OPERADS AND TOPOLOGICAL CONFORMAL FIELD THEORIES

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Contents

1. Introduction

The aim of these lecture notes is to give the reader an idea about one important aspect of collaboration between pure mathematics and theoretical physics. This aspect is the theory of operads which originated in the works of topologists studying H-spaces and structured ring spectra. A standard modern textbook on operads is the book [32] where plenty of further references could be found.

It was not my intention to give a complete treatment of all relevant topics; therefore none of the results discussed will be given detailed proofs. Most of the time the proofs will only be outlined or omitted altogether. Still, it is hoped that these lectures will provide a stimulus to the readers to delve deeper into this new and fascinating subject.

Even though an attempt has been made to provide a broad, though not detailed, coverage of the subject, some of the important and relevant themes are not mentioned. Among them are:

- Operator product expansion and vertex algebras. There are many good books on this subject, for example [25], [24].
- Renormalization. This is covered in all books on quantum field theory, a succinct account is found in [9]. Particularly recommended is the excellent modern treatment by Costello [7].
- Supergeometry. One could get acquainted with this subject by reading e.g. [33].
- Quantization of gauge systems [22].
- Minimal models for algebras over operads and modular operads. [4, 35].

2. Finite dimensional integrals and Feynman graphs

2.1. **Feynman integrals and graphs.** In this lecture we will consider the problem of evaluating the integral

$$\int_{\mathbb{R}^n} e^{-B(x,x)/2+S(x)}dx.$$

Here $B(x,x)$ is a positive definite symmetric bilinear form on $\mathbb{R}^n$ and

$$S(x)=\sum_{m\geq 3} g_m S_m(x^{\otimes m})/m!.$$

$S_m \in S^m V^*$. In other words, S_m is a homogeneous polynomial function on V of degree m. Of course, such an integral can diverge but we understand it in the formal sense, in other words, it takes values in the ring $\mathbb{R}[[g_1,\ldots,g_n,\ldots]]$. It turns out that the answer is formulated combinatorially in terms of graphs. This relationship between asymptotic expansions of integrals and combinatorics of graphs underlies all further links between field theory and the theory of operads.

We start by reminding the notion of a graph. Our description here will be somewhat informal, in order to avoid overburdening the reader with the precise details. A more complete description is contained in [16].

Definition 2.1. *A graph Γ is a one-dimensional cell complex. From a combinatorial perspective Γ consists of the following data:*

1. *A finite set, also denoted by Γ, consisting of the* half-edges *of Γ.*
2. *A partition $V(\Gamma)$ of Γ. The elements of $V(\Gamma)$ are called the* vertices *of Γ.*
3. *A partition $E(\Gamma)$ of Γ into sets having cardinality equal to two. The elements of $E(\Gamma)$ are called the* edges *of Γ.*

We say that a vertex $v \in V$ has valency n if v has cardinality n. The elements of v are called the *incident half-edges* of v. We will consider only those graphs whose internal vertices have valency ≥ 3.

There is an obvious notion of isomorphism for graphs. Two graphs Γ and Γ' are said to be isomorphic if there is a bijection $\Gamma \to \Gamma'$ which preserves the structures described by items (1–3) of Definition 2.1.

We will now temporarily leave graphs and become interested in computing certain type of integrals. Among those the simplest are as follows.

Example 2.2. *Consider the integral*

$$I=\int_{-\infty}^{\infty} e^{-x^2/2}x^{2m}dx.$$

Integrating by parts, using induction and the well-known identity $\int_\infty^\infty e^{-x^2/2}dx = \sqrt{2\pi}$ *we obtain*

$$I = \sqrt{2\pi}(2m-1)(2m-3)\dots 1 = \sqrt{2\pi}\frac{(2m)!}{2^m m!}.$$

Furthermore, it is clear that $I = \int_{-\infty}^{\infty} e^{-x^2/2}x^{2m+1}dx = 0$ *for any* m *as the integrand is an odd function.*

Note that $\frac{(2m)!}{2^m m!}$ is the number of splittings of the set $1, 2, \dots, 2m$ into pairs. The set of such splittings is acted upon by the permutation group S_{2m} and the stabilizer of an element is isomorphic to the semidirect product of S_m and $(\mathbb{Z}/2)^m$. This example admits the following generalization, known as the *Wick lemma*.

Proposition 2.3. *Let* f_1, f_N *be linear functions on* V*. Denote by* $\langle f_1, \dots, f_N\rangle_0$ *the ratio*

$$\frac{\int_{\mathbb{R}^n} f_1(x)\dots f_N(x)e^{-B(x,x)/2}dx}{\int_{\mathbb{R}^n} e^{-B(x,x)/2}dx}$$

Then $\langle f_1, \dots, f_N\rangle_0 = 0$ *if* N *is odd and*

$$\langle f_1, \dots, f_N\rangle_0 = \sum B^{-1}(f_{i_1}, f_{i_2}), \dots, B^{-1}(f_{i_{N-1}}, f_{i_N}) \tag{2.1}$$

where the summation is extended over all partitions of the set $1, \dots, N$ *into pairs* $(i_1, i_2), \dots, (i_{N-1}, i_N)$*.*

Proof. It is clear that the integrand is an odd function for N odd and therefore the integral vanishes in this case. Let N be even. By linear change of variables we could reduce B to a diagonal form $B = x_1^2 + \dots + x_n^2$; then $B^{-1}(x_i, x_j) = \delta_{ij}$. Note that since both sides of (2.1) are symmetric multilinear functions on in the variables f_i it is sufficient to check it for the case when $f_1 = \dots = f_N = f$. In this case the right hand side of (2.1) is equal to $\frac{(2m)!}{2^m m!}B^{-1}(f,f)^N$. Furthermore, by a linear change of variables preserving the quadratic form B and taking f to one of the coordinate functions x_i (up to a factor) one can reduce the left hand side of (2.1) to a one dimensional integral and thus to Example 2.2. □

Now consider a more general situation, i.e. the ratio of integrals

$$\langle f_1, \dots f_N\rangle := \frac{\int_{\mathbb{R}^n} f_1 \dots f_N e^{-B(x,x)/2+S(x)}dx}{\int_{\mathbb{R}^n} e^{-B(x,x)/2}dx}.$$

The expression $\langle f_1, \dots f_N\rangle$ takes values in $\mathbb{R}[[g_1, \dots, g_n, \dots]]$; one may or may not be able to set the parameters g_i to be equal to actual numbers. For example, the integral $\int_{-\infty}^{\infty} e^{-x^2/2-gx^4}$ actually converges for any number g whereas $\int_{-\infty}^{\infty} e^{-x^2/2+gx^4}$ only makes sense as a formal power series in g.

Remark 2.4. *The expression* $\langle f_1, \dots f_N\rangle$ *is known in quantum field theory as an* expectation value *of the observable* $f_1 \dots f_N$*.*

Let $\mathbf{n} = (n_3, n_4, \dots)$ be any sequence of nonnegative integers which is eventually zero. Denote by $G(N, \mathbf{n})$ the set of isomorphism classes of graphs which have N 1-valent vertices labeled by the numbers $1, \dots, N$ and n_i unlabeled i-valent vertices. The labeled vertices are called external, the unlabeled ones internal.

Then for any graph $\Gamma \in G(N, \mathbf{n})$ we construct a multilinear function F_Γ which associates to $f_1, \dots, f_N$ a number $F_\Gamma(f_1 \dots, f_N)$ as follows. We attach to any 1-valent vertex of Γ labeled by i the vector f_i; to any m-valent vertex the tensor S_m. We then take the tensor product of these tensors and take contractions along edges using the form B^{-1}. The we have the following result.

Theorem 2.5.

$$\langle f_1, \dots f_N\rangle = \sum_{\mathbf{n}} (\prod_i g_i^{n_i}) \sum_{\Gamma \in G(N,\mathbf{n})} |Aut(\Gamma)|^{-1} F_\Gamma(f_1 \dots, f_N) \tag{2.2}$$

where $Aut(\Gamma)$ is the (finite) group of automorphisms of Γ which fix the external vertices.

Proof. The proof follows from Wick's lemma. Note that (2.2) is a special case of (2.1) for $\mathbf{n} = 0$. We can regard every i-valent vertex of Γ as a collection of i 1-valent vertices sitting close to each other. Every such graph with k vertices corresponds to a summand in $(S)^k/k!$ in the expansion of e^S. Furthermore, each graph $\Gamma \in G(N, \mathbf{n})$ determines precisely $|Aut(\Gamma)|^{-1} \prod i!^{n_i} \prod n_i!$ different pairings of these 1-valent vertices. This finishes the proof. □

Remark 2.6. *The function F_Γ is called the Feynman amplitude of the graph Γ. Particularly important special case is when Γ has no external vertices (in which case it is called a vacuum graph). Then F_Γ can be paired with any symmetric tensor S giving rise to a number. Namely, the m-valent part of S is associated with m-valent vertices of Γ. We then take the tensor product of these tensors over all vertices of Γ and contract along the edges using the inverse of the given scalar product. These inverses are called* propagators *in physics. The resulting number is the Feynman amplitude of Γ.*

Note that the Feynman amplitude is defined unambiguously because S is a symmetric tensor and because the bilinear form B is symmetric. Therefore it does not matter in which order we multiply and contract our tensors. If S and B were arbitrary, we would have to specify the ordering on the half-edges and edges of Γ in order for the Feynman amplitude to be well-defined. If S is cyclically symmetric, then it could be paired with so-called ribbon graphs. *Further generalization is possible when V is a super-vector (or $\mathbb{Z}/2$)-graded vector space. Then the integral would also have to be understood in the super-sense. For the relevant discussion see* [10].

Here's another version of Feynman's theorem; it is in this form that it is used most frequently by physicists. Below $b(\Gamma)$ stands for the number of edges minus the number of vertices of a graphs Γ. Note that $b(\Gamma)$ is the first Betti number of Γ. Consider the following quantity

$$\langle f_1 \dots f_N \rangle_h := \frac{\int_{\mathbb{R}^n} f_1, \dots, f_N e^{-B(x,x)/2 - \frac{1}{h}S(x)} dx}{\int_{\mathbb{R}^n} e^{-B(x,x)/2} dx}.$$

Proposition 2.7.

$$\langle f_1, \dots f_N \rangle_h = \sum_{\mathbf{n}} (\prod_i) g_i^{n_i}) \sum_{\Gamma \in G(N, \mathbf{n}} \frac{h^{b(\Gamma)}}{|Aut(\Gamma)|} F_\Gamma(f_1 \dots, f_N) \tag{2.3}$$

where $Aut(\Gamma)$ is the (finite) group of automorphisms of Γ which fix the external vertices.

