SUSLIN AND NON-SUSLIN SUBRINGS OF $\mathbb{R}$
— SOLUTION OF THE ERDÖS RING PROBLEM

J MARTIN LINDSAY AND SERGEY UTEV

Abstract. Erdős' ring problem is: can a proper subring of $\mathbb{R}$ which is Borel measurable (or Suslin) have positive Hausdorff dimension? The Artin-Schreier theory of real closed fields is combined with some standard geometric measure theory to prove that it cannot. However, both (non-Borel) proper subrings and proper subfields of dimension one are shown to exist—without appeal to the continuum hypothesis. It is also shown that $\mathbb{R}$ is not algebraic over any proper Suslin subring.

0. INTRODUCTION

As a consequence of Steinhaus' Theorem on difference sets, proper subgroups of $\mathbb{R}$ contain no Lebesgue measurable sets of positive Lebesgue measure. Our first purpose here is to prove the following result.

Theorem 0.1. Proper subrings of $\mathbb{R}$ contain no Suslin sets of positive Hausdorff dimension.

Suslin is an analytic condition— weaker than Borel measurability but stronger than Lebesgue measurability—enjoying the natural property of being preserved under Borel maps, unlike its sister conditions ([Fa5], [Ma2], [Rog]; see also [Dud]—where the term analytic is used—for a nice account).

Two proofs of Theorem 0.1 are given. The first uses a result from the Artin-Schreier theory of real closed fields, combined with some standard results from geometric measure theory for which either of the first two above-mentioned texts is a good reference. In the second proof the field theory is replaced by direct analytic arguments. This proof also entails polynomial relationships between (sets of) $\sigma$-compact subgroups of $\mathbb{R}$ having positive Hausdorff dimension, given in the final section.

In the following section, which is purely algebraic, an abundant source of (non-Suslin) subrings of Hausdorff dimension one is established, via nontrivial valuations on the field $\mathbb{R}$. The Artin-Schreier result is then combined with the failure of its countable extension to yield one-dimensional proper subfields of $\mathbb{R}$. In neither case is the continuum hypothesis invoked. In the last section we return to the Suslin hypothesis. The main result is an extension of the Artin-Schreier Corollary: $\mathbb{R}$ cannot be algebraic over a proper Suslin subring $A$—in particular $\mathbb{R}$ is not countably generated as an $A$-module.

Our motivation for considering these questions arose from interest in the structure of sets of periodic points for a class of functional equations. In studying both sure and random algebraic structures in $\mathbb{R}$ we have found measures of size more sensitive than Hausdorff dimension to be useful ([LU]).

P. Erdős and B. Volkmann began the study of dimensional possibilities for subsets of $\mathbb{R}$ sharing some of its algebraic structure. They describe $\sigma$-compact subgroups of $\mathbb{R}$ having Hausdorff dimension $d$, for each $d$ strictly between 0 and 1 and, using the continuum hypothesis, also a proper subgroup of dimension 1 ([ErV]). A. Beck gave an example of a proper subgroup of dimension 1 which is generated as a
group by a compact set ([Bec]). D. Kahnert also examined subgroups of \( \mathbb{R} \) from the dimensional point of view. For example he obtains, for each \( d \) strictly between 0 and 1, zero-dimensional Borel subgroups \( G \) and \( H \) satisfying \( \dim(G + H) = d \) ([Ka 1], [Ka 2]).

For subrings \( R \) of \( \mathbb{R} \), K.J. Falconer showed that if \( R \) is Suslin then its Hausdorff dimension must lie in the set \([0, 1/2] \cup \{1\}\), giving two proofs. Both use Fourier transforms (see [Ma 2], Chapter 12); the first proof ([Fa 2]) is based on the author’s bounds on the dimension of exceptional sets of orthogonal projections in \( \mathbb{R}^2 \) ([Fa 1]); in the second ([Fa 4]) the result is deduced from estimates on the Hausdorff dimension and measures of distance sets in \( \mathbb{R}^n \). Invoking the continuum hypothesis R.O. Davies constructed subrings of all possible dimensions in ([Da 3]). Connections between dimension conjectures on distance sets and sets of Furstenburg type (see [Wol]), and Erdős’ ring problem, have been explored in a very recent paper by N.H. Katz and T. Tao ([KaT]).

