finally, we recall what it means for a fusion system  $\mathcal{F}$  to be  $H$ -free for some finite group  $H$ . A group  $G$  is  $H$ -free if there are no subgroups  $K, L$  of  $G$ , with  $L \trianglelefteq K$

and  $K/L \cong H$ . Now, for an arbitrary fusion system  $\mathcal{F}$  on  $S$ , we appeal to [3, Proposition 4.3]. That is, to any  $Q \in \mathcal{F}^{fc}$ , there corresponds, up to isomorphism, a unique finite group  $L_Q^{\mathcal{F}}$  having  $N_S(Q)$  as Sylow  $p$ -subgroup, such that  $C_{L_Q^{\mathcal{F}}}(Q) = Z(Q)$  and  $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_S(Q)}(L_Q^{\mathcal{F}})$ .

**Definition 2.16.** Let  $H$  be a finite group. A fusion system  $\mathcal{F}$  on  $S$  is  $H$ -free if  $L_Q^{\mathcal{F}}$  is  $H$ -free for all  $Q \in \mathcal{F}^{fc}$ .

*Remark 2.17.* Let  $\mathcal{F}$  be a fusion system on  $S$ .

- (1) By [7, Proposition 6.1], if  $L_Q^{\mathcal{F}}$  is  $H$ -free for all  $Q \in \mathcal{F}^{fc}$ , then  $\mathcal{F}$  is  $H$ -free. Hence, Definition 2.16 is equivalent to [7, Definition 1.1].
- (2) If  $\mathcal{F}$  has a normal centric subgroup  $Q$ , then  $\mathcal{F}$  is said to be *constrained*. In this case [3, Proposition 4.3] implies that there is a finite group  $G$  with Sylow  $p$ -subgroup  $S$  such that  $C_G(Q) = Z(Q)$  and  $\mathcal{F} = \mathcal{F}_S(G)$ .

### 3. GLAUBERMAN'S AND THOMPSON'S THEOREMS FOR FINITE GROUPS

In this section, we recall two classical theorems, one of Glauberman and one of Thompson, as presented in [6]. Firstly, [6, Theorem 14.10] states:

**Theorem G** (Glauberman, 1969). *Let  $G$  be a finite group and  $S$  a Sylow  $p$ -subgroup of  $G$ . If  $x \in S \cap Z(N_G(J(S)))$  and*

- (a)  $x \in (Z(S))^p$  or
- (b)  $p$  is odd or
- (c)  $G$  is  $S_4$ -free,

*then  $x$  is weakly closed in  $S$  with respect to  $G$ .*

Glauberman uses this theorem, along with [6, Lemma 4.5], to prove [6, Theorem 14.11], a slightly stronger version of a theorem of Thompson (cf. [12]).

**Theorem T** (Thompson, 1964). *Let  $G$  be a finite group and let  $S$  be a Sylow  $p$ -subgroup of  $G$ . Suppose that  $p$  is odd or that  $G$  is  $S_4$ -free. If  $C_G(Z(S))$  and  $N_G(J(S))$  are both  $p$ -nilpotent, then  $G$  is  $p$ -nilpotent.*

Note that [6, Lemma 4.5] also generalizes to fusion systems, but we leave it as an exercise for the reader since it is not needed to prove the fusion system version of Theorem T.

### 4. GLAUBERMAN'S AND THOMPSON'S THEOREMS FOR FUSION SYSTEMS

In this section, we prove the fusion system analogues of Theorem G and Theorem T.

**Theorem 4.1.** *If  $\mathcal{F}$  is a fusion system on  $S$ , then  $(Z(S))^p \cap Z(N_{\mathcal{F}}(J(S))) \leq Z(\mathcal{F})$ . Furthermore, if  $p$  is odd or  $\mathcal{F}$  is  $S_4$ -free, then  $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S)))$ .*

*Proof.* Suppose that the theorem is false and take a counterexample  $\mathcal{F}$  with a minimal number  $|\mathcal{F}|$  of morphisms. We shall show that  $\mathcal{F}$  is constrained. That is, there exists an  $\mathcal{F}$ -centric subgroup of  $S$  that is normal in  $\mathcal{F}$ . For then  $\mathcal{F} = \mathcal{F}_S(G)$  for some finite group  $G$  having  $S$  as a Sylow  $p$ -subgroup. Assuming we have done this, let  $x \in Z(N_{\mathcal{F}}(J(S))) = S \cap Z(N_G(J(S)))$ . Suppose that  $p$  is odd or  $\mathcal{F}$  is  $S_4$ -free or  $x \in (Z(S))^p$ . If  $\mathcal{F}$  is  $S_4$ -free, then so is  $G$ . By Theorem G, it follows that  $x$  is weakly closed in  $S$  with respect to  $G$ , i.e.  $x \in Z(\mathcal{F})$ , a contradiction. This will prove the theorem.

