

VARIETIES OF NILPOTENT ELEMENTS FOR SIMPLE LIE ALGEBRAS

UNIVERSITY OF GEORGIA VIGRE ALGEBRA GROUP¹

ABSTRACT. Let G be a simple algebraic group over $k = \mathbb{C}$, or $\overline{\mathbb{F}}_p$ where p is good. Set $\mathfrak{g} = \text{Lie } G$. Given $r \in \mathbb{N}$ and a faithful (restricted) representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, one can define a variety of nilpotent elements $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \{x \in \mathfrak{g} : \rho(x)^r = 0\}$. In this paper we determine this variety when ρ is an irreducible representation of minimal dimension or the adjoint representation.

1. Introduction

1.1. Let G be a simple algebraic group scheme defined over $\overline{\mathbb{F}}_p$ and $\mathfrak{g} = \text{Lie } G$ with p th power map $[p]$. Early work in the 1980s, by Jantzen [Jan1], Friedlander and Parshall [FP1] [FP2], and subsequent results by Suslin, Friedlander and Bendel [SFB1, SFB2] can be used to show that the spectrum of the cohomology ring for the first Frobenius kernel $H^\bullet(G_1, k)$ can be identified with the restricted nullcone $\mathcal{N}_1(\mathfrak{g}) := \{x \in \mathfrak{g} : x^{[p]} = 0\}$. In concrete terms, this algebraic variety can be identified with a set of matrices where each matrix in the set when multiplied by itself p times is zero. Even though the algebraic variety $\mathcal{N}_1(\mathfrak{g})$ plays a crucial role in the cohomology theory for restricted Lie algebras, only recently has a concrete description of this variety been given via orbit closures. Carlson, Lin, Nakano and Parshall [CLNP] have computed the restricted nullcone in the case when the underlying field has good characteristic. The description of the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ is given as the closure of a certain Richardson orbit. In particular, the restricted nullcone is an irreducible variety. The underlying methods employed in [CLNP] heavily relied on a conjecture of Jantzen on the support variety of Weyl modules [Jan2] which was verified by Nakano, Parshall and Vella [NPV].

Now assume that G is a simple algebraic group over an arbitrary algebraically closed field k . Let $\mathcal{N}(\mathfrak{g})$ be the set of nilpotent elements in \mathfrak{g} . The variety $\mathcal{N}(\mathfrak{g})$ is often referred to as the (ordinary) nullcone. The computation of the restricted nullcone motivates us to formulate more general problem. Let $\Psi : G \rightarrow \text{GL}(V)$ be a faithful representation of G . The derivative of Ψ , $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, is a faithful (restricted) representation of \mathfrak{g} . Define

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \{x \in \mathfrak{g} : \rho(x)^r = 0\}.$$

Then $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is an algebraic variety contained in $\mathcal{N}(\mathfrak{g})$ which is invariant under the action of G . Since there are only finitely many G -orbits on $\mathcal{N}(\mathfrak{g})$, it is reasonable to try to describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ as the union of closures of G -orbits. The varieties $\mathcal{N}_{r,\rho}(\mathfrak{g})$ might be relevant in the

¹The members of the UGA VIGRE Algebra Group are David J. Benson, Phil Bergonio, Brian D. Boe, Leonard Chastkofsky, Bobbe Cooper, G. Michael Guy, Jo Jang Hyun, Jerome Jungster, Graham Matthews, Nadia Mazza, Daniel K. Nakano, and Kenyon Platt.

future investigation of the cohomology for the quantum groups at roots of unity and for affine Lie algebras.

Our method in approaching this problem involves extending ideas used in the work of Nakano and Tanisaki [NT] on realizing orbit closures as support varieties. In that paper the authors show how to compute orbit closures in $\mathfrak{gl}(V)$ after intersecting with \mathfrak{g} . For fields of characteristic zero, one can then use the Jacobson-Morozov theorem, weighted Dynkin diagrams, as well as the knowledge about the characters of irreducible representations to calculate these varieties. In characteristic $p > 0$ one has to be more careful, and use tables given in [Law1, Law2] as a replacement for the Jacobson-Morozov theorem.

The main results in this paper are the computation of $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when ρ is an irreducible representation of minimal dimension or the adjoint representation of \mathfrak{g} when the characteristic of the field is either zero or a good prime. When \mathfrak{g} is a classical Lie algebra (i.e. the root system is of type A_l , B_l , C_l or D_l), the minimal dimension representation is the standard matrix representation of the Lie algebra on column vectors. For these algebras the results on the minimal representation and the adjoint representation are presented in Section 3. Our results for the exceptional Lie algebras are given in Section 4. As a corollary of our work, when the characteristic of the field is p , one can set $r = p$ to recover all the calculations of the restricted nullcone in [CLNP]. It should be noted that the computations in this paper do not make use of the verification of the Jantzen conjecture.

2. Intersections

2.1. Notation: Let G be a simple algebraic group defined over k . Let T be a maximal torus of G . The root system Φ with respect to the pair (G, T) is identified with a subset of the set of weights $X(T)$. Let Φ^+ be a set of positive roots with the negative roots being denoted by Φ^- . The set of simple roots determined by Φ^+ is $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Our convention is to use the ordering of simple roots given in [Jan3] following Bourbaki.

Throughout this paper we will always assume that p is a good prime for Φ or zero. A list of good primes is provided below.

- Φ of type A_l , all primes;
- Φ of type B_l, C_l, D_l , $p \geq 3$;
- Φ of type E_6, E_7, F_4, G_2 , $p \geq 5$;
- Φ of type E_8 , $p \geq 7$.

Let W be the Weyl group and h be the Coxeter number for Φ . For $1 \leq i \leq \ell$, let ω_i be the fundamental dominant weights. If $J \subset \Delta$ then let $P_J = L_J \times U_J$ be the parabolic subgroup determined by J with Levi factor L_J . Set $\mathfrak{u}_J = \text{Lie } U_J$.

Let $\mathcal{N}(\mathfrak{g})$ be the variety of nilpotent elements of \mathfrak{g} which is often called the nullcone. The group G acts on $\mathcal{N}(\mathfrak{g})$ via conjugation and $\mathcal{N}(\mathfrak{g})$ has finitely many G -orbits [L]. Furthermore, the nullcone is an irreducible variety of dimension equal to $|\Phi|$.

When k is an algebraically closed field and the characteristic is a good prime the classification and structures of these orbits coincide with the orbit theory for complex simple Lie algebras (see [Car, CM, Hum]). If $J \subseteq \Delta$ then $G \cdot \mathfrak{u}_J$ is a closed, irreducible subvariety of $\mathcal{N}(\mathfrak{g})$ and there exists a unique dense open G -orbit in $G \cdot \mathfrak{u}_J$. Orbits which arise in this way are the Richardson orbits in \mathfrak{g} .

2.2. Let G be a simple algebraic group over an algebraically closed field k . Set $\mathfrak{g} = \text{Lie } G$. Let $\Psi : G \hookrightarrow \text{GL}(V)$ be a faithful finite-dimensional representation of G . The derivative $\rho = d\Psi : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ is also a faithful finite-dimensional representation of \mathfrak{g} , with $N = \dim V$. The goal of this section is to provide formulas which describe intersecting nilpotent orbits of $\mathfrak{gl}(V)$, and their closures with \mathfrak{g} .