The last remark we need to make is that the right hand sides of the formulas (2.1) and (2.2) make sense without the assumption that the form B is positive definite. The following argument, allows one to make sense (formally) of the left hand sides as well.

We need to make sense of integrals of the form

$$\int_{\mathbb{R}^n} f_1 \dots f_N e^{-B(x,x)/2} dx \tag{2.4}$$

where B is a nondegenerate symmetric bilinear form and f_i's are linear functions on $\mathbb{R}^n$. Suppose now that B is *not* positive definite. A linear change of variables allows one to consider only the case when the quadratic function $B(x,x)$ has the form $\sum_{i=1}^k x_i^2 - \sum_{i=k+1}^n x_i^2$. We will introduce the function $g(t)$ as

$$g(t) := \int_{\mathbb{R}^n} f_1 \dots f_N e^{-\frac{1}{2}[\sum_{i=1}^k (x_i)^2 + \sum_{i=k+1}^n (tx_i)^2]} dx.$$

Then $g(t)$ is well-defined for nonzero real t and we can analytically continue g for arbitrary nonzero $t \in \mathbb{C}$; thus our integral (2.4) equals $g(i)$. For example, $\int_{-\infty}^{\infty} e^{\frac{1}{2}x^2} dx$ will be equal to $-i\sqrt{2\pi}$.

Remark 2.8. *The formal manipulation described above is known in physics as the* Wick rotation. *It is easy to check that* $\int_{\mathbb{R}^n} f_1 \dots f_N e^{-\frac{1}{2}B(x,x)}dx$ *as defined above obeys the standard rules of integration (i.e. the change of variables formula and integration by parts still hold) although the integral itself exists only formally. Theorem 2.5 will continue to hold.*

The geometric meaning of the Wick rotation is as follows. Consider $f_1, \dots, f_N, S$ and B as functions defined on $\mathbb{C}^n$. Choose a real slice in $\mathbb{C}^n$, i.e. a real subspace V such that $V \otimes_{\mathbb{R}} V \cong \mathbb{C}$ and such that the function B is a sum of squares on V. Then perform integration over V. The resulting power series does not depend on the choice of a real slice (by the uniqueness of the analytic continuation. Note that we the integral in both the numerator and denominator of $\langle f_1 \dots f_N \rangle$ could be complex but the ratio is always real. This is a small example of a situation often encountered in physics – the intermediate calculations might make only formal sense (like $\infty - \infty = 0$) but the end result is correct nonetheless.

3. Formal structure of quantum field theory

Quantum field theory is a vast and complicated subject whose prerequisites are classical field theory, including special relativity, and quantum mechanics. Our modest goal in this lecture is to describe the formal logical structure of quantum mechanics and quantum field theory from the point of view of Feynman integrals.

3.1. **Classical Mechanics.** One starts with classical mechanics which studies the motion of a particle (or a system of particles) subject to some force field. The position of our system corresponds to a point in a certain manifold – the *configuration space*. For example, the configuration space of system of n free point particles is $\mathbb{R}^{3n}$.

We are interested in the evolution of the state in the configuration space. Let $x(t)$ be the corresponding trajectory with $x(0)$ and $x(1)$ be its initial and final points. One introduces an *action functional* $S[x] = \int_0^1 L(x, \dot{x})dt$; here L is a function depending on x and $\dot{x}$; it is called the *Lagrangian* of the system. Sometimes L is also allowed to depend on the higher derivatives of x. The Lagrangian determines the dynamics of the system. For example, for a particle of mass m in a potential field with potential energy U the Lagrangian has the form $L = \frac{m\dot{x}^2}{2} - U(x)$.

The (classical) trajectory of the system is the minimum of the functional S, it is the so-called *least action principle*. This trajectory is, therefore, obtained from the equation $\delta S = 0$ which leads to the well-known Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial L}{\partial x} = 0.$$

For example $L = \frac{m\dot{x}^2}{2} - U(x)$ leads to the Newton law $m\ddot{x} = -U'(x)$.

3.2. **Classical field theory.** The situation here is similar. The 'position' of a field is a point in a certain infinite-dimensional space, typically a space of functions or sections of a vector bundle. The evolution of our system is a trajectory in this infinite-dimensional configuration space. These trajectories are usually functions of several space variables and one time variable. One introduces a Lagrangian on this space of fields; as above, it should be local, i.e. it should depend on the fields and their partial derivatives. The resulting Euler-Lagrange equations will then describe the dynamics of these classical fields.

One of the simplest examples is given by the relativistic free scalar field the Lagrangian of which has the form

$$L = 1/2(\partial_t\phi\partial_t\phi - \partial_{x_1}\phi\partial_{x_1}\phi - \partial_{x_2}\phi\partial_{x_2}\phi - \partial_{x_3}\phi\partial_{x_3}\phi - m^2\phi^2).$$

Using standard formalism of summation over repeated indices it could be rewritten as $L = 1/2(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$. The corresponding equation of motion is the so-called Klein-Gordon equation:

$$\Box\phi + m^2\phi = 0,$$

where $\Box\phi := \frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x_1^2}\phi + \frac{\partial^2}{\partial x_2^2}\phi + \frac{\partial^2}{\partial x_3^2}\phi$.

3.3. **Quantum theory.** In quantum mechanics the *space of states* of a system is a Hilbert space V, usually taken to be the space of L_2-functions on the configuration space. An *observable* is then a self-adjoint operator on V. The dynamics of a quantum system is described by a self-adjoint operator H called a *Hamiltonian* which may or may not depend on the time t. The equation of motion is the *Schrödinger equation*

$$\psi(t) = -\frac{i}{h} H\psi(t).$$

Assuming that H does not depend on t the general solution to the Schrödinger equation could be written as

$$\frac{d}{dt}\psi(t) = e^{-\frac{i}{h}Ht}\psi(0).$$

The operator $e^{-\frac{i}{h}H}$ is called the *evolution operator*; is is unitary since H is self-adjoint.

The problem of *quantization* is: given a classical system (described by a certain Lagrangian on a configuration space) construct the corresponding quantum system, i.e. specify a self-adjoint operator H (or the corresponding unitary operator U) on the space of L_2-functions on the configuration space. It is important to stress that this problem does not have a canonical and unique solution and finding it is a physical rather than a mathematical problem in that the result has to be verified by the experimental data. The following result gives a 'solution' to the quantization problem via *Feynman path integral.*

Theorem 3.1. *The integral kernel of the operator U is given by the following formula*

$$U(x_1, x_2) = \int e^{\frac{i}{h}S} Dx(t), \tag{3.1}$$

where S is the classical action functional

One has to make a few comments concerning the above statement. First of all, its claim is that the operator U acts on a function f as $U(f) = \int U(x, x_1) f(x_1) dx_1$. Furthermore, the integral (3.1) is taken over the space of all paths starting at x_1 and ending at x_2.

Formula (3.1) is problematic in that the space of paths does not have a natural measure which makes the functional integral in it converge. One (formal) way out of it is to do the Wick rotation as described in the finite-dimensional case in the first lecture. Next, we don't really expect any actual 'proof' of the above theorem. It could be justified by showing that the result agrees with the one obtained from the so-called 'canonical' quantization which we do not discuss. Really, the above result is best regarded as an axiom.

The foregoing was the Feynman integral approach to quantum mechanics; it is widely regarded as satisfactory. Unfortunately, quantum mechanics does not incorporate in a consistent way the relativity theory. The point is that if we consider a system of *interacting* particles we have to allow the possibility of creation and annihilation of new particles. Thus, even if we initially start with a single particle its quantum 'trajectory' is in fact not a curve: it could split into several curves, some of which could later merge etc. In other words, we have a *graph*, not a curve. In order to be consistent with the Feynman integral formulation of quantum mechanics we would have to integrate not over paths but over graphs. Should we then consider all possible (infinitely many) configurations of graphs? The introduction of a measure on this space also poses very serious problems. This leads to another point of view – so-called *secondary quantization* according to which a particle should be considered to generate a field theory in it own right. This way our 'configuration space' is infinite-dimensional from the start. Further, the corresponding Lagrangian will have a quadratic part (corresponding to non-interacting particles) and the higher-degree part (which corresponds to interactions). The corresponding classical field could then be quantized according to the Feynman recipe. It has to be mentioned that this interpretation of a quantum particle as a classical field goes by the name 'wave-particle duality'.

3.4. **Perturbation expansion of Feynman integrals.** Let us consider an interacting massive scalar field theory determined by the action functional

$$S[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - U(\phi))d^{n+1}x.$$

Here ϕ is a function defined on $\mathbb{R}^{n+1}$ and $U(\phi)$ is a polynomial function:$U(\phi) = U_3\phi^3 + U_4\phi^4 + \ldots + U_k\phi^k$. The term $U(\phi)$ is called the interaction term. Denote by S_0 the following action functional of a free field:

$$S_0[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)d^{n+1}x.$$

We need to compute integrals of the form

$$\langle f_1 \ldots f_l\rangle := \frac{\int f_1 \ldots f_l e^{\frac{i}{\hbar}S} D\phi}{\int e^{\frac{i}{\hbar}S_0} D\phi},$$

the so-called l-point function of our theory.

To be able to apply the formalism of Lecture 1 we perform the Wick rotation $x_0 = it$ to convert the Feynman integral to the Euclidean form

$$\langle f_1 \ldots f_l\rangle_{Eu} := \frac{\int f_1 \ldots f_l e^{-\frac{1}{\hbar}S_{Eu}D\phi}}{\int e^{-\frac{1}{\hbar}S_{0Eu}} D\phi}.$$

Here

$$S_{0Eu}[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}((\partial_t\phi)^2 + \ldots + (\partial_{x_n}\phi)^2 + m^2\phi^2)d^{n+1}x;$$

$$S_{Eu}[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}((\partial_t\phi)^2 + \ldots + (\partial_{x_n}\phi)^2 - m^2\phi^2 - U(\phi))d^{n+1}x.$$

Note that

$$S_{0Eu}[\phi] = \int_{\mathbb{R}^{n+1}} ((\partial_t^2\phi)\phi + \ldots + (\partial_{x_n}^2\phi)\phi - (m\phi)^2)d^{n+1}x,$$

and so $S_{0Eu}[\phi] = \langle A\phi, \phi\rangle$ where the operator A is the linear operator

$$A\phi = (\Delta\phi - m^2)/2$$

and

$$\langle\phi, \psi\rangle = \int_{\mathbb{R}^{n+1}} \phi\psi d^{n+1}x.$$

To find the inverse to the bilinear form $-(A\phi, \phi)$ we need to invert the operator $-A$. Its inverse will be the integral operator with kernel $G(x - y)$, the fundamental solution of $-A$, i.e. the solution of the differential equation

$$(-\Delta G + m^2G)/2 = \delta(x).$$

It is easy to check that $n = 0$ (quantum mechanics) $G = \frac{2e^{m|x|}}{m}$. Note that for $n > 0$ (QFT case) the Green function G has a singularity at 0; logarithmic for $n = 1$ and polynomial for $n > 1$. It is because of these singularities that the quantum field theory is vastly more complicated than quantum mechanics.