Concerning proper subfields \( F \) of \( \mathbb{R} \), B. Volkmann proved that if \( d := \dim F \) is strictly between 0 and 1 then, for any open interval \( I \), \( H^d(F \cap I) \in [0, \infty) \) ([Vol]). Here \( H^d \) denotes (outer) \( d \)-dimensional Hausdorff measure, which is proportional to outer \( d \)-dimensional Lebesgue measure for positive integers \( d \) (coinciding when \( d = 1 \)). P. Erdős deduced the existence of subfields \( F \) of first category satisfying \( H^1(F) > 0 \), by assuming the continuum hypothesis ([Er 2]). We have also found reference to a second unpublished work of R.O. Davies ([Da 4]).

1. Proof of Theorem 0.1

We first note an immediate simplification. By a theorem of Davies ([Da 1]) any Suslin subset of \( \mathbb{R}^n \) having Hausdorff dimension greater than \( d \) contains a compact set with the same property ([Fa 5] Theorem 3.6, or [Ma 2] Theorem 8.13). Thus Theorem 0.1 is equivalent to (both Erdős’ conjecture and) the following.

**Theorem 0.1’.** Let \( K \) be a compact subset of \( \mathbb{R} \) having positive Hausdorff dimension. Then the ring generated by \( K \) is \( \mathbb{R} \).

The proof separates into two parts—one geometric/analytic and the other entirely algebraic. Recall Steinhaus’ Theorem: if \( S \subset \mathbb{R} \) is Lebesgue measurable with positive Lebesgue measure, then the set \( S - S \) contains an interval around 0 of positive length ([Ste]). In particular, proper subgroups of \( \mathbb{R} \) contain no Lebesgue measurable sets of positive Lebesgue measure.

**Proposition 1.1.** Let \( A \) be a subgroup of \( \mathbb{R} \) containing a compact set of positive Hausdorff dimension. Then there is \( n \in \mathbb{N} \) and \( z_1, \ldots, z_n \in \mathbb{R} \) such that

\[
z_1A + \cdots + z_nA = \mathbb{R}.
\]

**Proof.** Let \( K \) be a compact set contained in \( A \) of positive Hausdorff dimension. Then, since Hausdorff dimension for subsets of Euclidean spaces enjoys the relation \( \dim X \times Y \geq \dim X + \dim Y \) ([Mar]—see [Fa 5] Section 3.3, or [Ma 2] Theorem 8.10), \( \dim K^n > 1 \), for sufficiently large \( n \). By a theorem of Mattila ([Mat]) it follows that there is a line \( \mathcal{L} \) in \( \mathbb{R}^n \) passing through the origin (in fact there are plenty!) such that the image of \( K^n \) under the orthogonal projection onto \( \mathcal{L} \) has positive 1-dimensional Hausdorff measure (see [Fa 5] Theorem 6.9, or [Ma 2] Corollary 9.8 and Theorem 8.9). Composing the projection with a linear bijection \( \mathcal{L} \to \mathbb{R} \) gives a linear functional \( \varphi : \mathbb{R}^n \to \mathbb{R} \) such that the compact set \( \varphi(K^n) \) has positive Lebesgue measure. By linearity \( \varphi(A^n) \) is a subgroup of \( \mathbb{R} \), and so by Steinhaus’ Theorem it equals \( \mathbb{R} \). Since all such linear functionals are of the form \( x \mapsto z : x \) the result follows. \( \Box \)
Proposition 1.2. Let $A$ be a subring of $\mathbb{R}$. If $\mathbb{R}$ is finitely generated as an $A$-module then $A$ equals $\mathbb{R}$.