Since there are finitely many G orbits on $\mathcal{N}(\mathfrak{g})$, we can let $S := \{z_1, z_2, \dots, z_t\}$ be a set of orbit representatives of G on $\mathcal{N}(\mathfrak{g})$. There are two partial orderings related to orbit closures that we need to introduce. First, if $x, y \in S$ then $x \preceq y$ if and only if $\overline{G \cdot x} \subseteq \overline{G \cdot y}$. The other ordering of relevance is the dominance ordering on partitions of N denoted by \trianglelefteq . If λ is a partition of N then let \mathcal{O}_λ be the orbit in $\mathcal{N}(\mathfrak{gl}(V))$ corresponding to the orbit of the Jordan block of size λ . We have $\overline{\mathcal{O}_\mu} \subseteq \overline{\mathcal{O}_\lambda}$ if and only if $\mu \trianglelefteq \lambda$. For $z \in S$, let $\epsilon(z)$ be the partition of N corresponding to the orbit containing $\rho(z) \in \mathcal{N}(\mathfrak{gl}(V))$.

For $r \in \mathbb{N}$, set $\mathcal{N}_r(\mathfrak{gl}(V)) = \{x \in \mathfrak{gl}(V) : x^r = 0\}$. We note that under the identification of \mathfrak{g} with $\rho(\mathfrak{g})$, we have

$$(2.2.1) \quad \mathcal{N}_{r,\rho}(\mathfrak{g}) \cong \mathfrak{g} \cap \mathcal{N}_r(\mathfrak{gl}(V)).$$

Now suppose that $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ factors as $\rho = \phi \circ \iota$, where $\phi : \mathfrak{gl}_m(k) \hookrightarrow \mathfrak{gl}(V)$ is a faithful representation for $\mathfrak{gl}_m(k)$ and $\iota : \mathfrak{g} \hookrightarrow \mathfrak{gl}_m(k)$ is a Lie algebra monomorphism. Note that this condition means that the action of \mathfrak{g} on V can be extended to an action of $\mathfrak{gl}_m(k)$ on V . Then it follows that under the identification of \mathfrak{g} in $\mathfrak{gl}_m(k)$ under ι one has

$$(2.2.2) \quad \mathcal{N}_{r,\rho}(\mathfrak{g}) \cong \mathfrak{g} \cap \mathcal{N}_{r,\phi}(\mathfrak{gl}_m(k)).$$

This formula will be useful when we study these nilpotent varieties for the adjoint representations for classical Lie algebras.

2.3. We first show what happens when we intersect orbits in $\mathcal{N}(\mathfrak{gl}(V))$ with \mathfrak{g} .

Proposition. *Let $z \in S$.*

- (a) *If $\mu \notin \{\epsilon(z_1), \epsilon(z_2), \dots, \epsilon(z_t)\}$ then $\mathcal{O}_\mu \cap \mathfrak{g} = \emptyset$.*
- (b) *$\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g} = \bigcup_{j \in I} G \cdot z_j$ where $I = \{j : \epsilon(z_j) = \epsilon(z)\}$.*

Proof. (a) Let $y \in \mathcal{O}_\mu \cap \mathfrak{g}$. Then y is conjugate to x_μ under $\text{GL}(V)$. Moreover, $y \in \mathcal{N}(\mathfrak{g})$ so y is conjugate under G to some z_j where $j = 1, 2, \dots, t$. Hence, z_j is conjugate to x_μ under $\text{GL}(V)$, and $x_\mu = x_{\epsilon(z_j)}$. It follows that $\mu = \epsilon(z_j)$ for some j , which contradicts our hypothesis. Consequently, $\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g} = \emptyset$.

(b) The set $\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g}$ is G -stable, so

$$\mathcal{O}_{\epsilon(z)} \cap \mathfrak{g} = \bigcup_{i \in I} G \cdot z_i$$

for some $I \subseteq S$. But from part (a) we have seen that if $y \in \mathcal{O}_{\epsilon(z)} \cap \mathfrak{g}$ then $\epsilon(z) = \epsilon(z_j)$ for some j . So this condition completely characterizes I . \square

2.4. The problem of intersecting closures of orbits in $\mathcal{N}(\mathfrak{gl}(V))$ with \mathfrak{g} is more delicate. We need to take into account the dominance ordering and the ordering on orbits in $\mathcal{N}(\mathfrak{g})$. For this purpose, we introduce two indexing sets. First, let $\epsilon(S) = \{\epsilon(z_j) : j = 1, 2, \dots, t\}$. For $\mu \vdash N$ set

$$\Omega(\mu) = \{\gamma \vdash N : \gamma \text{ is maximal with respect to } \preceq \text{ such that } \gamma \in \epsilon(S) \text{ and } \gamma \preceq \mu\}.$$

Furthermore, for $\gamma \vdash N$ let

$$\Pi(\gamma) = \{j : z_j \text{ is maximal with respect to } \preceq \text{ such that } z_j \in S \text{ and } \epsilon(z_j) \preceq \gamma\}.$$

Proposition. *Let $\mu \vdash N$ and \mathcal{O}_μ be the corresponding orbit in $\mathfrak{gl}(V)$. Then*

$$\overline{\mathcal{O}_\mu} \cap \mathfrak{g} = \bigcup_{\gamma \in \Omega(\mu)} \bigcup_{j \in \Pi(\gamma)} \overline{G \cdot z_j}.$$

Proof. First observe that

$$\overline{\mathcal{O}_\mu} \cap \mathfrak{g} = \left(\bigcup_{\sigma \preceq \mu} \mathcal{O}_\sigma \right) \cap \mathfrak{g} = \bigcup_{\sigma \preceq \mu} (\mathcal{O}_\sigma \cap \mathfrak{g}) = \bigcup_{\sigma \preceq \mu: \sigma \in \epsilon(S)} (\mathcal{O}_\sigma \cap \mathfrak{g})$$

The last equality follows by Proposition 2.3(a). Now

$$\bigcup_{\sigma \preceq \mu: \sigma \in \epsilon(S)} (\mathcal{O}_\sigma \cap \mathfrak{g}) = \left(\bigcup_{\sigma \preceq \mu: \sigma \in \epsilon(S)} \mathcal{O}_\sigma \right) \cap \mathfrak{g} \subseteq \left(\bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}_\gamma} \right) \cap \mathfrak{g} = \bigcup_{\gamma \in \Omega(\mu)} (\overline{\mathcal{O}_\gamma} \cap \mathfrak{g}).$$

On the other hand,

$$\bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}_\gamma} \cap \mathfrak{g} = \left(\bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}_\gamma} \right) \cap \mathfrak{g} \subseteq \overline{\mathcal{O}_\mu} \cap \mathfrak{g}.$$

This shows that $\overline{\mathcal{O}_\mu} \cap \mathfrak{g} = \bigcup_{\gamma \in \Omega(\mu)} \overline{\mathcal{O}_\gamma} \cap \mathfrak{g}$.

Next we need to analyze $\overline{\mathcal{O}_\gamma} \cap \mathfrak{g}$ where $\gamma \in \epsilon(S)$. From Proposition 2.3, we have

$$\overline{\mathcal{O}_\gamma} \cap \mathfrak{g} = \left(\bigcup_{\sigma \preceq \gamma} \mathcal{O}_\sigma \right) \cap \mathfrak{g} = \bigcup_{\sigma \preceq \gamma} (\mathcal{O}_\sigma \cap \mathfrak{g}) = \bigcup_{\sigma \in \mathcal{X}} \left[\bigcup_{i \in \mathcal{Y}(\sigma)} G \cdot z_i \right]$$

where $\mathcal{X} = \{\sigma \in \lambda(S) : \sigma \preceq \gamma\}$ and $\mathcal{Y}(\sigma) = \{i : \epsilon(z_i) = \sigma\}$. Since $\overline{\mathcal{O}_\gamma} \cap \mathfrak{g}$ is closed, we can express the last union as

$$\bigcup_{\sigma \in \mathcal{X}} \left[\bigcup_{i \in \mathcal{Y}(\sigma)} G \cdot z_i \right] = \bigcup_{j \in \Pi(\gamma)} \overline{G \cdot z_j}.$$

□

3. Classical Groups

3.1. Partitions. Let G be a classical group with underlying irreducible root system of type A_l , B_l , C_l , or D_l . Set $N = l + 1$ (resp. $2l + 1$, $2l$, $2l$) when Φ is of type A_l (resp. B_l , C_l , D_l). For classical groups, the nilpotent orbits are parameterized by certain partitions of N . If $X = A$, (resp. B , C , D), let $\mathcal{P}_X(N)$ be the set of partitions of N parameterizing the set of nilpotent orbits for A_l (resp. B_l , C_l , D_l). A precise description of $\mathcal{P}_X(N)$ [CM, Thm. 5.1.2–5.1.4] is given as follows.