We can formulate the rules for computing the perturbation expansion for correlation functions of a Euclidian QFT. To compute the amplitude F_Γ of a graph Γ with labeled external vertices one applies the following procedure:

- Assign the function f_i to each external vertex.
- assign the Green function $G(x_i - x_j)$ to each edge connecting the ith and jth internal vertices and $G(f_i - x_j)$ to the edge connecting the ith external vertex and jth internal vertex.
- Let $G_\Gamma(f_1, \ldots\ f_l, x_1, \ldots)$ be the product of all these functions.

- Finally set

$$F_\Gamma(f_1,\ldots,f_l)=\prod_j U_{v(j)}\int G_\Gamma(f_1,\ldots,f_l,x_1,\ldots)dx_1dx_2\ldots,$$

where $v(j)$ is the valency of the jth vertex.

The correlation function $\langle f_1\ldots f_l\rangle$ if the equal to

$$\sum_\Gamma \frac{h^{b(\Gamma)}}{|Aut(\Gamma)|}F_\Gamma(f_1,\ldots,f_l).$$

Here the summation is extended over all graphs Γ whose internal vertices have valencies ≥ 3 and $b(\Gamma)$ is the number of edges minus the number of vertices of Γ.

Remark 3.2. *Note that for a QFT case ($n>0$) the Green functions have singularities at* 0 *which makes the amplitude of graphs with loops formally undefined. In good cases the amplitudes still make sense as distributions. In worse (typical) cases one has to deal with divergences which gives rise to* renormalization. *This difficulty also presents itself when one attempts to canonically quantize a classical field theory, i.e. introduce a suitable Hilbert space, Hamiltonian operators etc.*

What is the significance of QFT for pure mathematics? The general philosophy is that given a mathematical object (say, a smooth or holomorphic manifold) one associates a classical field theory with it and then quantizes this field theory. The obtained expectation values of natural observables of the theory (i.e. functions on the space of fields or corresponding quantum operators) will then be invariants of the original geometric object.

Example 3.3.

(1) *Chern-Simons theory on a 3-dimensional manifold M is one of the simplest field theories to formulate. Our space of fields will be connections in an U_n-bundle over M and the action functional will have the form:*

$$S_{cs}(A)=\int_M Tr(AdA+2/3A^3).$$

(2) *Of great interest is also Yang-Mills theory. Let M be a smooth manifold with a Riemannian metric g. Since g determines an identification between the tangent and the cotangent bundles of M it gives a way to identify covariant tensors with contravariant ones. In particular, we can pair any two-forms ω,ψ: in coordinates we have $\omega=\omega_{ij}dx^idx^j;\psi=\psi_{ij}dx^idx^j$ and $g=g_{ij}dx^idx^j$, then*

$$\langle\omega,\psi\rangle=\omega_{ij}g^{ik}g^{jl}\omega_{kl}.$$

Furthermore, using the volume form dV (which is determined by g) we can define a global scalar product:

$$(\omega,\psi)=\int_M\langle\omega,\psi\rangle dV.$$

Let our space of fields be again the space of connections on a U_n-bundle over M, for a connection A denote by F_A its curvature form, it is indeed a two-form on A with values in U_n. The Yang-Mills functional has the form:

$$S_{YM}[A]=(F_A,F_A).$$

This functional for $n=1$ and M being the four-dimensional spacetime describes electrodynamics (in vacuum). The corresponding quantum field theory for general 4-manifolds is related to Donaldson invariants.

(3) *A particularly interesting example, related to many topics of current interest is the so-called $N=2$ supersymmetric Σ-model where the space of fields is (very roughly) the space of maps from a Riemann surface to a fixed spacetime manifold. One then* twists *this theory to obtain another one, whose correlators do not depend on the metric in*

the target manifold (in fact there are two such twists: A and B). This is the theory of topological strings. We cannot even briefly touch this subject but mention that this theory is a source of mirror symmetry *and is, to a great extent, the motivation for much of the developments covered in our course. A comprehensive introduction into these ideas is the book* [23]*.*

There are many good books which give an in-depth introduction to quantum field theory, I found [37] to be one of the most lucid. Another source, particularly useful for mathematicians is [8].

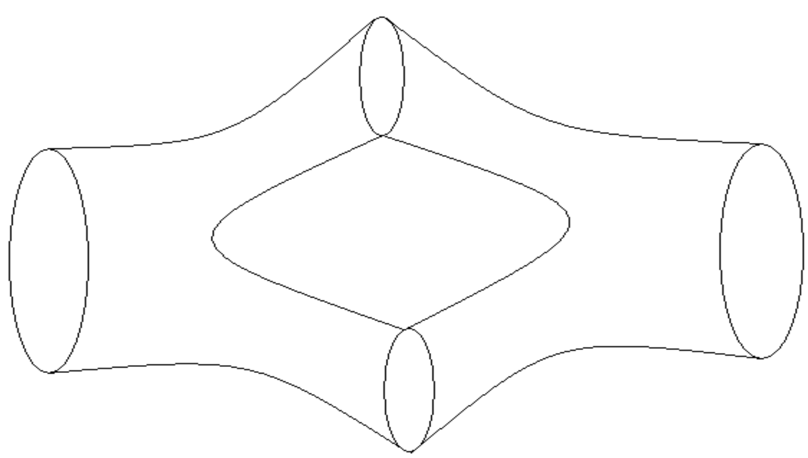
4. String theory

This lecture is devoted to reviewing string theory as an instance of QFT. A comprehensive modern course in string theory is [38] while [42] is a very readable introduction not assuming almost any background. A mathematical introduction to string theory is given in [11].

The formal structure of string theory resembles that of quantum mechanics except that particles are regarded as linear, rather than point objects. Most of string theory is done in *first* quantization which might account for some of its deficiencies. We will not even attempt to describe the approaches to second quantization of strings or *string field theory* (cf. [41]) although that was one of the first examples of operadic structures appearing in physics.

In string theoretic approach a particle is described by a circle (closed string) or an interval (open string). A string propagates through spacetime sweeping a two-dimensional surface – its *worldsheet*. Various elementary particles observed in nature (photon, electron etc.) correspond to various excited states of a string which are analogous to quantum mechanical states of a harmonic oscillator. Among these states one finds the *graviton* state, which corresponds to the particle generating the gravitational field. In this sense string theory predicts gravity which is considered by some as evidence for its validity. This picture is in marked contrast with the QFT picture where elementary particles corresponds to irreducible representations of the Lorentz group.

What makes string theory an attractive alternative to QFT is that it is free from the so-called *ultraviolet divergences*, which are due to the singularities of the Green functions. The related circumstance is that the interaction of strings is described by the topology of the worldsheet. One string could split into two which could then merge again etc.



The analogue of the path integral for strings will be the integral over all two-dimensional surfaces. What makes this much more feasible than the analogous problem for interacting point-particles is that there are much fewer topological configurations of two-dimensional surfaces than those of graphs; recall that any surface is classified topologically by its *genus*. This, in conjunction with a vast group of symmetries of string theory, makes most of the divergences disappear.

In this lecture we consider very briefly the theory of quantum bosonic string and how one come naturally to the critical dimension 26. One should note that most physicists consider the bosonic string to be unrealistic since it does not contain *fermionic states*, i.e. those states which are antisymmetric with respect to the permutation of two identical particles. Since there are known elementary particles that are fermions, the bosonic string cannot accurately describe nature. Another objection against the bosonic string is the presence of *tachyon states* corresponding to particles traveling faster than light. These difficulties are overcome in *superstring* theory which is considerably more involved than the bosonic string theory, yet shares many essential features with the latter.

4.1. **Classical theory.** We start with a description of the Dirichlet action functional of which the string action will be a special case. Let Σ and M be manifolds of dimensions d and n and with non-degenerate metrics h and g respectively. Our space of *fields* will be the space of smooth maps $\phi : \Sigma \to M$. Consider the following action functional:

$$S[\phi, g] = \int_\Sigma |d\phi|^2 \sqrt{|\det h|} d^d x.$$

Let us explain this notation. Let V and W be two vector spaces with nondegenerate bilinear forms h and g. Then the space of all linear maps $V \to W$ also has a bilinear form: for two linear maps f and ϕ we define $\langle f, \phi \rangle$ as

$$\langle f, \phi \rangle = Tr(f^* \circ \phi),$$

where $f^* : W \to V$ is the adjoint map determined by the forms g and h.

In coordinates: let $(a^i_{i'})$ and $b^j_{j'}$ be the matrices of f and ϕ respectively. We would like to multiply these matrices and take a trace. This is, of course, impossible, but we can use the bilinear forms h and g to lower and raise indices appropriately so that the multiplication becomes possible. We have:

$$\langle f, \phi \rangle = h^{j'i'} a^i_{i'} b^j_{j'} g_{ij}.$$

It follows that we can define the norm of a map f as $|f| := \sqrt{\langle f, f \rangle}$.

Coming back to our manifold situation we see that for $\phi : \Sigma \to M$ the expression $|d\phi|^2$ is a well-defined function on Σ which we can integrate against the canonical measure on Σ determined (up to a sign) by the metric h; specifically it is given by the formula $\sqrt{|\det h|} d^n x$. In coordinates we have:

$$S_\Sigma(\phi) = \int_\Sigma \sqrt{|\det h|} h^{\alpha\beta} \partial_\alpha \phi^\mu \partial_\beta \phi^\nu g_{\mu\nu} dx^d. \tag{4.1}$$

Note the following symmetries of the Dirichlet action:

- Diffeomorphisms in the source space:
$$\phi'^\mu(x') = \phi^\mu(x);$$
$$\frac{\partial x'^c}{\partial x'^a} \frac{\partial x'^d}{\partial x'^b} h_{cd}(x') = h_{ab}(x).$$
- Invariance with respect to diffeomorphisms in the target space preserving the form g:
$$\lambda(\phi'(x)) = \phi(x).$$
- Two-dimensional Weyl invariance. If $d = 2$ then S does not change under a conformal scaling of the metric h:
$$\phi'(x) = \phi(x);$$
$$h'_{ab}(x) = e^{c(x)} h_{ab}(x)$$
for an arbitrary function $c(x)$.