Since  $\mathcal{F}$  is a counterexample, if  $p$  is odd or  $\mathcal{F}$  is  $S_4$ -free, then we may choose an element  $x \in S$  that is central in  $N_{\mathcal{F}}(J(S))$  but not in  $\mathcal{F}$ ; otherwise choose  $x \in (Z(S))^p$  such that  $x$  is central in  $N_{\mathcal{F}}(J(S))$  but not in  $\mathcal{F}$ . Note that  $x \in Z(N_{\mathcal{F}}(J(S))) \leq Z(S)$ . It follows that if  $x \in T$  for a subgroup  $T$  of  $S$ , then  $x \in Z(T)$  and hence  $x \in J(T)$  by (2.13). Let  $\mathcal{C}$  be the conjugation family for  $\mathcal{F}$  consisting of the  $(\mathcal{F}, J)$ -well-placed subgroups of  $S$ . Observe that if  $Q \in \mathcal{C}$ , then  $J(N_S(Q)) \in \mathcal{C}$ . There exist a subgroup  $Q \in \mathcal{C}$  containing  $x$  and a morphism  $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$  such that  $\varphi(x) \neq x$ ; that is,  $x \notin Z(N_{\mathcal{F}}(Q))$ . Let

$$\mathcal{J} = \{D \in \mathcal{C} \mid x \in D = J(D), x \notin Z(N_{\mathcal{F}}(D))\}.$$

We claim that  $\mathcal{J} \neq \emptyset$ . Suppose that  $J(N_S(Q)) \notin \mathcal{J}$ . By (2.14),  $J(J(N_S(Q))) = J(N_S(Q))$  and since  $x \in Z(S)$  we have  $x \in J(N_S(Q))$ . Thus  $x \in Z(N_{\mathcal{F}}(J(N_S(Q))))$ . Next we show that for any subgroup  $D \in \mathcal{C}$ ,

$$(4.2) \quad \text{if } x \notin Z(N_{\mathcal{F}}(D)) \text{ and } x \in Z(N_{\mathcal{F}}(J(N_S(D)))) \text{, then } D \triangleleft \mathcal{F}.$$

Set  $\mathcal{G} = N_{\mathcal{F}}(D)$ , a fusion subsystem of  $\mathcal{F}$  on  $N_S(D)$ . Then,  $N_{\mathcal{G}}(J(N_S(D)))$  is a fusion subsystem of  $N_{\mathcal{F}}(J(N_S(D)))$  on  $N_S(D)$ . By Lemma 2.8,  $x \in Z(N_{\mathcal{F}}(J(N_S(D)))) \cap N_S(D) \leq Z(N_{\mathcal{G}}(J(N_S(D))))$ . But  $x \notin Z(\mathcal{G})$  and so  $Z(\mathcal{G}) < Z(N_{\mathcal{G}}(J(N_S(D))))$ . Furthermore, if  $x \in (Z(S))^p$ , then  $x \in (Z(N_S(D)))^p$ . By the minimality of  $|\mathcal{F}|$ , we have  $\mathcal{F} = \mathcal{G} = N_{\mathcal{F}}(D)$ , i.e.  $D \triangleleft \mathcal{F}$ , which proves (4.2). This shows for  $Q$  as above that  $Q \triangleleft \mathcal{F}$ , and then by Lemma 2.9, we get  $J(Q) \triangleleft \mathcal{F}$ . Thus,  $J(Q) \in \mathcal{J}$ . This shows that  $\mathcal{J} \neq \emptyset$ .

Choose  $D \in \mathcal{J}$  such that  $D$  is maximal with respect to inclusion and such that  $d(D)$  is maximal. Observe that this assumption along with (2.14) implies that if  $D \leq E$  such that  $J(E) \in \mathcal{J}$ , then

$$(4.3) \quad d(D) = d(J(E)) = d(E)$$

and

$$(4.4) \quad \mathcal{A}(D) \leq \mathcal{A}(E), \text{ which implies that } D = J(D) \leq J(E).$$

We shall show that  $D$  is an  $\mathcal{F}$ -centric subgroup of  $S$  that is normal in  $\mathcal{F}$ . First we show that  $x \in Z(N_{\mathcal{F}}(J(N_S(D))))$ . Suppose that  $x \notin Z(N_{\mathcal{F}}(J(N_S(D))))$ . Then,  $J(N_S(D)) \in \mathcal{J}$  because  $x \in Z(S)$  implies that  $x \in J(N_S(D))$ . So, by (4.4),  $D \leq J(N_S(D))$ . The maximality of  $D$  with respect to inclusion in  $\mathcal{J}$  implies that  $D = J(N_S(D))$ . Consequently,

$$N_S(D) \leq N_S(N_S(D)) \leq N_S(J(N_S(D))) = N_S(D),$$

which yields  $N_S(D) = S$  and  $J(S) \in \mathcal{J}$ , contradicting the original assumption that  $x \in Z(N_{\mathcal{F}}(J(S)))$ . Therefore, (4.2) implies that  $D \triangleleft \mathcal{F}$ .