- $\mathcal{P}_A(N)$: all partitions of N .
- $\mathcal{P}_B(N)$: partitions of N such that even parts occur with even multiplicity.
- $\mathcal{P}_C(N)$: partitions of N such that odd parts occur with even multiplicity.
- $\mathcal{P}_D(N)$: partitions of N such that even parts occur with even multiplicity.

A *very even partition* in $\mathcal{P}_D(N)$ is a partition of N with only even parts. For $\lambda \in \mathcal{P}_X(N)$, let \mathcal{O}_λ be the corresponding nilpotent orbit with the following additional proviso in the case of type D . When Φ is of type D , for very even partitions λ , there are two orbits corresponding to λ . The two orbits will be denoted by \mathcal{O}_λ^I and \mathcal{O}_λ^{II} . Finally, for a given $X = B$, C , or D if $\lambda \in \mathcal{P}_A(N)$, let $\lambda_X \in \mathcal{P}_X(N)$ denote the X -collapse of λ . The partition given by the X -collapse λ_X is the unique maximal partition in $\mathcal{P}_X(N)$ dominated by λ [CM, Lem. 6.3.3].

3.2. Minimal Representation: For classical simple algebraic groups, one has a realization of these groups via matrices which preserve certain symmetric and skew symmetric forms. One also has a standard representation for these groups given by the action of the group on column vectors. This irreducible representation is the minimal dimensional non-trivial representation for G . The standard representation induces an embedding $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, where $\dim V := N = l + 1$ (resp. $N = 2l + 1$, $N = 2l$, $N = 2l$) for type A_l (resp. B_l , C_l , D_l). The following result shows that when ρ is the standard representation, the closed subvariety $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is an irreducible variety in types A_l , B_l , or C_l . An explicit description of this variety is also provided below.

Theorem. *Let G be a simple algebraic group of classical type and ρ be the minimal representation. For $r \geq 0$, express $N = dr + s$ where $0 \leq s \leq r - 1$, and set $\lambda_{N,r} = (r^d, s)$.*

- (a) *If Φ is of type A then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_{N,r}}$.*
- (b) *If Φ is of type X where X is B or C then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,r})_X}$.*
- (c) *If Φ is of type D then*
 - (i) $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,r})_X}$ *when $(\lambda_{N,r})_X$ is not very even;*
 - (ii) $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,r})_X}^I \cup \overline{\mathcal{O}}_{(\lambda_{N,r})_X}^{II}$ *when $(\lambda_{N,r})_X$ is very even.*

Proof. Set $\lambda := \lambda_{N,r} = (r^d, s)$ to be the partition with d parts of size r and one part of size d . Then λ is maximal in the dominance ordering of partitions of N which have parts of size at most r . First assume that Φ is type A . The variety $\mathcal{N}_{r,\rho}(\mathfrak{g})$ in this situation consists of matrices with Jordan block sizes at most r . Therefore, $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_\lambda$.

Now assume that G has type B , C , or D . For the sake of notation (in this proof), for $\lambda \in \mathcal{P}_A(N)$, let $\mathcal{O}_\lambda^{\mathfrak{gl}(V)}$ be the corresponding nilpotent orbit in $\mathfrak{gl}(V)$. Since the X -collapse λ_X is the unique maximal partition in $\mathcal{P}_X(N)$ dominated by λ , we have by Proposition 2.4,

$$(3.2.1) \quad \overline{\mathcal{O}_\lambda^{\mathfrak{gl}(V)}} \cap \mathfrak{g} = \bigcup_{j \in \Pi(\lambda_X)} \overline{G \cdot z_j}.$$

Since $\mathcal{N}_{r,\rho}(\mathfrak{g}) \cong \mathfrak{g} \cap \mathcal{N}_r(\mathfrak{gl}(V))$, one can use the computation in type A along the formula above to deduce that $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda_X}}$ as long as λ_X is not very even in type D . In the case that λ_X is very even and in type D , we can conclude using the same reasoning that $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda_X}^I} \cup \overline{\mathcal{O}_{\lambda_X}^{II}}$. \square

3.3. Adjoint Representation, characteristic zero: In this subsection we determine $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when the characteristic of k is zero and ρ is the adjoint representation, using a tensor-product realization of the adjoint representation.

Theorem. *Let \mathfrak{g} be a classical Lie algebra of type $X = A, B, C$, or D over an algebraically closed field k of characteristic 0. Let ρ be the adjoint representation of \mathfrak{g} . Let N be the dimension of the standard representation of \mathfrak{g} . Given a positive integer r , set $a = \lfloor (r+1)/2 \rfloor$, and $\lambda = \lambda_{N,a}$.*

- (a) *If $X = A$ or C , then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda_X}}$.*
- (b) *If $X = B$ or D and $r \geq 2N - 1$, then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda_X}}$.*
- (c) *If $X = B$ or D and r is odd with $r \leq 2N - 3$, write $N - (a + 1) = d'(a - 1) + s'$ where $0 \leq s' < a - 1$, and let $\lambda' = (a + 1, (a - 1)^{d'}, s')$. Then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda_X}} \cup \overline{\mathcal{O}_{\lambda'_X}}$.*
- (d) *If $X = B$ or D and r is even with $r \leq 2N - 2$, write $N - (a + 1) = d'a + s'$ where $0 \leq s' < a$, and let $\lambda'' = (a + 1, a^{d'}, s')$. Then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda''_X}}$.*

In our notation, if μ is very even (in type D) then $\overline{\mathcal{O}_\mu} = \overline{\mathcal{O}_\mu^I} \cup \overline{\mathcal{O}_\mu^{II}}$.

Proof. We first do the computation for $\mathfrak{gl}(V)$, after extending the adjoint representation from \mathfrak{g} to $\mathfrak{gl}(V)$. We then apply (2.2.2) to the embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, along with the collapse argument in (3.2.1), to obtain the theorem for general classical \mathfrak{g} .

Assume first that $\mathfrak{g} = \mathfrak{gl}(V)$ with $\dim V = N$. It is well known that the adjoint representation of \mathfrak{g} is naturally isomorphic to $V \otimes V^*$ where $V^* = \text{Hom}_k(V, k)$ is the contragredient (dual) module.

Let $x \in \mathfrak{g}$ be nilpotent, corresponding to the partition $\mu = (\mu_1, \dots, \mu_t) \vdash N$. Given $v \in V$, $w \in V^*$ we have

$$(3.3.1) \quad x^r \cdot (v \otimes w) = \sum_{k=0}^r \binom{r}{k} x^k v \otimes x^{r-k} w.$$

Then $x \in \mathcal{N}_{r,\rho}(\mathfrak{g})$ if and only if $x^r(v \otimes w) = 0$ for all $v \in V, w \in V^*$.