We now set $d = 2$ and assume that h is a Lorentzian metric in which case $|\det h| = -\det h$. In that case the action (4.1) is called the Polyakov action for the bosonic string. Its diffeomorphism invariance implies that the internal motion and geometry of the string has *no physical meaning*. The invariance with respect to the symmetries in the target spaces translates into the usual Poincaré invariance whereas the Weyl invariance crucially leads one to considering *moduli spaces of Riemann surfaces*.

Let us now vary the functional S_Σ with respect to h. First of all, note observe the following formula for the derivative of a determinant:

$$(\det A)' = \det A \,\mathrm{Tr}(A' A^{-1}),$$

which could be derived from the formula $\det A = e^{\mathrm{Tr}\, log A}$.

Furthermore denote by γ the metric on Σ induced from g by the map ϕ. Then (4.1) could be rewritten as follows.

$$S[\gamma] = \int_\Sigma \sqrt{-\det h}\,\mathrm{Tr}(h^{-1}\gamma)dx.$$

We have:

$$\begin{aligned}
\delta_h S &= \int_\Sigma [\frac{1}{2}(-\det h)^{-1/2}(\det h)\,\mathrm{Tr}(\delta h h^{-1})\,\mathrm{Tr}(h^{-1}\gamma) - \sqrt{-\det h}\,\mathrm{Tr}(h^{-1}\delta h h^{-1}\gamma)]dx \\
&= \int_\Sigma \sqrt{-\det h}[\frac{1}{2}Tr(\delta h h^{-1})\,\mathrm{Tr}(h^{-1}\gamma) - \mathrm{Tr}(h^{-1}\delta h h^{-1}\gamma)]dx \\
&= \int_\Sigma \sqrt{-\det h}\,\mathrm{Tr}\,\delta(h^{-1})\left(\gamma - \frac{1}{2}h\,\mathrm{Tr}(h^{-1}\gamma)\right)
\end{aligned}$$

Here we used the identity $\delta(h^{-1}) = -h^{-1}\delta h h^{-1}$.

It follows that for a critical metric (the solution of the equation $\delta_h S = 0$) we have

$$\gamma - \frac{1}{2}h\,\mathrm{Tr}(h^{-1}\gamma) = 0.$$

In other words, the critical metric h is proportional to the induced metric γ. This implies

$$\mathrm{Tr}(h^{-1}\gamma) = 2\frac{\sqrt{-\det\gamma}}{\sqrt{-\det h}}.$$

Plugging this into the Polyakov action we get

$$S = 2\int_\Sigma \sqrt{-\gamma}d^d x.$$

The latter action (up to a factor) is called the *Nambu-Goto action*. It is, therefore equivalent (classically) to the Polyakov action. Its geometric meaning is simply twice the area of the worldsheet.

Let us now consider the case of a *free* string, i.e. when the surface Σ is either a cylinder $S^1 \times \mathbb{R}$ (closed string) or $I \times \mathbb{R}$ (open string). One also imposes suitable boundary conditions in the case of open or closed strings. We will not discuss these boundary conditions.

We will fix the standard flat metric $dt^2 - dx^2$ on $\mathbb{R} \times S^1$ and the standard Lorentzian metric $g = diag(-1, 1)$. We take M to be the flat Minkowski space. In that case the string Lagrangian will have the form

$$L = \partial_t\phi^\mu\partial_t\phi^\mu - \partial_x\phi^\mu\partial_x\phi^\mu.$$

The corresponding Euler-Lagrange equations are:

$$(\partial_t^2 - \partial_x^2)\phi^\mu = 0$$

In other words, each field ϕ^μ satisfies the Klein-Gordon massless equation. It is not hard to solve these equations; let us introduce the *light-cone coordinates*:

$$\sigma^+ = t + x, \sigma^- = t - x.$$

Denote ∂_+, ∂_- the corresponding partial derivatives. It is clear that

$$\partial_+ = \partial_t + \partial_x; \partial_+ = \partial_t + \partial_x.$$

These equations can be rewritten as

$$\partial_+\partial_-\phi^\mu = 0$$

whose general solutions are

$$\phi^\mu = \phi^\mu_L(\sigma^+) + \phi^\mu_R(\sigma^-).$$

4.2. **Quantization.** We now discuss quantization of interacting strings and see how moduli spaces of Riemann surfaces arise in this context. Our treatment of this important subject will be very sketchy since the complete treatment relies on many deep facts of both physics and algebraic geometry which we cannot discuss here.

First of all, any complex structure on a 2-dimensional surface gives rise to a Riemannian metric; indeed one takes locally the metric $dzd\bar{z}$ and then glues those with a partition of unity. Conversely, any Riemannian metric gives rise to a complex structure. It is clear that if two Riemannian metrics are conformally equivalent (i.e. differ at each point by a factor) then the corresponding complex structures are also equivalent; indeed, an easy calculation shows that the metrics $dzd\bar{z}$ and $dz'd\bar{z}'$ are conformally equivalent if and only if the coordinate change $z' = z'(z)$ is holomorphic.

It follows that the moduli space of complex structures on a 2-dimensional surface Σ is isomorphic to the space of all conformal classes of metrics on S modulo the group $Diff$ of all diffeomorphisms of S.

The Polyakov path integral has the form:

$$\int e^{-S[\phi,g]} D\phi Dg.$$

Here the integral is taken over all Riemannian metrics g on a surface Σ and over all fields ϕ. We have taken the Euclidean version of the Feynman integral, the Minkowskian version could be reduced to it via the Wick rotation.

This integral is not quite right since it contains an enormous overcounting since the configurations ϕ, g and ϕ', g' which are Diff×Weyl equivalent represent the same physical configuration. Thus, we need to take precisely one element from each equivalence class; this is called *gauge-fixing*. The conclusion is that we are left with integrating over the moduli space of Riemann surfaces.

One final remark: this is still not quite right: we should have checked that the Feynman 'measure' on the space of metrics, not just the action functional is invariant with respect to Diff×Weyl. If this is not so it indicates at the presence of a 'quantum anomaly'. It turns out that the anomaly is indeed present unless the dimension of the target space is 26.

5. Topological Quantum field theories and Frobenius Algebras

In this lecture we start to apply the physical ideas and considerations expounded in the previous lectures to build an abstract version of quantum field theory of which the simplest version is *topological quantum field theory*. A very detailed exposition could be found in the book [27].

Let us recall the description of a quantum particle. Let **H** be its space of states and H the Hamiltonian operator. Then the propagation of the state is given by the evolution operator $e^{-\frac{i}{\hbar}H}$. This is the $0+1$-dimensional field theory. Here 0 refers to the dimension of the particle itself and 1 is the additional time dimension.

To make this theory *topological* we require that the evolution operator depend only on the topology of the interval. That is the same as asking that it do not depend on t. This, in turn is equivalent to H being identically zero. Thus, topological quantum mechanics is completely trivial and uninteresting.

We now move to the dimension 1+1; a suitable generalization also exists in higher dimensions. The situation is now much more interesting because there are many topologically inequivalent two-manifolds. Below when we say '2-dimensional surface' we shall always mean 'an oriented

2-dimensional surface', all homeomorphisms will be tacitly assumed to preserve the orientation. Let us consider the following category $\mathcal{C}$.

Definition 5.1.

- *The objects of $\mathcal{C}$ are finite sets, thought of as disjoint unions of circles S^1.*
- *A morphism from the set I to the set J is a 2-dimensional surface (or 2-cobordism) whose boundary components parametrized and partitioned into two classes: incoming and outgoing; the incoming components are labeled by the set I and the outgoing components components are labeled by the set J. Two such morphisms are regarded to be equal if there exists a homeomorphism between the corresponding surfaces respecting the labelings on the set of boundary components.*
- *The composition $f \circ g$ of two morphisms is defined by glueing the outgoing boundary components of g to the incoming components of f.*

Note that the unit morphism corresponds to the cylinder connecting two circles. The category $\mathcal{C}$ has a monoidal structure, cf. *MacLane*: the product of two objects or morphisms is their disjoint union. Without going into a protracted discussion of monoidal categories we simply say that a monoidal structure is the abstraction of the notion of a cartesian product on sets or of a tensor product of vector spaces in that it is a bilinear operation together with a suitable coherent associativity isomorphism. Our monoidal structure is also *symmetric*, or commutative.

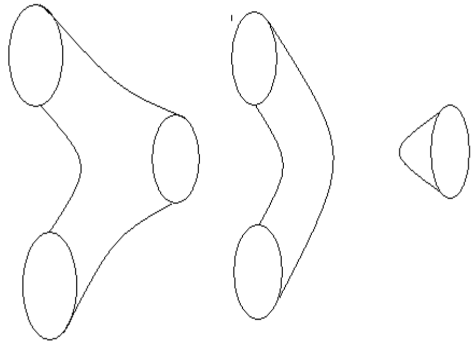
Definition 5.2. *A (closed) topological field theory (closed TFT for short) is a monoidal functor F from $\mathcal{C}$ to the category of vector spaces over $\mathbb{C}$.*

Note that the requirement that F be monoidal simply means that F takes disjoint unions of circles and the corresponding cobordisms into tensor products of vector spaces and their linear maps. It turns out that a closed TFT is equivalent to to a purely algebraic collection of data called a commutative *Frobenius algebra*.