Proof. Let $\mathbb{R}$ be finitely generated as an $A$-module, so that (1.1) holds for some $n \in \mathbb{N}$ and $z \in \mathbb{R}^n$. It follows that $F$, its field of fractions, is a proper subfield of $\mathbb{C}$ of finite codimension. Therefore, by a result of Artin and Schreyer ([ArS]—see below and [Jae] Chapter VI, Theorem 17), $F + i F = \mathbb{C}$. Since $F \subset \mathbb{R}$ this obviously implies that $F = \mathbb{R}$. In particular there is $b \in A_{\neq 0}$ such that $b z_i \in A$ for each $i$. The proof is completed by noting that

$$\mathbb{R} = b\mathbb{R} = b z_1 A + \cdots + b z_n A \subset A.$$  

Proof of Theorem 0.1'. Immediate from the two propositions. 

2. Alternative proof

This alternative proof, in which field theory is replaced by direct analytic arguments, has consequences which may be of independent interest (see Proposition 4.5 below).

Proposition 2.1. A finite direct sum of $\sigma$-compact proper subgroups of $\mathbb{R}$ is proper.

Proof. Suppose that $\mathbb{R}$ is the direct sum of (nontrivial) $\sigma$-compact subgroups $A_1, \ldots, A_n$. Thus $A_1 + \cdots + A_n = \mathbb{R}$, and if $a \in A_1 \times \cdots \times A_n \subset \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 0$ then $a = 0$. It follows that each $A_i$ satisfies $\mathbb{Q}A_i = A_i$. We establish the proposition by showing that $A_1 = \mathbb{R}$.

For each $i$ let $(K^{(l)}_i)_{l \geq 1}$ be an increasing sequence of compact subsets with union $A_i$. By assumption the following map is bijective: $f := \sigma \mid_{A_1 \times \cdots \times A_n}$ where $\sigma : \mathbb{R}^n \to \mathbb{R}$ is the linear functional $x \mapsto \sum_{i=1}^n x_i$. Its surjectivity implies that $\mathbb{R} = \bigcup_{l=1}^\infty (K^{(l)}_1 + \cdots + K^{(l)}_n)$—in particular, there is $m \in \mathbb{N}$ such that $K^{(m)}_1 + \cdots + K^{(m)}_n$ has positive Lebesgue measure. By Steinhaus’ Theorem

$$K_1 + \cdots + K_n \supset [-\delta, \delta]$$

for some $\delta > 0$, where $K_i = K^{(m)}_i - K^{(m)}_i$ for $i = 1, \ldots, n$.

Set $A = A_1$ and $K = K_1$. Since $A$ is a group it contains the compact set $K$ and the proof is complete once we have shown that $K$ contains an interval of positive length, since that implies that $A = \mathbb{R}$. Now, for any sequences $(v_l)_{l \geq 1}$ in $\mathbb{Q}$ tending to infinity,

$$\mathbb{R} = \bigcup_{l=1}^\infty v_l (K_1 + \cdots + K_n) = f \left( \bigcup_{l=1}^\infty (v_l K_1 \times \cdots \times v_l K_n) \right).$$

In particular, by the bijectivity of $f$,

$$\bigcup_{l=1}^\infty v_l K = A. \quad (2.1)$$

Now let $a \in A_{>0}$ and suppose that the following is false:

$$(\mathbb{Q} \cap [0, \varepsilon]) a \subset K \text{ for some } \varepsilon > 0. \quad (2.2)$$

In this case there is a sequence of positive rationals $(v_l)_{l \geq 1}$ tending to $+\infty$ and such that $v_l^{-1} a \notin K$ for each $l$. Since that would contradict (2.1), this shows that (2.2) does hold. By the density of $\mathbb{Q}$ and the compactness of $K$,

$$[0, \varepsilon a] = (\mathbb{Q} \cap [0, \varepsilon]) a \subset K,$$

and so the proof is complete. 

Corollary 2.2. Let $A$ be a $\sigma$-compact subring of $\mathbb{R}$. If $\mathbb{R}$ is finitely generated as an $A$-module, then $A = \mathbb{R}$. 