Choose a Jordan basis for V with respect to x , and index the basis vectors corresponding to the largest $(\mu_1 \times \mu_1)$ block as $v_0, v_1 = xv_0, \dots, v_{\mu_1-1} = x^{\mu_1-1}v_0$ where $x^{\mu_1}v_0 = 0$. The largest Jordan block of x acting on V^* has Jordan basis $w_i = (-1)^i v_{\mu_1-1-i}^*$, $0 \leq i \leq \mu_1 - 1$. Clearly (3.3.1) is 0 for all v, w if and only if it is 0 for $v = v_0, w = w_0$. When $v = v_0, w = w_0$,

the terms in the sum in (3.3.1) are linearly independent (or 0); thus every term must be 0. Thus for each k ($0 \leq k \leq r$), either $x^k v_0 = 0$ or $x^{r-k} w_0 = 0$; equivalently, $\max(k, r-k) \geq \mu_1$. The key term is $k = \mu_1 - 1$, in the sense that if the condition is satisfied for this k , then it is satisfied for all larger and smaller k . So the condition is reduced to simply $r - (\mu_1 - 1) \geq \mu_1$; i.e., $\mu_1 \leq (r+1)/2$. There is evidently a unique maximal (with respect to \trianglelefteq) partition μ of N satisfying this condition. Set $a = \min(\lfloor (r+1)/2 \rfloor, N)$ and $\mu = \lambda_{N,a}$. Then $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_\mu$. Note that the answer is the same for $\mathfrak{g} = \mathfrak{sl}(V)$ because, as \mathfrak{g} -module, $V \otimes V^* \cong k \oplus \mathfrak{g}$.

In types B, C , and D , the natural representation V of \mathfrak{g} is self-dual (because \mathfrak{g} preserves a non-degenerate bilinear form on V), so the adjoint representation ρ occurs as a summand of $V \otimes V$. Assume \mathfrak{g} is of type C (resp. B or D). Using the fact that \mathfrak{g} preserves a skew-symmetric (resp. symmetric) form, and following through the identifications of $\mathfrak{gl}(V)$ with $V \otimes V^*$ and of V^* with V , one finds that \mathfrak{g} corresponds to $S^2(V)$, the degree 2 symmetric algebra on V (resp. $\Lambda^2(V)$, the degree 2 exterior algebra on V). Thus ρ can be realized as the standard action of \mathfrak{g} on $S^2(V)$ (resp. $\Lambda^2(V)$). Note that this action lifts to $\mathfrak{gl}(V)$. Thus it suffices to determine the order r nilpotent elements for the action ρ of $\mathfrak{gl}(V)$ on $S^2(V)$ (resp. $\Lambda^2(V)$), and then intersect with \mathfrak{g} .

Consider first the obvious representation ϕ of $\mathfrak{gl}(V)$ on $S^2(V)$. Let $x \in \mathfrak{gl}(V)$ be nilpotent, corresponding to a partition $\mu \vdash N$. The action of x^r on $S^2(V)$ is again given by (3.3.1), where now $v, w \in V$. Let v_0, \dots, v_{μ_1-1} again be the portion of a Jordan basis for V corresponding to the largest block of x . Then $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ if and only if (3.3.1) is 0 when $v = w = v_0$. In this case, (3.3.1) becomes

$$(3.3.2) \quad x^r \cdot (v_0 \otimes v_0) = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} 2 \binom{r}{k} x^k v_0 \otimes x^{r-k} v_0 + \delta_{r,\text{even}} \binom{r}{r/2} x^{r/2} v_0 \otimes x^{r/2} v_0,$$

where $\delta_{r,\text{even}}$ is 1 if r is even and 0 otherwise. The terms written are all linearly independent (if they are non-zero), so $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ if and only if $\mu_1 \leq (r+1)/2$. This condition characterizes a unique maximal partition $\mu = \lambda_{N,a}$ as before. This completes the proof in type C .

Assume now that ϕ is the representation of $\mathfrak{gl}(V)$ on $\Lambda^2(V)$. Let $x = x_\mu \in \mathfrak{gl}(V)$ be nilpotent element corresponding to a partition $\mu \vdash N$, and let v_0, \dots, v_{μ_1-1} as above be the portion of a Jordan basis for V corresponding to the $\mu_1 \times \mu_1$ block of x . Also let w_0, \dots, w_{μ_2-1} be the portion of the Jordan basis for V corresponding to the $\mu_2 \times \mu_2$ block of x . Then $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ if and only if

$$(3.3.3) \quad 0 = x^r \cdot (v_0 \wedge w_0) = \sum_{k=0}^r \binom{r}{k} x^k v_0 \wedge x^{r-k} w_0$$

and

$$(3.3.4) \quad \begin{aligned} 0 = x^r \cdot (v_0 \wedge v_1) &= \sum_{k=0}^r \binom{r}{k} x^k v_0 \wedge x^{r+1-k} v_1 \\ &= \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \left[\binom{r}{k} - \binom{r}{r+1-k} \right] x^k v_0 \wedge x^{r+1-k} v_0. \end{aligned}$$

The terms in (3.3.3) are linearly independent or 0, so either $k \geq \mu_1$ or $r - k \geq \mu_2$, which reduces to $r - (\mu_1 - 1) \geq \mu_2$; i.e.,

$$(3.3.5) \quad \mu_1 + \mu_2 \leq r + 1.$$

The terms in (3.3.4) are linearly independent or 0, and the difference of binomial coefficients is never 0, so $\mu_1 \leq r + 1 - \lfloor r/2 \rfloor$; i.e.,

$$(3.3.6) \quad \mu_1 \leq \lfloor (r + 3)/2 \rfloor.$$

When r is odd, there may be two maximal partitions μ of N which satisfy (3.3.5) and (3.3.6). The first is $\lambda_{N,a}$ where $a = (r + 1)/2$ as above. The second, μ' (which only occurs when $r \leq 2N - 3$), has $\mu'_1 = (r + 3)/2 = a + 1$. Write $N - (a + 1) = d'(a - 1) + s'$ with $0 \leq s' < a - 1$. Then $\mu' = (a + 1, (a - 1)^{d'}, s') = \lambda'$ (as in (c) of the statement of the theorem).

Assume r is even. We will show there is a unique maximal partition μ satisfying (3.3.5) and (3.3.6). When $(r + 2)/2 > N$ we are forced to take $\mu_1 \leq N \leq r/2 = a$, and then $\mu = \lambda_{N,a}$. Assuming $(r + 2)/2 \leq N$, we may take $\mu_1 = (r + 2)/2 = a + 1$. Write $N - (a + 1) = d'a + s'$ with $0 \leq s' < a$. Then $\mu = (a + 1, a^{d'}, s') = \lambda''$ (as in (d) of the statement of the theorem). (Note that $\lambda'' \supseteq \lambda_{N,a}$. For instance, when r is small enough, $\lambda'' = ((r+2)/2, (r/2)^{d'}, s') \supseteq \lambda_{N,a} = ((r/2)^{d'+1}, s'+1)$. The situation for larger r is simpler.) This completes the verification of the theorem in types B and D . \square

We remark that there are instances in the previous theorem where $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is reducible and has three irreducible components. Consider the case D_4 where $N = 8$ and $r = 3$. Then $\lambda = (2^4)$ and $\lambda' = (3, 1^5)$. Since λ is very even and $\lambda' \in \mathcal{P}_D(8)$ we have $\mathcal{N}_{3,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{(2^4)}^I \cup \overline{\mathcal{O}}_{(2^4)}^{II} \cup \overline{\mathcal{O}}_{(3,1^5)}$.

3.4. Adjoint Representation, prime characteristic: We now use the tensor-product techniques of the previous subsection to determine $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when the characteristic of k is a prime p and ρ is the adjoint representation. The issue that we need to deal with is that certain binomial coefficients are now zero in k . First, we need a lemma describing binomial coefficients upon reduction modulo p .