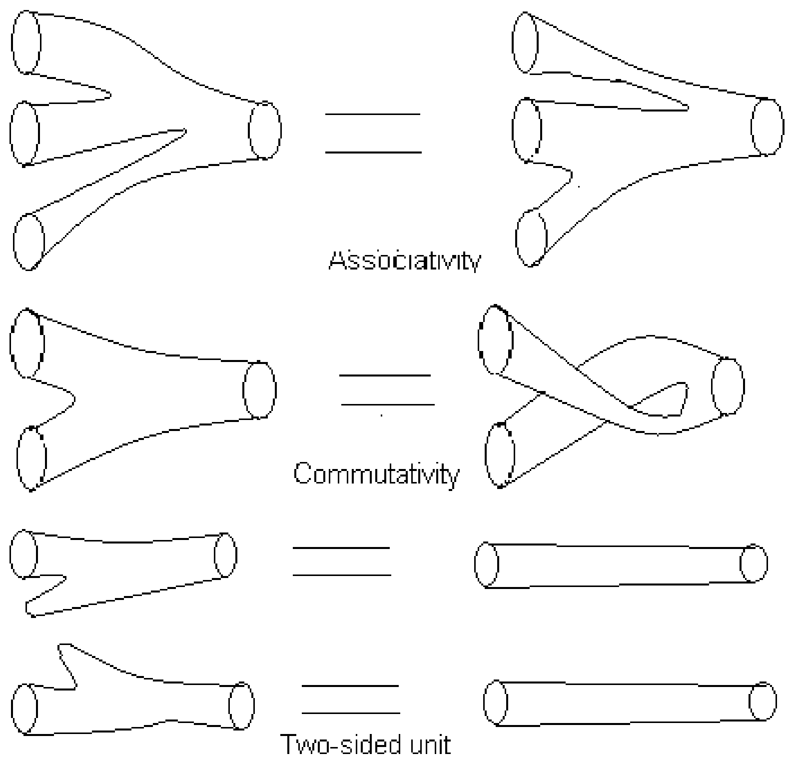
Definition 5.3. *A Frobenius algebra is a (unital) associative algebra A possessing a non-degenerate symmetric scalar product $\langle,\rangle$ which is invariant in the sense that $\langle ab, c\rangle = \langle a, bc\rangle$ for any $a, b, c \in A$.*

A unital Frobenius algebra has a trace $\mathrm{Tr}(a) := \langle a, 1\rangle$; it is clear that there is a 1-1 correspondence between invariant scalar products and traces. An example of a Frobenius algebra is given by a group algebra of a finite group, it will be commutative if the group is commutative. The corresponding trace is the usual augmentation in the group algebra.

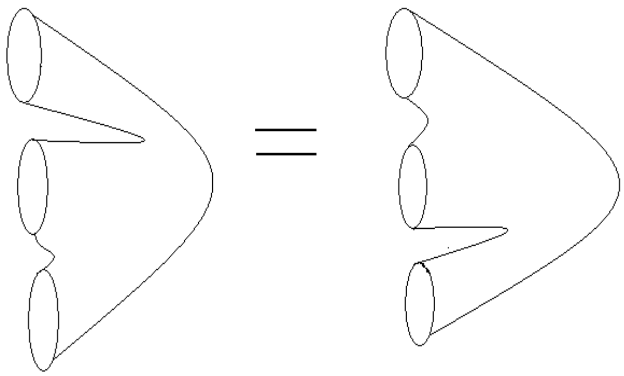
Suppose that one has a closed TFT F. We construct a Frobenius algebra A according to the following recipe. The underlying space of A will be $F(S^1)$. The multiplication map $A \otimes A \to A$, the unit map $\mathbb{C} \to A$ and the scalar product $A \otimes A \to \mathbb{C}$ are obtained by applying F to the following cobordisms where the incoming boundaries are positioned on the left and the outgoing ones on the right:



The following pictures show that the constructed multiplication is associative, commutative and has a two-sided unit; the first of these pictures is known as a pair of pants



The following picture proves the invariance condition:



Theorem 5.4. *The above construction gives a* $1-1$ *correspondence between isomorphism classes of commutative Frobenius algebras and isomorphism classes of closed TFT's.*

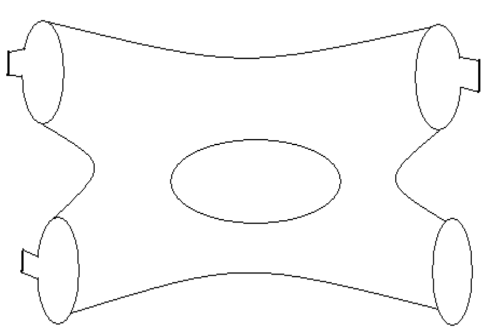
Proof. To construct a closed TFT out of a Frobenius algebra note that any 2-dimensional surface with boundary could be sewn from pairs of pants. To finish the proof one needs to show that the resulting functor *does not depend* on the choice of the pants decomposition of a surface. Informally speaking, that means that there are no further relations in a closed TFT besides associativity, commutativity and the invariance condition. This is done in [27] using Morse theory; another proof (of a differently phrased but equivalent result) is given in [5]. □

What about *open strings*? It turns out, that there is an analogous theorem which relates them to *noncommutative Frobenius algebras.*

Consider the category $\mathcal{OC}$ whose objects are unions of intervals I and whose morphisms are 2-dimensional surfaces with boundary components. We require that a set of intervals – open boundaries – are embedded in the union of all boundaries. The complement of the open boundaries are *free* boundaries; the latter can be either circles or intervals. The open boundaries are parametrized and partitioned into *incoming* and outgoing open boundaries.

The composition is defined by glueing at the open boundary intervals. Clearly the unit morphism between two intervals is represented by a rectangle connecting them. The following picture illustrates this definition.

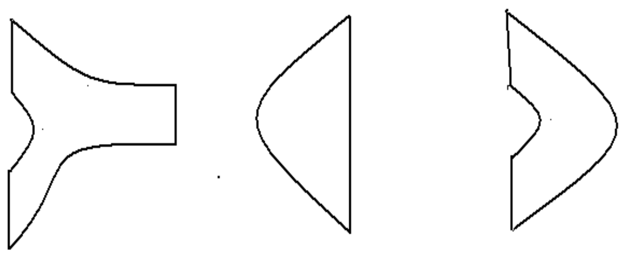
Here the incoming open boundaries are painted red whereas the outgoing open boundaries are green. The free boundaries are not colored.



Note that the category $\mathcal{OC}$ is monoidal with disjoint union determining the monoidal structure.

Definition 5.5. *An open TFT is a monoidal functor from $\mathcal{OC}$ to the category of vector spaces.*

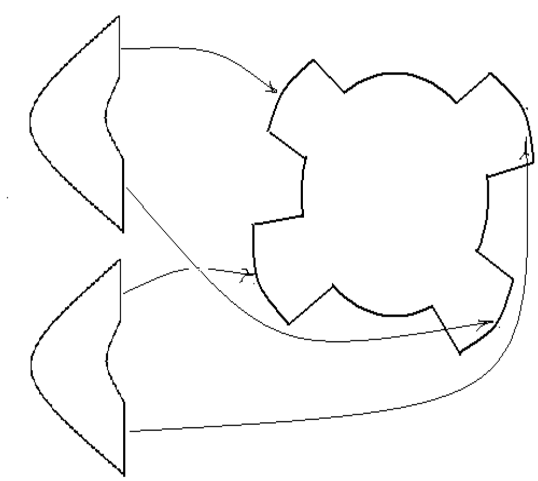
Let us now construct a (generally noncommutative) Frobenius algebra A from an open TFT F. The underlying space of A will be the result of applying F to I. The multiplication map, the unit map and the scalar product are obtained by applying F to the following pictures.



It is straightforward to prove the associativity, the unit axiom and the invariance property. Note that that the product is not necessarily commutative. This is because there is no orientation-preserving homeomorphism of a disc which fixes one point on its boundary and switches two points.

Theorem 5.6. *The above construction gives a* $1-1$ *correspondence between isomorphism classes of Frobenius algebras and isomorphism classes of open TFT's.*

To prove this theorem one has, first of all, decompose any 2-dimensional surface with boundary into pairs of 'flat pants', i.e. discs with three intervals embedded in the boundary circle. It is clear that we could obtain a disc with any number of free boundaries (this would correspond to taking the iterated product in the corresponding Frobenius algebra. Further, glueing those free boundary intervals (which corresponds to composing with the disc with two open boundaries) one can build any 2-surface with any number of free boundaries. For example, glueing four boundary intervals of a disc as indicated in the picture below, one obtains a torus with one free boundary:



Again, one has to prove that no other relations besides associativity and invariance are present; for this see [36] or [5].

Note that one can also meaningfully consider an *open-closed theory* which combines both open and closed glueing. Restricting to its closed (open) sector will give a closed (open) TFT. The corresponding algebraic structure is a pair of two Frobenius algebras, a map between them and a compatibility condition known as 'Cardy condition'. These issues are treated in detail in [36].

Finally we mention another generalization of the notion of TFT; the functor on the cobordism category can take values in the category of graded, $\mathbb{Z}/2$-graded (or super-) vector spaces or the corresponding categories of complexes. The above results readily generalize; the relevant algebraic structures are (super-) graded or differential graded Frobenius algebras, commutative or not.

An example of a graded Frobenius algebra is given by a cohomology ring of a manifold; the invariant scalar product being given by the Poincaré duality form.

Another example is given by the Dolbeault algebra of a Calabi-Yau manifold. The Dolbeault algebra of any complex manifold M has the form

$$\bigoplus_i \Omega^{0,i},$$

where $\Omega^{(0,i)}$ is the space of $(0,i)$ differential forms, i.e. forms which could be locally written as $f^{l_1\ldots l_i} d\bar{z}_{l_1} \wedge \ldots d\bar{z}_{l_i}$ where $f^{l_1\ldots l_i}$ is a smooth function. It is in fact a complex with respect to the $\bar{\partial}$-differential.

A Calabi-Yau manifold possesses a top-dimensional holomorphic form ω; wedging with this form following by integration over the fundamental cycle of M determines a trace on the Dolbeault algebra making it into a kind of Frobenius algebra (albeit infinite dimensional). The homotopy category of differential graded modules over this 'Frobenius' algebra is equivalent to the derived category of coherent analytic sheaves on M according to a recent result of J. Block [3]. This leads to an algebraic approach to the construction of Gromov-Witten invariants on Calabi-Yau manifolds.

6. Higher structures, moduli spaces and operads

Recall the observation that we made in Lecture 4: a topological field theory is nothing but a Frobenius algebra. This observation is extremely fruitful and we will try to generalize and build on it.

Recall the definition of the category $\mathcal{C}$: its objects are disjoint of circles and the objects are topological cobordisms. This is a category of sets; we can turn it into a linear category by taking linear spans of the sets of morphisms. Thus, we obtain a category whose sets of morphisms are vector spaces and compositions are linear maps. We denote the category thus obtained by $l\mathcal{C}$. The passage from $\mathcal{C}$ to $l\mathcal{C}$ is quite general and is similar to the passing from a group to its group algebra. We then consider *linear* functors from $l\mathcal{C}$ to the category of vector spaces (or graded vector spaces or complexes of vector spaces). Here by linear functors we mean those functors that map spaces of morphisms in $l\mathcal{C}$ *linearly* into spaces of morphisms of vector spaces. It is clear that such functors are in 1-1 correspondence with all functors from $\mathcal{C}$ to vector spaces; these are thus topological field theories.