3
Proof. If \( n \in \mathbb{N} \) and \( z \in \mathbb{R}^n \) are such that \( \mathbb{R} = z_1 A + \cdots + z_n A \), with \( n \) minimal, then \( \mathbb{R} \) is the direct sum of the \( \sigma \)-compact subgroups \( z_1 A, \ldots, z_n A \). The result follows. \( \square \)

Alternative proof of Theorem 0.1'. The subring of \( \mathbb{R} \) generated by a set \( K \) is
\[
\bigcup_{m=1}^{\infty} \bigcup_{p \in \mathcal{P}_m} p(K^m),
\]
where \( \mathcal{P}_m \) is the ring of polynomials in \( m \) variables with integer coefficients and zero constant terms, and so is \( \sigma \)-compact if \( K \) is compact. Therefore Proposition 1.1 and Corollary 2.2 yield the result. \( \square \)

3. Algebraic ramifications

The Artin-Schreier result is their “beautiful characterisation of real closed fields”, which appears as the culminating theorem of Jacobson’s three volume series Lectures in Abstract Algebra. We quote it next for interest.

Theorem 3.1 ([ArS]). Let \( F \) be an algebraically closed field and let \( E \) be a proper subfield which has finite codimension in \( F \). Then \( E \) is real closed and \( F = E(\sqrt{-1}) \).

Remark. The case of codimension two yields to elementary arguments.

Corollary 3.2 (Artin-Schreier). \( \mathbb{R} \) has no proper subfields of finite codimension.

By contrast the following result is contained in [Bia].

Proposition 3.3 (Bialynicki-Birula). \( \mathbb{R} \) has proper subfields of countable codimension.

An elementary fact is noted next for ease of reference.

Lemma 3.4. Let \( T = \bigcup_{n \geq 1} f_n(S_n) \) where \( (f_n : S_n \subset \mathbb{R}^{d_n} \to \mathbb{R}^d)_{n \geq 1} \) is a sequence of locally Lipschitz maps. Then
\[
\sup_{n \geq 1} \dim S_n \geq \dim T.
\]

Proof. First note that \( \dim T = \sup_{n \geq 1} \dim f_n(S_n) \). Since dimension is nonincreasing under Lipschitz maps ([Fa 5] Lemma 1.8, or [Ma2] Theorem 7.5), and it is easy to see that the same holds for locally Lipschitz maps, the result follows. \( \square \)

Algebraic substructures of \( \mathbb{R} \) enjoy certain dimensional relationships which will be exploited later.

Proposition 3.5.

(a) Let \( A \) be a subgroup of \( \mathbb{R} \) and let \( R \) be the subring it generates. Then \( \dim A^n \geq \dim R \) for some \( n \in \mathbb{N} \).

(b) Let \( A \) be a subring of \( \mathbb{R} \) and let \( F \) be its field of fractions. Then \( \dim A^2 \geq \dim F \).

(c) Let \( A \) be a subring of \( \mathbb{R} \) and let \( I \) be a nontrivial ideal of \( A \). Then \( \dim I = \dim A \).

Proof. (a) Apply Lemma 3.4 (with \( T = R \)) to the countable family of locally Lipschitz maps
\[
A^{n_1} \times \cdots \times A^{n_k} \subset \mathbb{R}^d, \quad (x^{(1)}, \cdots, x^{(k)}) \mapsto x_1^{(1)} \cdots x_{n_1}^{(1)} + \cdots + x_1^{(k)} \cdots x_{n_k}^{(k)},
\]
where \( k \) and \( n_1, \ldots, n_k \) vary in \( \mathbb{N} \) and \( d = n_1 + \cdots + n_k \), to obtain \( \frac{1}{2} \dim A^{2m} \geq \dim A^m \geq \frac{1}{2} \dim R \) for some \( m \), by Marstrand’s dimension relation.

(b) In this case apply Lemma 3.4 (with \( T = F \)) to the locally Lipschitz maps
\[
A \times A \neq 0 \to \mathbb{R}, \quad (x, y) \mapsto x/y,
\]
the result follows. \( \square \)
to obtain \( \dim A^2 = \dim A \times A_{\neq 0} \geq \dim F \).

(c) For any \( z \in I_{\neq 0} \), \( A \subset z^{-1}I \) and the map \( x \mapsto z^{-1}x \) is bi-Lipschitz, thus
\[
\dim A \leq \dim z^{-1}I = \dim I \leq \dim A.
\] □

It is convenient to note a basic algebraic fact here.