Lemma. *Let p be prime, $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and assume $p^m \leq r < p^{m+1}$.*

- (a) $p \mid \binom{p^m}{k}$ for $0 < k < p^m$.
- (b) $p \nmid \binom{r}{k}$ for $2^n p^m \leq r \leq 2^{n+1} p^m$, $k = 2^n p^m$.
- (c) $p \nmid \left[\binom{r}{k} - \binom{r}{k-1} \right]$ for $2^n p^m \leq r \leq 2^{n+1} p^m - 2$, $k = 2^n p^m$.
- (d) $p \nmid \left[\binom{r}{k} - \binom{r}{k-1} \right]$ for $r = 2^{n+1} p^m - 1$ or $2^{n+1} p^m$ when $k = 2^n p^m + 1$ and $p \neq 2$.

Proof. The general fact we need is the following: if $r = ap^m + b$ with $0 \leq b < p^m$, and $k = cp^m + d$ with $0 \leq d < p^m$ then $\binom{r}{k}$ is congruent to $\binom{a}{c} \binom{b}{d}$ modulo p . This can be seen by looking at the identity $(x + 1)^r = (x^{p^m} + 1)^a (x + 1)^b$ over \mathbb{F}_p and comparing coefficients of $x^k = (x^{p^m})^c x^d$.

(a) We have $a = 1$, $b = 0$, $c = 0$ and $d = k$, so p divides $\binom{b}{d}$. (b) This is a special case of the more general statement that if $k = cp^m \leq r < p^{m+1}$ then p does not divide $\binom{r}{k}$. One

can use the fact that $c \leq a < p$ and $d = 0$. (c) To compute $\binom{r}{k-1}$, note that $d = p^m - 1$ and $b < p^m - 1$ so $\binom{r}{k-1}$ is zero modulo p . Use (b) to see that $\binom{r}{k}$ is nonzero modulo p . For part (d), there are two cases. If $r = 2^{n+1}p^m - 1$ then $\binom{r}{k}$ is congruent to $-\binom{2^{n+1}-1}{2^n}$ and $\binom{r}{k-1}$ is congruent to $\binom{2^{n+1}-1}{2^n}$. Since $r < p^{m+1}$ it follows that $2^{n+1} - 1 < p$ and these binomial coefficients are nonzero modulo p . Their difference is twice the first one, which is still nonzero modulo p . For the second case when $r = 2^{n+1}p^m$, $\binom{r}{k}$ is zero modulo p and $\binom{r}{k-1}$ is congruent to $\binom{2^{n+1}}{2^n}$ modulo p . \square

The theorem below gives a concrete description of $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for a classical Lie algebra over an algebraically closed field of characteristic $p > 0$ where ρ is the adjoint representation.

Theorem. *Let \mathfrak{g} be a classical Lie algebra of type $X = A, B, C,$ or D over an algebraically closed field k of characteristic $p > 0$, where p is a good prime for \mathfrak{g} . Let ρ be the adjoint representation of \mathfrak{g} and $\text{std} : \mathfrak{g} \hookrightarrow \mathfrak{gl}_N(k)$ the standard representation.*

- (a) *When $1 \leq r < p$, $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is the union of the closures of the orbits given by the same partitions as in characteristic zero.*
- (b) *When $p^m \leq r < p^{m+1}$, $m > 0$,*

$$(3.4.1) \quad \mathcal{N}_{r,\rho}(\mathfrak{g}) = \mathcal{N}_{p^m,\rho}(\mathfrak{g}) = \mathcal{N}_{p^m,\text{std}}(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{N,p^m})_X}.$$

Proof. In view of Section 2, it suffices to consider the representation of $\mathfrak{gl}(V)$ on $V \otimes V^*$ for $X = A$, on $S^2(V)$ for $X = C$, and on $\Lambda^2(V)$ for $X = B, D$. (In all but the first case, we are assuming $p > 2$, so the identifications of the adjoint representation in the previous subsection remain valid in characteristic p .) Let $x \in \mathfrak{gl}(V)$ be nilpotent, associated to the partition $\mu \vdash N$.

Let ϕ be the representation of $\mathfrak{gl}(V)$ on $V \otimes V^*$. Then $\phi(x^r)$ is given by (3.3.1). As in characteristic zero, choose a Jordan basis for V with respect to x , labelled as before. Then $\phi(x^r) = 0$ if and only if every term in $x^r \cdot (v_0 \otimes w_0)$ is zero. In characteristic p , this may happen because $\binom{r}{k}$ may be 0 in k . However, when $1 \leq r < p$ this does not occur, and the analysis is identical to that in characteristic zero, with the answer given by the same partition(s) μ .

Suppose that $r = p^m$, $m > 0$. All the binomial coefficients $\binom{p^m}{k} = 0$ for $0 < k < p^m$, so

$$x^{p^m} \cdot (v_0 \otimes w_0) = x^{p^m} v_0 \otimes w_0 + v_0 \otimes x^{p^m} w_0,$$

which is zero if and only if $p^m \geq \mu_1$. Thus $\mu = \lambda_{N,p^m}$. This verifies the last two equalities in (3.4.1) (when $X = A$).

Assume that $p^m \leq r < p^{m+1}$ and choose $n \geq 0$ such that $2^n p^m \leq r \leq 2^{n+1} p^m$. We will show that $\mathcal{N}_{r,\phi}(\mathfrak{gl}(V)) = \mathcal{N}_{2^n p^m,\phi}(\mathfrak{gl}(V))$. By induction on n , this will verify the first equality in (3.4.1), and thus prove the theorem (in type A). Since \supseteq is obvious, it remains to check \subseteq . Let $x \in \mathcal{N}_{r,\phi}(\mathfrak{gl}(V))$ and consider the term in $x^r \cdot (v_0 \otimes w_0)$ with $k = 2^n p^m$ (recall (3.3.1)):

$$\binom{r}{2^n p^m} x^{2^n p^m} v_0 \otimes x^{r-2^n p^m} w_0 = 0.$$

By the lemma, the binomial coefficient is not zero. Since $r - 2^n p^m \leq 2^{n+1} p^m - 2^n p^m = 2^n p^m$, we must have $2^n p^m \geq \mu_1$. This proves $x \in \mathcal{N}_{2^n p^m, \rho}(\mathfrak{g})$, and hence the inclusion $\mathcal{N}_{r, \phi}(\mathfrak{gl}(V)) \subset \mathcal{N}_{2^n p^m, \phi}(\mathfrak{gl}(V))$.

Consider $X = C$ and let ϕ be the representation of $\mathfrak{gl}(V)$ on $S^2(V)$, with $p > 2$, and with notation as in characteristic zero. Then $\phi(x^r) = 0$ if and only if every term of (3.3.2) is zero. When $r < p$, the conditions are the same as in characteristic zero (because none of the binomial coefficients are zero). When $r = p^m$, (3.3.2) reduces to

$$x^{p^m} \cdot (v_0 \otimes v_0) = 2v_0 \otimes x^{p^m} v_0,$$

which is zero if and only if $p^m \geq \mu_1$. Thus $\mu = \lambda_{N, p^m}$, and this verifies the last two equalities in (3.4.1) for $X = C$. A similar argument to the one given in the previous paragraph shows that $\mathcal{N}_{r, \phi}(\mathfrak{gl}(V)) = \mathcal{N}_{2^n p^m, \phi}(\mathfrak{gl}(V))$ when $2^n p^m \leq r \leq \min(2^{n+1} p^m, p^{m+1} - 1)$. One needs to look at the term $k = r - 2^n p^m$ in (3.3.2).

Finally, let $X = B$ or D and let ϕ be the representation of $\mathfrak{gl}(V)$ on $\Lambda^2(V)$, with $p > 2$. Then $\phi(x^r) = 0$ if and only if (3.3.3) and (3.3.4) hold. When $r < p$, the resulting conditions on μ are identical to the characteristic zero results, which proves (a).