We are going to define a certain 'derived' version of TFT's of various flavors. To this end consider the *topological* category $\mathcal{C}onf$ whose objects are again the disjoint unions of parametrized circles but whose morphisms are Riemann surfaces, i.e. 2-dimensional surfaces with a choice of a complex structure (equivalently, a choice of a conformal class of a Riemannian metric). Two morphisms are regarded to be equal if the corresponding Riemann surfaces are biholomorphically equivalent. In terms of metrics this can be phrased as follows: each conformal class of a metric contains a unique representative of constant curvature -1; two such are then considered equivalent if there exists a diffeomorphism of the surface taking one to another. [This is reminiscent of our discussion of the Diff×Weyl invariance of the Polyakov action.] As before, the category $\mathcal{C}onf$ is symmetric monoidal. We have the following result.

Proposition 6.1. *The category $\pi_0\mathcal{C}onf$ whose objects are the same as those of $\mathcal{C}$ and whose morphisms are π_0 of the spaces of morphisms of $\mathcal{C}$ is equivalent to the category $\mathcal{C}$.*

Proof. It is well-known that the moduli spaces of Riemann surfaces of a fixed genus are connected. Therefore the set of connected components of these moduli spaces is labeled by the genus. Two Riemann surfaces are homeomorphic if and only if they have the same genus. □

The analogous results hold for open and open-closed analogues of the category $\mathcal{C}onf$. One needs to use the fact that moduli spaces of Riemann surfaces with boundaries having the same genus and the same number of boundary components is connected and that these two numbers form a complete topological invariant of a 2-dimensional surface.

We would like to consider the monoidal functors from $\mathcal{C}onf$ to vector spaces. This leads to the notion of *conformal field theory* (CFT). More precisely, the notion of a CFT should also include the *complex-analytic structure* on the moduli space of Riemann surfaces. A result of Huang [24] states that the notion of a CFT is more or less equivalent to the notion of a *vertex operator algebra*. We will take a different route, replacing the topological spaces by chain complexes (e.g. singular chain complexes and the category $\mathcal{C}onf$ – by the corresponding differential graded category $dg\mathcal{C}onf$ (a differential graded category is the category whose sets of morphisms are complexes and the compositions of morphisms are compatible with the differential). Another possibility would be to find a *cellular* or *simplicial* model for the moduli spaces such that the glueing maps are simplicial or cellular. Taking, somewhat ambiguously, $dg\mathcal{C}onf$ to mean one of these dg categories, make the following definition.

Definition 6.2. *A closed topological conformal field theory (TCFT) is a monoidal functor $dg\mathcal{C}onf$ into the category of vector spaces.*

As before, this definition could be modified in several ways. Firstly, we can consider *open* TCFT's or, more generally, open-closed TCFT's. Secondly, we can consider the graded, $\mathbb{Z}/2$-graded vector spaces or complexes of vector spaces. The image of S^1 under TCFT is called the *state space*, if it has grading then it is usually called in physics literature *ghost number* and the differential on it (if present) is called the *BRST operator*.

One could consider other versions of field theories related to the category $\mathcal{C}onf$ and its open or open-closed analogues. Namely, instead of considering a dg category one could take its *homology* which will result in a graded category and consider monoidal functors from t into vector spaces. It is natural to call such functors *cohomological field theories* but this term has already been reserved for a slightly different notion.

Note that the moduli spaces we are considering are *non-compact*. Indeed, imagine the holomorphic sphere $\mathbb{C}P^1$ embedded into $\mathbb{C}P^2$ as the locus of the equation $xy = \epsilon z$ where x, y, z are the homogeneous coordinates in $\mathbb{C}P^2$ and $\epsilon \neq 0$. When ϵ approaches zero our sphere degenerates into a singular surface that is topologically a wedge of two spheres. There is a natural compactification of the moduli space $\mathscr{M}_g$ of smooth Riemann surfaces of genus g obtained by adding surfaces with simple double points. This compactification $\overline{\mathscr{M}}_g$ is called the *Deligne-Mumford compactification* and it is known to be a smooth orbifold, in particular, it has Poincaré duality in its rational cohomology. There is also the corresponding notion $\overline{\mathscr{M}}_{g,n}$ for Riemann surfaces with n marked points.

The spaces $\overline{\mathscr{M}}_{g,n}$ form a category $DM\mathcal{C}$. Its objects are disjoint unions of points (thought of as infinitesimal circles) and its morphisms are the Deligne-Mumford spaces of surfaces whose marked points are partitioned into two classes – incoming and outgoing. The composition is simply the glueing of surfaces at marked points. The monoidal structure is given by the disjoint unions of points and surfaces. Denote the *homology* of this category by $hoDM\mathcal{C}$.

Then the *cohomological field theory* in the terminology of Kontsevich-Manin is a monoidal functor $hoDM\mathcal{C}$ to the category of vector spaces. We will return to this notion in the later lectures, for now we'll just say that they are related to many topics of much curent interest such as *Frobenius manifolds* and *mirror symmetry.* These topics are the subject of [34].

The categories $\mathcal{C}, \mathcal{C}onf, DM\mathcal{C}$ etc. are examples of *PROP's.* A PRO is a symmetric monoidal category whose objects are identified with the set of natural numbers and the tensor product on them is given by addition. A PROP is a PRO together with a right action of the permutation group S_m and a left action of S_n on the set $Mor(m,n)$ compatible with the monoidal structure and the compositions of morphisms. A good discussion of PROP's is found in [1]. One of the simplest examples of PROP's is an *endomorphism PROP* for which $Mor(m,n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$ where V is a vector space. An *algebra* over a PROP P is a morphism of RPOP's from P into a suitable endomorphism PROP which is the same as a monoidal functor from P to vector spaces.

At this point we change the viewpoint slightly and will consider *operads* rather then PROP's. To be sure, PROPs are more general than operads and certain structures (e.g. bialgebras) cannot be described by operads. However, for the purposes of treating such objects as TCFT's operads are adequate and their advantage is that they are considerably smaller than PROP's. Informally speaking, an operad retains only part of the information encoded in a PROP: about the morphisms with only one output. Consequently, the composition is only partially defined. Here's the definition of an operad in vector spaces; a similar definition makes sense in any symmetric monoidal category.

Definition 6.3. *An operad $\mathcal{O}$ is a collection of vector spaces $\mathcal{O}(n)$, $n \geq 0$ supplied with actions of permutation groups S_n and a collection of composition morphisms for $1 \leq i \leq n$:*

$$\mathcal{O}(n) \otimes \mathcal{O}(n') \to \mathcal{O}(n + n' - 1)$$

given by $(f,g) \mapsto f \circ_i g$ satisfying the following properties:

- *Equivariance: compositions are equivariant with respect to the actions of the permutation groups.*
- *Associativity: for each $1 \leq j \leq a, b, c,\ f \in \mathcal{O}(a) h \in \mathcal{O}(c)$*

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+c-1} g, 1 \leq i \leq j \\ f \circ_j (g \circ_{i-j+1} h), j \leq i < b+j \\ (f \circ_{i-b+1} h) \circ g, j+b \leq i \leq a+b-1 \end{cases}.$$

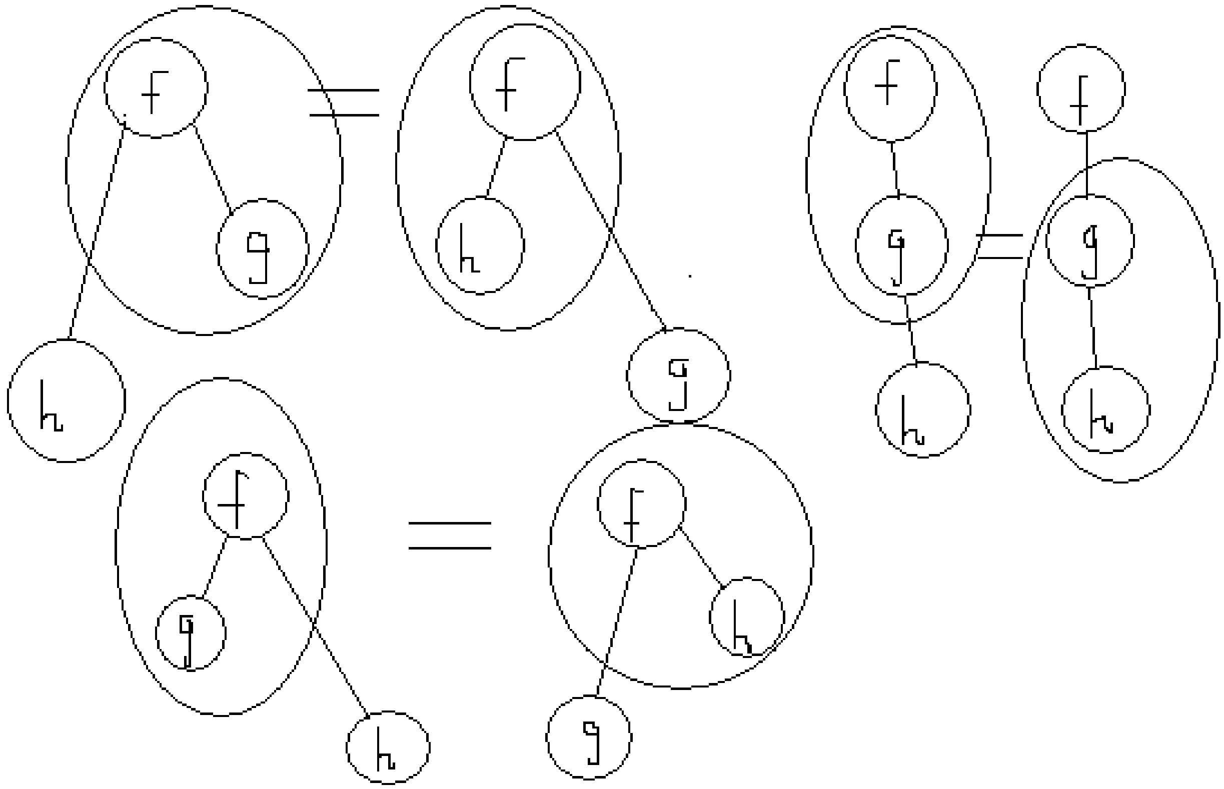
- *Unitality. There exists $e \in \mathcal{O}(1)$ such that*

$$f \circ_i e = e \text{ and } e \circ_1 g = g.$$

It is useful to visualize this definition thinking of the operad of *trees.* A tree is an oriented graph without loops, such that each vertex has no more than one outgoing edge and at least on incoming edge. The edges abutting only one vertex are called *external.* There is a unique outgoing external edge called the *root*, the rest of the external edges are called the *leaves.*

Set $Trees(n)$ to be the set of trees with n leaves. This will be an operad of *sets*; to pass from it to an operad of vector spaces on simply takes the linear span of everything in sight.