**Lemma 3.6.** Let \( A \) be a subring of \( \mathbb{R} \). If \( \mathbb{R} \) is countably generated as an \( A \)-module then \( \mathbb{R} \) is algebraic over \( A \).

**Proof.** Let \( F \) be the field of fractions of \( A \). Then \( [\mathbb{R} : F] \leq \aleph_0 \)—in particular \( F \) has the cardinality of the continuum. If \( \mathbb{R} \) did contain an element \( z \) which is transcendental with respect to \( F \) then the uncountable set \( \{(z - f)^{-1} : f \in F\} \) would be linearly independent over \( F \). Therefore \( \mathbb{R} \) is an algebraic extension of \( F \); it follows that \( \mathbb{R} \) is algebraic over \( A \) too. □

Our next result establishes the existence of one-dimensional proper subrings and subfields of \( \mathbb{R} \)—without appeal to the continuum hypothesis. Field valuations are used for the former, and the failure of a countable form of the Artin-Schreier Corollary (cf. Theorem 4.3 and its Corollary below) is exploited for the latter.

**Theorem 3.7.**

(a) \( \mathbb{R} \) has a proper subring \( A \) with the following properties:

(i) \( H^1(A) = \infty \);

(ii) \( \mathbb{R} \setminus A \subset 1/A_{\neq 0} \);

(iii) \( \bigcup_{n \geq 0} z^n A = \mathbb{R} \) for all \( z \in \mathbb{R} \setminus A \).

In particular \( A \) is one-dimensional and has \( \mathbb{R} \) as its field of fractions.

(b) \( \mathbb{R} \) has proper subfields of positive outer Lebesgue measure.

**Proof.** (a) Let \( A \) be the ring of a nontrivial valuation of height one on the field \( \mathbb{R} \). Then (ii) holds and (by [Bou] Chapter VI, Section 4, Proposition 6) \( A \) is a maximal proper subring of \( \mathbb{R} \). Let \( z \in \mathbb{R} \setminus A \). Then \( z^n A = z^{n+1} z^{-1} A \subset z^{n+1} A \) and since \( A \) is proper the inclusion is strict. By maximality therefore the subring \( \bigcup_{n \geq 0} z^n A \) must equal \( \mathbb{R} \) and, by countable subadditivity, \( H^1(z^n A) > 0 \) for some \( n \in \mathbb{N} \). Thus (iii) holds and \( H^1(A) > 0 \). The above inclusion implies that \( H^1(A) \leq |z| H^1(A) \); taking \( z \) of modulus less than one therefore shows that (i) also holds.

(b) By the Białynicki-Birula Proposition, \( \mathbb{R} \) has a proper subfield \( A_0 \) of countable codimension. In particular, by Lemma 3.6, \( \mathbb{R} \) is an algebraic extension of \( A_0 \). For each \( n \in \mathbb{N} \) let \( A_n = A_0(z_1, \ldots, z_n) \), where \( (z_n)_{n \geq 1} \) is a basis for \( \mathbb{R} \) over \( A_0 \). Thus \( A_0, A_1, \ldots \) form an increasing sequence of subfields with union \( \mathbb{R} \); moreover each \( A_n \) is a finite extension of \( A_0 \) and so cannot equal \( \mathbb{R} \) by the Artin-Schreier Corollary. The result therefore follows by countable subadditivity. □

**Remark.** The existence of plenty of nontrivial valuations of height one on the field \( \mathbb{R} \) may be seen as follows. Let \( v_0 \) be a \( p \)-adic valuation on \( \mathbb{Q} \) and let \( F_1 \) be a maximal transcendental extension of \( \mathbb{Q} \) in \( \mathbb{R} \). A Zorn’s Lemma argument shows that there is a valuation \( v_1 \) on \( F_1 \) with value group \( \mathbb{Z} \) extending \( v_0 \). By [Bou] VI §3 Proposition 5, there is a valuation \( v \) on \( \mathbb{R} \), with value group \( \Gamma \) say, and a monomorphism of ordered groups \( \phi : \mathbb{Z} \rightarrow \Gamma \) satisfying \( v|_{\mathbb{R}} = \phi \circ v_1 \). Since \( \mathbb{R} \) is an algebraic extension of \( F_1 \), [Bou] VI §8 Corollary 1 implies that the valuation \( v \) has the same height as \( \phi \circ v_1 \)—namely one. In fact the valuation \( v \) may be chosen to be a literal extension of \( v_0 \) with value group \( \mathbb{Q} \).