When $r = p^m$ we get

$$x^{p^m} v_0 \wedge w_0 + v_0 \wedge x^{p^m} w_0 = 0 = x^{p^m} v_0 \wedge x v_0 + v_0 \wedge x^{p^m+1} v_0,$$

which is true if and only if $p^m \geq \mu_1$. Thus $\mu = \lambda_{N, p^m}$, and this verifies the last two equalities in (3.4.1) for $X = B$ or D .

Assume $2^n p^m \leq r \leq \min(2^{n+1} p^m, p^{m+1} - 1)$. We wish to show that $\mathcal{N}_{r, \phi}(\mathfrak{gl}(V)) = \mathcal{N}_{2^n p^m, \phi}(\mathfrak{gl}(V))$. Since we need to focus on the higher of the powers of x in (3.3.4) (that being the ‘‘factor’’ which must be 0), let us rewrite (3.3.4) as

$$(3.4.2) \quad x^r \cdot (v_0 \wedge v_1) = \sum_{k=\lceil \frac{r+2}{2} \rceil}^{r+1} \left[\binom{r}{k} - \binom{r}{k-1} \right] x^k v_0 \wedge x^{r+1-k} v_0.$$

When $r < 2^{n+1} p^m - 1$, the term $k = 2^n p^m$ occurs in the sum, and its coefficient is nonzero by the lemma, so we deduce that $2^n p^m \geq \mu_1$, as desired. When $r = 2^{n+1} p^m - 1$ or $2^{n+1} p^m$, the term $k = 2^n p^m + 1$ occurs, and its coefficient is nonzero, so we deduce that $\mathcal{N}_{r, \phi}(\mathfrak{gl}(V)) \subset \mathcal{N}_{2^n p^m+1, \phi}(\mathfrak{gl}(V))$. But we have already shown (inductively on r) that the latter set equals $\mathcal{N}_{2^n p^m, \phi}(\mathfrak{gl}(V))$, so we are done. \square

3.5. Adjoint Representation, via Roots and Jacobson-Morozov: In this subsection we present an alternate method to compute $\mathcal{N}_{r, \rho}(\mathfrak{g})$ when the characteristic of k is 0, \mathfrak{g} is classical, and ρ is the adjoint representation, using roots and the Jacobson-Morozov Theorem.

Embed $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, with $\dim V = N$ via the standard representation. Let $x \in \mathfrak{gl}(V)$ be nilpotent. As usual, x corresponds to a partition $\mu = (\mu_1, \dots, \mu_t) \vdash N$. Using the Jacobson-Morozov Theorem, x belongs to a standard \mathfrak{sl}_2 -triple whose semisimple element $h_\mu \in \mathfrak{sl}(V)$ is a diagonal matrix, with diagonal blocks $D(\mu_1), \dots, D(\mu_t)$, where $D(\mu_i) = \text{diag}(\mu_i - 1, \mu_i - 3, \dots, -\mu_i + 3, -\mu_i + 1)$ [CM]. In fact, the semisimple element of the \mathfrak{sl}_2 -triple in \mathfrak{g} corresponding to x has a W -conjugate \tilde{h}_μ in the dominant chamber, with

diagonal entries a permutation of those of h_μ , in the following form:

$$\tilde{h}_\mu = \begin{cases} \text{diag}(a_1, \dots, a_{l+1}), & a_1 \geq \dots \geq a_{l+1} \geq 0 \text{ in type } A_l \\ \text{diag}(0, a_1, \dots, a_l, -a_1, \dots, -a_l), & a_1 \geq \dots \geq a_l \geq 0 \text{ in type } B_l \\ \text{diag}(a_1, \dots, a_l, -a_1, \dots, -a_l), & a_1 \geq \dots \geq a_l \geq 0 \text{ in type } C_l \\ \text{diag}(a_1, \dots, a_l, -a_1, \dots, -a_l), & a_1 \geq \dots \geq a_{l-1} \geq |a_l| \text{ in type } D_l. \end{cases}$$

The weighted Dynkin diagram associated to x is obtained by applying the simple roots to \tilde{h}_μ [CM], and thus the \mathfrak{sl}_2 -weights of the adjoint representation of \mathfrak{g} are obtained by applying all the roots to \tilde{h}_μ (together with l copies of the weight 0 coming from \mathfrak{h}). When these weights are organized into irreducible representations of \mathfrak{sl}_2 , the dimensions of these irreducible \mathfrak{sl}_2 -modules form the partition $\lambda(x) = (\lambda_1, \dots, \lambda_n) \vdash \dim \mathfrak{g}$ corresponding to the nilpotent element $\rho(x)$ in the adjoint representation. In particular, $\lambda_1 = 1 + \alpha_0(\tilde{h}_\mu)$, where α_0 is the highest root of \mathfrak{g} . Write roots in terms of the standard ε_i basis of \mathfrak{h}^* .

In type A_l , $\alpha_0 = \varepsilon_1 - \varepsilon_{l+1}$, so $\alpha_0(\tilde{h}_\mu) = (\mu_1 - 1) - (-\mu_1 + 1) = 2\mu_1 - 2$ and $\lambda_1 = 2\mu_1 - 1$. Now $\rho(x)^r = 0 \iff \lambda_1 \leq r \iff \mu_1 \leq (r+1)/2$. This is the same condition as the one obtained in Section 3.3, so we get the same maximal partition $\lambda_{N,a}$ where $a = \lfloor (r+1)/2 \rfloor$.

Next let us consider type C_l . Then $\alpha_0 = 2\varepsilon_1$, so $\alpha_0(\tilde{h}_\mu) = 2(\mu_1 - 1)$ and again $\lambda_1 = 2\mu_1 - 1 \leq r$. As before, there is a unique maximal partition $\mu = \lambda_{N,a}$ satisfying this condition. Thus $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \overline{\mathcal{O}}_{\mu_C}$, where μ_C is the C -collapse of μ as in Section 3.1.

Assume now that \mathfrak{g} is of type B_l or D_l . The highest root is $\alpha_0 = \varepsilon_1 + \varepsilon_2$. We have

$$(3.5.1) \quad \lambda_1 = 1 + \alpha_0(\tilde{h}_\mu) = \begin{cases} 2\mu_1 - 1, & \text{if } \mu_2 = \mu_1, \\ 2\mu_1 - 2, & \text{if } \mu_2 = \mu_1 - 1, \\ 2\mu_1 - 3, & \text{if } \mu_2 \leq \mu_1 - 2; \end{cases}$$

in the first and second cases, $\varepsilon_2(\tilde{h}_\mu) = \mu_2 - 1$, whereas in the third case, $\varepsilon_2(\tilde{h}_\mu) = \mu_1 - 3$.

When r is odd, there may be two possible maximal partitions μ of N which satisfy this condition (arising from the first and third cases of (3.5.1)). The first is $\lambda_{N,a}$. The second (which only occurs when $r \leq 2N - 3$), is the λ' defined in Theorem 3.3(c).

When r is even, we get a unique maximal partition μ satisfying (3.5.1), arising from the second case. When $r \geq 2N$, $\mu = \lambda_{N,a}$; otherwise, $\mu = \lambda''$ as defined in Theorem 3.3(d).

4. Exceptional Groups

In this section we will provide tables listing the orbit closures which describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for exceptional groups in good characteristics. The first column consists of a list of orbits. The remaining columns indicate the characteristic of field under consideration. The column labelled ‘‘other’’ corresponds to those fields which have characteristic 0 or a good prime which is not listed separately. The cell values provide the r 's for which $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is the orbit closure of the orbit(s) listed in that row. For example: for type E_6 with ρ a minimal representation, $\mathcal{N}_{8,\rho}(E_6) = \overline{\mathcal{O}(D_5(a_1))}$ for $p \neq 7$ and for $p = 7$, $\mathcal{N}_{8,\rho}(E_6) = \overline{\mathcal{O}(E_6(a_3))}$. Here $\mathcal{O}(X)$ denotes the orbit corresponding to the Bala-Carter label X . In the tables, we have also indicated by (*) the non-Richardson orbits which arise. The even orbits can be read off from the table in [Car] and the odd Richardson orbits are listed in [Hi].