The operation $T_1 \circ_i T_2$ grafts the root of the tree T_2 to the ith leaf to the tree T_1. The associativity relations could be depicted as follows. Here the two trees which are composed first and encircled.



Remark 6.4. *One can omit any reference to the permutation groups in the definition of an operad thus getting a definition of a non-Σ-operad. Another variation is* non-unital *operads – the existence of a unit is not required.*

Remark 6.5. *In many ways operads behave like associative algebras. In fact, they* are *generalizations of associative algebras – their definition implies that the component $\mathcal{O}(1)$ of an operad is an associative algebra.*

Example 6.6. *We have seen one example of an operad – the operad of trees. Closely related to it are various operads constructed from moduli spaces of Riemann surfaces. Take, for example, the Deligne-Mumford operad whose nth space is the $\overline{\mathbb{M}}_{0,n+1}$, the compactified moduli space of Riemann surfaces with $n+1$ marked points. One views the first n marked point as inputs and the remaining on as the output; the composition maps are simply glueing at marked points. One can also consider uncompactified versions of this operad with either closed or open glueings. These are* topological *operads; to obtain a linear operad one takes its singular or cellular complex or its homology. Another example is the* endomorphism operad $\mathcal{E}(V)(n) := \mathrm{Hom}(V^{\otimes n}, V)$ *where V is a vector space (or a graded vector space etc.*

There is a functor from PROP's to operads forgetting part of the structure. Namely, if $\mathcal{P}$ is a PROP then the associated operad is $\mathcal{O}(n) := \mathcal{P}(n, 1)$. For example, we can speak about the *endomorphism operad* of a vector space $\mathcal{E}(V)(n) := \mathrm{Hom}(V^{\otimes n}, V)$. Conversely, one can write down a 'free' PROP generated by a given operad; these functors are adjoint.

Definition 6.7. *Let $\mathcal{O}$ be an operad. An algebra over $\mathcal{O}$ is a map of operads $\mathcal{O} \to \mathcal{E}(V)$ where V is a vector space (graded vector space etc.)*

Note the similarity of between the notions of an algebra over an operad and over a PROP. Indeed, an algebra over a PROP, freely generated by an operad | is the same as an $\mathcal{O}$-algebra which follows from the adjointness of the corresponding functors.

However there is one deficiency of operads compared to PROP's. For example, starting from the a pair of pants surface and forming iterated PROPic compositions with itself one can get a surface of an arbitrary genus whereas operadic compositions preserve the genus. The notion that is completely adequate for the description of various TCFT's is that of a *modular operad*, not merely an operad. The idea of a modular operad is that together with grafting operations one is allowed to form 'self-glueings'. We will discuss this notion in the future lectures.

7. Operads and their cobar-constructions; examples

In this lecture we will continue our study of operads; our main source is the foundational paper [17], a lot of useful examples and motivation could be found in the survey article [39].

We will start by defining the notion of a *free operad*; it is naturally formulated in the language of trees. We will call a Σ-module a collection $E(n), n \geq 0$ of right S_n-modules. There is an obvious forgetful functor from the category of operads to the category of Σ-modules; we will now explain how to construct a *left adjoint* to this forgetful functor; its value on a Σ-module E will be called the *free operad* on E.

It will be useful to regard a Σ-module E as a functor from the category of finite sets to the category of vector spaces. Namely, set

$$E(S) := E(n) \otimes_{\mathbb{C}[S_n]} \mathrm{Iso}([n], S),$$

where S is a finite set, $[n]$ is a set $1, 2, \ldots, n$ and $\mathrm{Iso}([n], S)$ is the set of bijections from $[n]$ into S.

Let T be a labeled tree. For a vertex v of T we denote by $in(v)$ the set of input edges of v. Consider the expression

$$E(T) := \bigotimes_v E(in(v)).$$

Here the tensor product is extended over all vertices of T. Informally we will call an element of $E(T)$ an E-decorated tree.

Definition 7.1. *Define the free non-unital operad ΦE on a Σ-module E by the formula*

$$FE(n) = \bigoplus_T E(T),$$

where $n \geq 0$ and the summation is taken over classes of isomorphism *of all labeled trees with n leaves.*

In order to validate this definition we have to specify the operadic maps $\circ_i$ in ΦE. Take two E-decorated trees. That means that we are given two normal trees T_1 and T_2 and two tensors e_{T_1} and e_{T_2} – their 'decorations'. Then

$$(T_1, e_{T_1}) \circ_i (T_2, e_{T_2}) = (T_1 \circ_i T_2, e_{T_1} \otimes e_{T_1}).$$

Furthermore, the action of the permutation group on $\Phi E(n)$ is by relabeling the inputs.

Remark 7.2. *Note that ΦE is indeed a non-unital operad. E.g. let $E(1) = \mathbb{C}$, the ground field and $E(k) = 0$ for $k \neq 1$. Then E-decorated trees will obviously have only bivalent vertices, and the tree without vertices (corresponding to the operadic unit) will not qualify as an E-decorated tree. In fact, it is easy to see that $\Phi E(1) = \mathbb{C}_+[x]$, the non-unital tensor (=polynomial) algebra on a single generator x.*

To obtain a free unital *operad from ΦE one should formally add an operadic unit to it, i.e. an element $e \in \Phi E(1)$ satisfying the axioms for the unit. We will denote the obtained operad FE.*

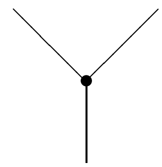
We will not prove that Φ is indeed a free functor. This is almost obvious although the rigorous proof is a little fussy. Given a Σ-module E we have to form all possible $\circ_i$-products using its elements but it's clear that any sequence of such $\circ_i$-products is encoded in a tree. For example, composing unary operations (elements in $E(1)$) results in bivalent trees. When one compose binary operations (elements in $E(2)$ the result will be binary trees etc.

Operads are in many ways similar to associative algebras, except the multiplication in them is conducted in a tree-like rather than a linear fashion. There is an analogue of an *ideal* in the operadic context:

Definition 7.3. *A (two-sided) ideal in an operad $\mathcal{O}$ is a collection $I(n)$ of S_n-invariant subspaces in each $\mathcal{O}(n)$ such that if $x \in I(n)$ and $a \in \mathcal{O}(k)$ then both $x \circ_i a$ and $a \circ_i$ belong to I whenever these compositions make sense.*

For an operad $\mathcal{O}$ and its ideal I one can define the quotient operad $\mathcal{O}/I$ in an obvious manner. If the case when $\mathcal{O}$ is free on a Σ-module E we say that $\mathcal{O}/I$ is generated by E and has the ideal of relations I.

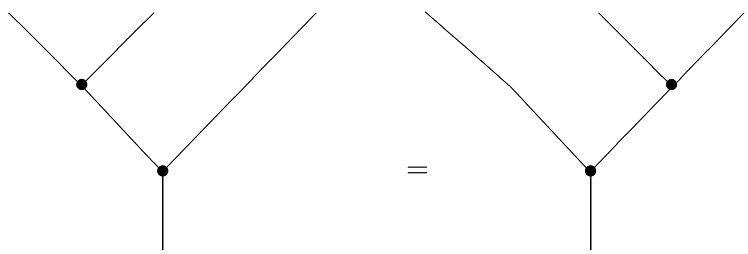
Example 7.4. (1) *One of the simplest linear operads is $\mathcal{C}om$ whose algebras are the usual commutative non-unital algebras. It is generated by on element $m \in \mathcal{C}om(2)$ represented by the fork*



having the trivial action of S_2 and subject by the associativity relation:

$$m \circ_1 m = m \circ_2 m$$

which can also be expressed pictorially as follows:



Note that the operad $\mathcal{C}om$ is the genus zero part of the closed TFT PROP.

(2) *The associative operads $\mathcal{A}ss$ has two generators in $\mathcal{A}ss(2)$:*

which are permuted by the generator of S_2; for both m_2 and m_1 the associativity condition $m_i \circ_1 m_i = m_i \circ_2 m_i$ also holds. Note that the operad $\mathcal{A}ss$ is the genus zero part of the open TFT PROP. The operad $\mathcal{C}om$ is obtained from $\mathcal{A}ss$ by quotienting out by the ideal $m_1 - m_2$.

(3) *The Lie operad $\mathcal{L}ie$ has one generator m in $\mathcal{L}ie(2)$ which is sent to minus itself by the generator of S_2. The relation that it satisfies is the Jacobi identity which could be expressed as the sum of three trees on three leaves being equal to zero.*

(4) *Recall that a* Poisson *algebra is a vector space V with a unit, a commutative and associative dot-product and a Lie bracket which are related by the compatibility condition:*

$$[a, bc] = [a, b]c + b[a, c] \text{for all } a, b, c \in V.$$

It follows that the operad $\mathscr{P}$ whose algebras are Poisson algebras is generated by two elements $\mathscr{P}(2)$ with relations determined by associativity, commutativity, the Jacobi identity and the compatibility condition.

7.1. **Cobar-construction.** Cobar-construction is one of the most interesting general constructions that can be performed on an operad; it is a generalization of the cobar-construction for associative algebras.

Definition 7.5. *operad $\mathcal{O}$ is admissible in the sense that $\mathcal{O}(k)$ is finite-dimensional for all k, $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) = \mathbb{C}$, the ground field.*

Remark 7.6. *Our definition of admissibility is a simplified version of that defined in* [17]*; there they assume that $\mathcal{O}(1)$ is a semisimple algebra and take tensor products over $\mathcal{O}(1)$. We could also have considered the case when $\mathcal{O}(1)$ is a nilpotent non-unital algebra. The general case (i.e. when no restrictions are imposed on $\mathcal{O}$) should involve a certain completion.*

Before giving the general definition of the cobar-construction let us discuss the notion of a *derivation* of an operad.

Definition 7.7. *A collection of homogeneous maps $f_n : \mathcal{O}(n) \to \mathcal{O}(n)$ is called a derivation if*

$$f_n(a \circ_i b) = f(a) \circ_i b + (-1)^{|f||a|} a \circ_i f(b).$$

for any a, b, n and i for which $a \circ_i b$ makes sense.