The simple example below shows that Proposition 2.1 fails unless some analytic regularity is imposed. Sets constructed from Hamel bases in this way are typically nonmeasurable ([Er1]).
Example. Let $H$ be a Hamel basis for $\mathbb{R}$ as a vector space over the field $\mathbb{Q} + \sqrt{2}\mathbb{Q}$. Then, as a vector space over the field $\mathbb{Q}$, $\mathbb{R}$ has the direct sum decomposition $V \oplus \sqrt{2}V$, in which $V$ is the rational-linear span of $H$. (By Lemma 3.4, $\dim V^2 \geq 1$.)

The following problem is suggested by Proposition 3.5 (a).

Problem. Find a subgroup of $\mathbb{R}$ which is $\sigma$-compact, has zero Hausdorff dimension, and yet generates $\mathbb{R}$ as a ring.

4. Suslin extensions

Using Davies’ Theorem (or the invariance of the Suslin property under Cartesian products and Borel maps) Proposition 1.2 together with the methods used in the proof of Proposition 1.1 yield the following strengthening of Proposition 1.1 and Theorem 0.1, for which we find several applications.

Proposition 4.1. Let $A_1, \ldots, A_n$ be subgroups of $\mathbb{R}$ and let $A$ be a subring of $\mathbb{R}$.

(a) If $A_1 \times \cdots \times A_n$ contains a Suslin set of dimension greater than one then $z_1A_1 + \cdots + z_nA_n = \mathbb{R}$ for some $z \in \mathbb{R}^n$.

(b) If $A^n$ contains a Suslin set of positive Hausdorff dimension, for some $m \in \mathbb{N}$, then $A = \mathbb{R}$.

Remark. Part (b) contrasts with Theorem 3.7. Davies has shown that, if the continuum hypothesis is assumed, Mattila’s projection theorem can fail even for subsets of the plane which are $H^s$-measurable for all $s$ ([Da2]). Thus the validity of part (a) may require some analytic condition too.

Corollary 4.2. Let $A$ be a Suslin subring of $\mathbb{R}$. If any Cartesian power of its field of fractions has positive Hausdorff dimension then $A$ equals $\mathbb{R}$.

Proof. Let $F$ be the field of fractions of $A$. Since $F = q(A \times (A \cap \mathbb{R}_{\neq 0}))$, where $q : \mathbb{R} \times \mathbb{R}_{\neq 0} \to \mathbb{R}$ is the continuous map $(x, y) \mapsto x/y$, $F$ is Suslin and so Proposition 4.1 (b) implies that $F = \mathbb{R}$. Thus, by Proposition 3.5 (b), $\dim A^2 \geq 1$. Since $A^2$ is Suslin too, we may again apply Proposition 4.1 (b) to deduce that $A$ equals $\mathbb{R}$. \qed

For Suslin subrings the Artin-Schreier Corollary has the following extension.

Theorem 4.3. $\mathbb{R}$ is not algebraic over any proper Suslin subring.

Proof. Suppose that $\mathbb{R}$ is algebraic over $A$, where $A$ is a Suslin subring with field of fractions $F$. By Corollary 4.2 it suffices to show that $F$ equals $\mathbb{R}$. We first establish that the degree of every simple extension of $F$ is bounded, that is, for some $n \in \mathbb{N}$,

$$[F(x) : F] \leq n \quad \text{for all } x \in \mathbb{R}.$$  \hspace{1cm} (4.1)