Finally, we show that  $D$  is  $\mathcal{F}$ -centric. Since  $D \triangleleft \mathcal{F}$ , it suffices to show that  $C_S(D) \leq D$ . First, we claim that  $J(DC_S(D)) \in \mathcal{J}$ . Indeed,  $J(DC_S(D)) \in \mathcal{C}$  since  $J(DC_S(D))$  is normal in  $S$ . Also,  $x \in J(DC_S(D))$  since  $x \in D$ . Thus, it remains to show that  $x \notin Z(N_{\mathcal{F}}(J(DC_S(D))))$ . By Lemmas 2.8 and 2.9,  $Z(N_{\mathcal{F}}(J(DC_S(D)))) \leq Z(N_{\mathcal{F}}(DC_S(D)))$ , and so it suffices to show that  $x \notin Z(N_{\mathcal{F}}(DC_S(D)))$ . Suppose that  $x \in Z(N_{\mathcal{F}}(DC_S(D)))$  and let  $\varphi \in \text{Hom}_{\mathcal{F}}(\langle x \rangle, S)$ . Since  $D \triangleleft \mathcal{F}$  and  $x \in D$ ,  $\varphi$  extends to an  $\mathcal{F}$ -automorphism of  $D$ . Moreover, by the extension axiom, this automorphism extends to an  $\mathcal{F}$ -automorphism of  $DC_S(D)$ . Hence  $\varphi \in N_{\mathcal{F}}(DC_S(D))$  and  $\varphi(x) = x$ . This implies that  $x \in Z(\mathcal{F})$ , a contradiction. Consequently,  $J(DC_S(D)) \in \mathcal{J}$ . Thus, by (4.3), we get  $d(D) = d(J(DC_S(D)))$ .

Let  $A \in \mathcal{A}(D)$ ,  $y \in C_S(D)$ . Then,  $|\langle A, y \rangle| \leq d(D) = |A|$ , so  $y \in A$ . Thus  $C_S(D) \leq D$  and hence  $D$  is  $\mathcal{F}$ -centric.  $\square$

**Theorem 4.5.** *Let  $\mathcal{F}$  be a fusion system on  $S$ . Assume that  $p$  is odd or that  $\mathcal{F}$  is  $S_4$ -free. If  $C_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(J(S)) = \mathcal{F}_S(S)$ , then  $\mathcal{F} = \mathcal{F}_S(S)$ .*

*Proof.* By Theorem 4.1, we have  $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S))) = Z(\mathcal{F}_S(S)) = Z(S)$ . Hence,  $\mathcal{F} = C_{\mathcal{F}}(Z(S)) = \mathcal{F}_S(S)$ .  $\square$

The following corollary strengthens Frobenius' theorem for fusion systems (see [8, Theorem 3.11]) when  $p$  is odd or the fusion system is  $S_4$ -free. The proof mimics that of Thompson in [12], where he considers only the case where  $p$  is odd.

**Corollary 4.6.** *Let  $\mathcal{F}$  be a fusion system on  $S$ . Assume that  $p$  is odd or that  $\mathcal{F}$  is  $S_4$ -free. The following are equivalent:*

- (1)  $\mathcal{F} = \mathcal{F}_S(S)$ .
- (2)  $\text{Aut}_{\mathcal{F}}(Q)$  is a  $p$ -group for every nontrivial characteristic subgroup  $Q$  of  $S$ .
- (3)  $N_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$  for every nontrivial characteristic subgroup  $Q$  of  $S$ .

*Proof.* It is immediate from the definitions that (1) implies (2). Assume that (2) implies (3) does not hold and let  $\mathcal{F}$  be a minimal counterexample. Fix a nontrivial characteristic subgroup  $Q$  of  $S$  such that  $N_{\mathcal{F}}(Q) \neq \mathcal{F}_S(S)$ . If  $\mathcal{F}$  is  $S_4$ -free, then by [7, Propositions 6.3, 6.4],  $N_{\mathcal{F}}(Q)/Q$  is  $S_4$ -free. Also, if  $I$  is any subgroup of  $S$  containing  $Q$  and such that  $I/Q$  is characteristic in  $S/Q$ , then  $I$  is characteristic in  $S$ . This shows that  $\text{Aut}_{N_{\mathcal{F}}(Q)/Q}(I/Q)$  is a  $p$ -group and so the hypotheses of (2) hold for the fusion system  $N_{\mathcal{F}}(Q)/Q$ . By the minimality of  $\mathcal{F}$ , we conclude that  $N_{\mathcal{F}}(Q)/Q = \mathcal{F}_{S/Q}(S/Q) = \mathcal{F}_S(S)/Q$ . Now, by Lemma 2.5,  $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(Q)}(Q) = SC_{N_{\mathcal{F}}(Q)}(Q) = SC_{\mathcal{F}}(Q)$ . Then, [7, Proposition 3.4] implies that  $N_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$ , a contradiction. Therefore, (2) implies (3). Finally, if we specialize  $Q$  to be  $Z(S)$  and  $J(S)$  in (3), we get  $N_{\mathcal{F}}(J(S)) = \mathcal{F}_S(S) = C_{\mathcal{F}}(Z(S))$  and by Theorem 4.5,  $\mathcal{F} = \mathcal{F}_S(S)$ . Thus, (3) implies (1).  $\square$

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