In order to obtain these results for exceptional groups we use the tables given in [Law1] on unipotent elements along with work of [McN] to make the transition to nilpotent elements. Recall that for a faithful representation $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, we have $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \mathfrak{g} \cap \mathcal{N}_r(\mathfrak{gl}(V))$. For ρ a minimal dimensional representation or the adjoint representation, given a nilpotent element x in \mathfrak{g} , one can read off the corresponding partition $\lambda(x)$ of $N := \dim V$ in [Law1]. We can then invoke Proposition 2.4 by comparing the dominance ordering with the ordering on orbit closures given by the Hasse diagrams (see [Car, p. 439–445]) for exceptional groups to express $\mathcal{N}_{r,\rho}(\mathfrak{g})$ as a union of orbit closures.

4.1. Minimal Representation: We first consider the minimal dimensional irreducible representations for the exceptional groups. For E_6 there are two irreducible representations of minimal dimension. They are dual to one another, but since $\mathcal{N}_{r,\rho}(\mathfrak{g}) = \mathcal{N}_{r,\rho^*}(\mathfrak{g})$, where ρ^* is the dual representation of ρ , it suffices to consider only one of them. For the minimal dimensional representations $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is irreducible except in four cases involving only E_6 and E_7 :

Type E_6 :

$$\mathcal{N}_{7,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(A_4 + A_1)} \cup \overline{\mathcal{O}(D_4)} \quad \text{for } p \neq 5, 7$$

Type E_7 :

$$\begin{aligned} \mathcal{N}_{7,\rho}(\mathfrak{g}) &= \overline{\mathcal{O}(A_4 + A_2)} \cup \overline{\mathcal{O}(D_4)} && \text{for } p \neq 7 \\ \mathcal{N}_{11,\rho}(\mathfrak{g}) &= \overline{\mathcal{O}(D_5 + A_1)} \cup \overline{\mathcal{O}(A_6)} && \text{for } p \neq 11 \\ \mathcal{N}_{17,\rho}(\mathfrak{g}) &= \overline{\mathcal{O}(E_6)} \cup \overline{\mathcal{O}(E_7(a_3))} && \text{for } p \neq 17, 13 \end{aligned}$$

The tables below describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when ρ is a minimal dimensional representation and \mathfrak{g} is an exceptional Lie algebra. The conventions are explained at the beginning of Section 4.

Type E_6 :

Orbits	other	p=13	p=11	p=7	p=5
1	1	1	1	1	1
A_1 (*)	2	2	2	2	2
A_2	3	3	3	3	3
$A_2 + 2A_1$	4	4	4	4	4
$D_4(a_1)$	5-6	5-6	5-6	5-6	
$A_4 + A_1$					5-7
$A_4 + A_1, D_4$	7	7	7		
$D_5(a_1)$	8	8	8		8
$E_6(a_3)$	9-10	9-10	9-10	7-10	9-10
D_5	11-12	11-12		11-12	11-12
$E_6(a_1)$	13-16		11-16	13-16	13-16
E_6	17+	13+	17+	17+	17+

Type E_7 :

Orbits	other	$p = 23$	$p = 19$	$p = 17$	$p = 13$	$p = 11$	$p = 7$	$p = 5$
1	1	1	1	1	1	1	1	1
A_1 (*)	2	2	2	2	2	2	2	2
A_2	3	3	3	3	3	3	3	3
$A_2 + 3A_1$	4	4	4	4	4	4	4	4
$D_4(a_1)$	5	5	5	5	5	5	5	
$A_3 + A_2 + A_1$	6	6	6	6	6	6	6	
$A_4 + A_2$								5–6
$A_4 + A_2, D_4$	7	7	7	7	7	7		7
$D_5(a_1) + A_1$	8	8	8	8	8	8		8
$E_6(a_3)$	9	9	9	9	9	9		9
$E_7(a_5)$	10	10	10	10	10	10		10
A_6							7–10	
$D_5 + A_1, A_6$	11	11	11	11	11		11	11
$E_7(a_4)$	12	12	12	12	12		12	12
$E_6(a_1)$	13–15	13–15	13–15	13–15			13–15	13–15
$E_7(a_3)$	16	16	16	16		11–16	16	16
$E_6, E_7(a_3)$	17	17	17			17	17	17
$E_7(a_2)$	18–21	18–21	18		13–21	18–21	18–21	18–21
$E_7(a_1)$	22–27	22		17–27	22–27	22–27	22–27	22–23
E_7	28+	23+	19+	28+	28+	28+	28+	24+

Type E_8 :

Orbits	other	$p = 53$	$p = 47$	$p = 43$	$p = 41$	$p = 37$	$p = 31$
1	1-2	1-2	1-2	1-2	1-2	1-2	1-2
$2A_1$ (*)	3	3	3	3	3	3	3
$4A_1$ (*)	4	4	4	4	4	4	4
$2A_2$	5	5	5	5	5	5	5
$2A_2 + 2A_1$ (*)	6	6	6	6	6	6	6
$D_4(a_1) + A_2$	7	7	7	7	7	7	7
$2A_3$ (*)	8	8	8	8	8	8	8
$A_4 + A_2 + A_1$	9	9	9	9	9	9	9
$A_4 + A_3$ (*)	10	10	10	10	10	10	10
$E_8(a_7)$	11-12	11-12	11-12	11-12	11-12	11-12	11-12
$A_6 + A_1$	13-14	13-14	13-14	13-14	13-14	13-14	13-14
$D_7(a_2)$	15	15	15	15	15	15	15
A_7 (*)	16	16	16	16	16	16	16
$E_8(b_6)$	17-18	17-18	17-18	17-18	17-18	17-18	17-18
$E_8(a_6)$	19-22	19-22	19-22	19-22	19-22	19-22	19-22
D_7 (*)							
$E_8(a_5)$	23-26	23-26	23-26	23-26	23-26	23-26	23-26
$E_8(b_4)$	27-28	27-28	27-28	27-28	27-28	27-28	27-28
$E_8(a_4)$	29-34	29-34	29-34	29-34	29-34	29-34	29-30
$E_8(a_3)$	35-38	35-38	35-38	35-38	35-38	35-36	
$E_8(a_2)$	39-46	39-46	39-46	39-42	39-40		
$E_8(a_1)$	47-58	47-53					
E_8	59+	53+	47+	43+	41+	37+	31+

Type F_4 :

Orbits	other	$p = 13$	$p = 7$
1	1	1	1
$A_1 (*)$	2	2	2
A_2	3	3	3
$A_2 + \widetilde{A}_1 (*)$	4	4	4
$F_4(a_3)$	5–6	5–6	5–6
B_3	7–8	7–8	
$F_4(a_2)$	9–10	9–10	7–10
$F_4(a_1)$	11–16	11–12	11–16
F_4	17+	13+	17+

Type G_2 :

Orbits	other
1	1
$A_1 (*)$	2
$G_2(a_1)$	3–6
G_2	7+

4.2. Adjoint Representation: We first remark that the adjoint representation for E_8 coincides with the minimal dimensional (non-trivial) irreducible representation. When ρ is the adjoint representation, $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is irreducible for exceptional Lie algebras in all but 2 cases. Again these cases occur in E_6 and E_7 .