To see this define $S_k := \{x \in \mathbb{R} : [F(x) : F] \leq k\}$ and note that $\bigcup_{k \geq 1} S_k = \mathbb{R}$ and $\mathbb{NS}_k = S_k$. Since $S_k$ is the image of $(\{1\} \times F^k \times \mathbb{R}) \cap f^{-1}(0)$ under the $(k + 2)$-fold coordinate projection, where $f : \mathbb{R}^{k+1} \times \mathbb{R} \to \mathbb{R}$ is the continuous map $(z, x) \mapsto \sum_{i=0}^k z_ix^{k-i}$, it is Suslin and so Lebesgue measurable. By countable subadditivity and Steinhaus’ Theorem $S_k - S_k \supset [-\delta, \delta]$ for some $k \in \mathbb{N}$ and $\delta > 0$. But, for $c = a - b \in S_k - S_k$,

$$[F(c) : F] \leq [F(a, b) : F] = [F(a, b) : F(a)][F(a) : F] \leq k^2,$$

so $c \in S_{k^2}$. Thus $\mathbb{R} = \bigcup_{[-\delta, \delta]} \cap \mathbb{NS}_{k^2} = S_{k^2}$—in other words (4.1) holds for $n = k^2$.

Now let $n \in \mathbb{N}$ be the smallest integer for which (4.1) holds. Choose $a \in \mathbb{R}$ such that $[F(a) : F] = n$, and suppose that $[\mathbb{R} : F] = \infty$. Then $[F(a, b) : F(a)] > 1$ for some $b \in \mathbb{R}$ and, since $a$ and $b$ are algebraic over $F$ and $\mathbb{R}$ has characteristic zero,
\[ F(a, b) = F(c) \] for some algebraic element \( c \) ([Jac] Chapter 1, Theorem 14) which implies that

\[ [F(c) : F] = [F(a, b) : F(a)][F(a) : F] > n, \]

contradicting (4.1). Thus \([ \mathbb{R} : F] \) must be finite and the result follows from the Artin-Schreier Corollary.

In particular we have a strengthening of Corollary 2.2 to contrast with the Białynicki-Birula Proposition.

**Corollary 4.4.** Let \( A \) be a Suslin subring of \( \mathbb{R} \). If \( \mathbb{R} \) is countably generated as an \( A \)-module then \( A \) equals \( \mathbb{R} \).

**Proof.** Immediate from Lemma 3.6.

We end with an interesting consequence of Proposition 2.1. Its proof is a straightforward induction.

**Proposition 4.5.** If \( A_1, \ldots, A_n \) are \( \sigma \)-compact subgroups of \( \mathbb{R} \) satisfying \( z_1 A_1 + \cdots + z_n A_n = \mathbb{R} \) for some \( z \in \mathbb{R} \) then there are elements \( p_i^{(k)} \) of \( \prod_{l \neq k} A_l \) \( (k = 1, \ldots, n; i = 1, \ldots, 2^{n-1}) \) such that

\[ p_1^{(1)} A_1 + \cdots + \left( \sum_{i=1}^{2^{n-2}} p_i^{(n-1)} \right) A_{n-1} + p_1^{(n)} A_n = \mathbb{R}. \]

**Remark.** Each labelling of the subgroups leads to such a representation.

Thus, for example, suppose that \( A \) and \( B \) are \( \sigma \)-compact subgroups of \( \mathbb{R} \)—such as those constructed in [ErV] and [Bec]—for which \( \dim A + \dim B > 1 \). Then, using Proposition 4.1 (a), Proposition 4.5 implies that there are elements \( a \in A \) and \( b \in B \) such that

\[ b A + a B = \mathbb{R}. \]

**Acknowledgements.** It is a pleasure to acknowledge valuable discussions with our colleagues Joel Feinstein, John Cremona, Ivan Fesenko and Michael Spiess.

**References**


[Da3] R.O. Davies, Rings of dimension d, unpublished (circa 1984) [referenced in Fa4], and in [Ma2].

[Da4] R.O. Davies, Fields of dimension d, unpublished (circa 1985) [referenced in [Fa3]].


SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM NG7 2RD, U.K.

E-mail address: jmi@maths.nott.ac.uk
E-mail address: sergey.utev@nottingham.ac.uk