Type E_6 :

$$\mathcal{N}_{5,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(2A_2)} \cup \overline{\mathcal{O}(A_2 + 2A_1)} \quad \text{for } p \neq 5$$

Type E_7 :

$$\mathcal{N}_{5,\rho}(\mathfrak{g}) = \overline{\mathcal{O}(A_2 + 3A_1)} \cup \overline{\mathcal{O}(2A_2)} \quad \text{for } p \neq 5$$

The tables below describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for \mathfrak{g} an exceptional Lie algebra and ρ the adjoint representation. For E_8 , the reader is referred back to the previous section for the table.

Type E_6 :

Orbits	other	$p = 19$	$p = 17$	$p = 13$	$p = 11$	$p = 7$	$p = 5$
1	1-2	1-2	1-2	1-2	1-2	1-2	1-2
$2A_1$	3	3	3	3	3	3	3
$3A_1$ (*)	4	4	4	4	4	4	4
$2A_2, A_2 + 2A_1$	5	5	5	5	5	5	
$2A_2 + A_1$ (*)	6	6	6	6	6	6	
$D_4(a_1)$	7-8	7-8	7-8	7-8	7-8		
$A_4 + A_1$	9-10	9-10	9-10	9-10	9-10		5-10
$E_6(a_3)$	11-14	11-14	11-14	11-12		7-14	11-14
D_5	15-16	15-16	15-16				15-16
$E_6(a_1)$	17-22	17-18			11-22	15-22	17-22
E_6	23+	19+	17+	13+	23+	23+	23+

Type E_7 :

Orbits	other	$p = 23$	$p = 19$	$p = 17$	$p = 13$	$p = 11$	$p = 7$	$p = 5$
1	1-2	1-2	1-2	1-2	1-2	1-2	1-2	1-2
$(3A_1)''$	3	3	3	3	3	3	3	3
$4A_1 (*)$	4	4	4	4	4	4	4	4
$A_2 + 3A_1, 2A_2$	5	5	5	5	5	5	5	
$2A_2 + A_1 (*)$	6	6	6	6	6	6	6	
$A_3 + A_2 + A_1$	7-8	7-8	7-8	7-8	7-8	7-8		
$A_4 + A_2$	9-10	9-10	9-10	9-10	9-10	9-10		5-10
$E_7(a_5)$	11-12	11-12	11-12	11-12	11-12			11-14
A_6	13-14	13-14	13-14	13-14			7-14	
$E_7(a_4)$	15-16	15-16	15-16	15-16				15-16
$E_6(a_1)$	17-18	17-18	17-18				15-18	17-18
$E_7(a_3)$	19	19				11-22	19-22	19-22
$E_7(a_2)$	20-25	20-22			13-26		23-26	23-24
$E_7(a_1)$	26-27			17-34	27-34	23-34	27-34	
E_7	28-35	23+	19+	35+	35+	35+	35+	25+

Type F_4 :

Orbits	other	$p = 19$	$p = 17$	$p = 13$	$p = 11$	$p = 7$	$p = 5$
1	1-2	1-2	1-2	1-2	1-2	1-2	1-2
$\widetilde{A}_1 (*)$	3	3	3	3	3	3	3
$A_1 + \widetilde{A}_1 (*)$	4	4	4	4	4	4	4
$A_2 + \widetilde{A}_1 (*)$	5	5	5	5	5	5	
$\widetilde{A}_2 + A_1 (*)$	6	6	6	6	6	6	
$F_4(a_3)$	7-10	7-10	7-10	7-10	7-10		5-10
$F_4(a_2)$	11-14	11-14	11-14	11-12		7-14	11-14
$F_4(a_1)$	15-22	15-18	15-16		11-22	15-22	15-22
F_4	23+	19+	17+	13+	23+	23+	23+

Type G_2 :

Orbits	other	$p = 7$
1	1-2	1-2
$A_1 (*)$	3	3
$\widetilde{A}_1 (*)$	4	4
$G_2(a_1)$	5-10	5-6
G_2	11+	7+

5. Applications

5.1. Let G be a simple algebraic group scheme over k where the characteristic of k is $p > 0$. The maximal ideal spectrum of the cohomology ring $H^{\bullet}(G_1, k)$ can be identified with the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$. The following results give a precise description of $\mathcal{N}_1(\mathfrak{g})$ in terms of closures of orbits. For the classical groups, one can use the partition labelling in order to describe the restricted nullcone closures as the closure of certain Richardson orbits. Given $N > 0$, express $N = dp + s$ where $0 \leq s \leq p - 1$ and recall that $\lambda_{N,p} = (p^d, s) \vdash N$. The first result is an immediate consequence of Theorem 3.2.

Theorem. *Let G be a simple classical algebraic group over k where $\text{char } k = p > 0$ is good.*

- (a) *If Φ is of type A_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{O}_{\lambda_{l+1,p}}$.*
- (b) *If Φ is of type B_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{O}_{(\lambda_{2l+1,p})_B}$.*

- (c) If Φ is of type C_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{2l,p})_C}$.
- (d) If Φ is of type D_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}}_{(\lambda_{2l,p})_D}$.

5.2. The following theorem provides a description of the restricted nullcone for exceptional Lie algebras. These results are easily deduced by setting $r = p$ and using the tables in Sections 4.1 and 4.2.

Theorem. *Let G be an exceptional algebraic group with p good. Then*

- (a) If Φ is of type E_6 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = E_6$ ($p \geq 13$), $E_6(a_1)$ ($p = 11$), $E_6(a_3)$ ($p = 7$), $A_4 + A_1$ ($p = 5$).
- (b) If Φ is of type E_7 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = E_7$ ($p \geq 19$), $E_7(a_1)$ ($p = 17$), $E_7(a_2)$ ($p = 13$), $E_7(a_3)$ ($p = 11$), A_6 ($p = 7$), $A_4 + A_2$, ($p = 5$).
- (c) If Φ is of type E_8 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = E_8$ ($p \geq 31$), $E_8(a_1)$ ($p = 29$), $E_8(a_2)$ ($p = 23$), $E_8(a_3)$ ($p = 19$), $E_8(a_4)$ ($p = 17$), $E_8(a_5)$ ($p = 13$), $E_8(a_6)$ ($p = 11$), $A_6 + A_1$ ($p = 7$).
- (d) If Φ is of type F_4 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = F_4$ ($p \geq 13$), $F_4(a_1)$ ($p = 11$), $F_4(a_2)$ ($p = 7$), $F_4(a_3)$ ($p = 5$).
- (e) If Φ is of type G_2 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = G_2$ ($p \geq 7$), $G_2(a_1)$ ($p = 5$).

6. VIGRE Algebra Group at the University of Georgia

This project was initiated during Fall 2003 under the Vertical Integration of Research and Education (VIGRE) Program sponsored by the National Science Foundation (NSF) at the Department of Mathematics at the University of Georgia (UGA). The VIGRE Algebra Group consists of 5 faculty members, 2 postdoctoral fellows and 5 graduate students. The group was led by David J. Benson, Brian D. Boe and Daniel K. Nakano. The email addresses of the members of the group are given below.

Faculty:

David J. Benson	djb@byrd.math.uga.edu
Brian D. Boe	brian@math.uga.edu
Leonard Chastkofsky	lenny@math.uga.edu
Jerome Jungster	jerome@math.uga.edu
Daniel K. Nakano	nakano@math.uga.edu

Postdoctoral Fellows:

Jo Jang Hyun	jhjo@math.uga.edu
Nadia Mazza	nmazza@math.uga.edu

Graduate Students:

Phil Bergonio	pbergonio@math.uga.edu
Bobbe Cooper	bcooper@math.uga.edu
G. Michael Guy	guy@math.uga.edu
Graham Matthews	matthews@math.uga.edu
Kenyon J. Platt	platt@math.uga.edu

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602