Note that a derivation of a free operad ΦE is determined by its value on the generating space E, moreover any map $E \to \Phi E$ could be extended to a derivation (this is analogous to the well-known property of a free associative algebra and could be proved similarly).

Remember that the space of *all* E-decorated trees is the free operad on T; it should be thought of as an analogue of a tensor algebra. Now consider trees T with precisely 2 vertices and the corresponding decorated trees $E(T)$; denote it by $\Phi^2(E)$. Under this analogy, this space corresponds to the subspace of 2-tensors in the tensor algebra. If one thinks of a decorated tree as an iterated composition of multi-linear operations then these trees correspond to composing precisely two operations. If E is an operad then operadic compositions $\circ_i$ determine a map $m : \Phi^2(E) \to E$. This map completely determines the operad and is analogous to the multiplication map $T^2(A) = A \otimes A \to A$ for an associative algebra A.

Introduce some notation: For a dg vector space V we will denote its dual V^* by $(V^*)^i = (V^{-i})^*$ and its shift $V[1]^i = V^{i+1}$. Then $V[1]$ and $V[1]$ will again be a dg vector spaces.

Let $\mathcal{O}$ be an admissible operad and consider the operad $\Phi\mathcal{O}^*[-1]$; introduce a derivation d in it which is induced by the linear map

$$\mathcal{O}^*[-1] \to \Phi^2(\mathcal{O}^*[-1]) = \left((\Phi^2)(\mathcal{O}[-1])\right)^*$$

that is dual to the structure map $m : \Phi^2(E) \to E$ defined above.

The above definition ensures that d is indeed a derivation but not that $d^2 = 0$. We will now reformulate the definition in a more visualizable way; the formula $d^2 = 0$ will then be straightforward to prove.

Let C be a tree without any external edges (sometumes called a *corolla*), with i leaves, consider $\mathcal{O}^*(i)$, this is the same as an $\mathcal{O}^*$-decorated corolla C. Its image under d will be a sum over all $\mathcal{O}^*$-decorated trees with only one internal edge such that upon contracting it one gets C back. Thus, the image of $\xi \in C(\mathcal{O})$ will be the a sum of elements of the form $\xi_l \otimes \xi_m$ with $\xi_l \in \mathcal{O}(l)^*, \xi_m \in \mathcal{O}(m)$ and the map is the dual to the structure map of $\mathcal{O}$. The case of a general decorated tree $T(\mathcal{O}^*)$ is handled by representing T as an iterated composition of corollas and using the Leibniz rule.

Thus, the image of a general decorated tree $T(\mathcal{O})^*$ will be a sum of decorated trees obtained by *expanding a vertex* of T, i.e.by partitioning the set of half-edges of one vertex of T into two subsets, taking them apart and joining by a new edge.

In this description we neglected to mention the *shift* $[-1]$; it introduces an additional sign in the formula for the differential. Namely, the vertices of our trees need to be linearly ordered (which would correspond to choosing a decomposition of our tree into corollas) and the result of expanding the ith vertex acquires the sign $(-1)^{i-1}$. This is a direct application of the Koszul sign rule.

Definition 7.8. *The operad $\Phi\mathcal{O}^*[-1]$ with the differential d defined above is called the* cobar-construction *of the admissible operad $\mathcal{O}$. It will be denoted by $\mathsf{C}\mathcal{O}$.*

Among the most interesting are the cobar-constructions of the commutative, associative and Lie operads. To describe these let us introduce the notion of an *operadic suspension*.

Definition 7.9. *Let Λ be the operad with $\Lambda(n) = \mathrm{Hom}(\mathbb{C}[-1]^{\otimes n}, \mathbb{C}[-1])$ (in other words Λ is the endomorphism operad of $\mathbb{C}[-1]$. For an arbitrary operad $\mathcal{O}$ define its shift $\mathcal{O}[-1]$ as the operad $\mathcal{O} \otimes \Lambda$. Note that $\mathcal{O}[-1](n) = \mathcal{O}[n-1] \otimes \epsilon_n$ where ϵ_n is the sign character of S_n. The $\mathcal{O}[-1]$-algebras on a space V are the same as $\mathcal{O}$-algebras on the space $V[-1]$. The operad $\mathcal{O}[n]$, $n \in \mathbb{Z}$ is defined similarly.*

Example 7.10.

- *The operad $\mathsf{C}\mathcal{C}om$ is quasi-isomorphic to the operad $\mathcal{L}ie[1]$ whose algebras with underlying space V are Lie algebras on $V[1]$.*
- *The operad $\mathsf{C}\mathcal{A}ss$ is quasi-isomorphic to the operad $\mathcal{A}ss[1]$ whose algebras with underlying space V are Lie algebras on $V[1]$.*
- *The operad $\mathsf{C}\mathcal{L}ie$ is the operad $\mathcal{C}om[1]$ whose algebras with underlying space V are Lie algebras on $V[1]$.*

8. More on the operadic cobar-construction; ∞-algebras

Our main source in this lecture continues to be [17].

8.1. **Cobar-constructions.** Here's the main result about the operadic cobar-construction.

Theorem 8.1. *For any admissible operad $\mathcal{O}$ there is a canonical quasi-isomorphism $\mathsf{CC}\mathcal{O} \to \mathcal{O}$.*

Remark 8.2. *One should view the map $\mathsf{CC}\mathcal{O} \to \mathcal{O}$ as a canonical cofibrant resolution of an admissible operad $\mathcal{O}$. Namely, there is a closed model category structure on non-unital operads such that free operads are cofibrant. If $\mathcal{O}$ is not free then its category of algebras (which is the category of operad maps $\mathcal{O} \to \mathcal{E}(V)$ where $\mathcal{E}(V)$ is the endomorphism operad of a dg space V) may be 'wrong'. The example to have in mind here is the space of maps $X \to Y$ for two topological spaces X and Y – if X is not a cell complex then this space may change its weak homotopy type if X or Y are replaced by weakly equivalent spaces. To get the 'correct' homotopy type of the mapping space one should replace X by its cellular approximation.*

Rather than giving a proof of this theorem we sketch a proof of the corresponding result for associative algebras and then indicate how it generalizes to operads.

Let A be a graded non-unital associative algebra; we'll assume that its graded components A^i are finite-dimensional and that $A^i = 0$ for $i \leq 0$. This is the analogue of the admissibility condition for operads. We want to stress here that these assumptions are adopted here for convenience only, the general case could be treated using suitable completions and we refer for a much more complete treatment to [20]. In fact one can generalize our constructions to arbitrary dg algebras or A_∞-algebras.

In the following definition the symbol T stands for the non-unital tensor algebra: $TV := \bigoplus_{i=1}^\infty V^i$. It is clear that any (graded) derivation of TV is determined by its restriction on V, and the latter could be an arbitrary linear map. Choose a basis $x_1, \ldots, x_n$ in V; then any derivation could be written uniquely as a linear combination $\sum_i f(x_1, \ldots, x_n)\partial_i$ where ∂_i are 'partial derivatives' with respect to x_i.

Definition 8.3. *The* cobar-construction *of A is the dg algebra $\mathsf{C}A := T(A^*[-1])$ whose differential d is the derivation whose restriction to the space of generators $A^*[-1]$ is the map of degree 1:*

$$A^*[-1] \to A^*[-1] \otimes A^*[-1]$$

which is dual to the multiplication map $A \otimes A \to A$.

Theorem 8.4. *There is a canonical quasi-isomorphism $\mathsf{CC}A \to A$.*

Proof. Note that $\mathsf{CC}A = T(T(A^*[-1])^*[-1])$ thus $(A^*[-1])^*[-1] \cong A$ is a direct summand in $\mathsf{CC}A$. The quasi-isomorphism mentioned in the statement of the theorem is the projection onto this direct summand; it is straightforward to prove that this projection is a chain map.

The complex $\mathsf{CC}A$ can be written as a double complex

$$(T(A^*[-1]))^* \to (T(A^*[-1])) * \otimes (T(A^*[-1]))^* \to \dots . \tag{8.1}$$

Note that the horizontal differential is the standard cobar differential whose cohomology is $Ext^*_{T(A^*[-1])}(\mathbb{C}, \mathbb{C})$ for $* > 0$. Note that

$$Ext^k_{TV}(\mathbb{C}, \mathbb{C}) = \begin{cases} \mathbb{C}, k = 0 \\ V^*, k = 1 \\ 0, k > 1 \end{cases} . \tag{8.2}$$

Consider the spectral sequence whose E_2-term is obtained by taking first the horizontal cohomology of (8.1), then vertical. By the previous calculation it collapses at the E_2-term giving the result. Note that the 'admissibility conditions' ensure the convergence of this spectral sequence. □

To generalize the above proof to operads we replace the algebraic bar-construction by the operadic bar-construction. The only nontrivial bit is the calculation of the cobar-cohomology of a free operad. Let us formulate the corresponding result.

Lemma 8.5. *Let E be a Σ-module for which $E(0) = E(1) = 0$ and all spaces $E(n)$ are finite-dimensional. Then the cohomology of the cobar-construction of the operad $\Phi(E)$ has E^* as its underlying Σ-module.*

We will not prove this lemma; it could be proved by a direct calculation as in [17] or, more conceptually, by establishing an analogue of the equation (8.2). Note that this equation follows from the resolution of a TV-module $\mathbb{C}$:

$$V \otimes TV \to TV \to \mathbb{C}$$

For the last formula to make sense one has to develop the abelian category of modules over an operad, free resolutions etc. As far as we know this has not been done.

Remark 8.6. *One can strengthen Lemma 8.5 to the statement that the operad $\mathsf{C}\Phi E$ is quasi-isomorphic to the* operad *E^* whose composition maps are all zero. This should be compared with the corresponding result for associative algebras: the cobar-construction of the non-unital tensor algebra is quasi-isomorphic to the algebra with zero multiplication. Conversely, the cobar-construction of the trivial algebra (or operad) is quasi-isomorphic to the free algebra (operad).*

8.2. **∞-algebras.** We now discuss $A_\infty-, C_\infty-$ and $L_\infty-$algebras.

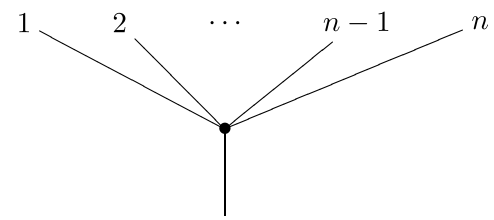
Definition 8.7.

- *An A_∞-structure on a dg vector space V is a $\mathsf{C}\mathcal{A}ss$-algebra structure on $V[1]$.*
- *An L_∞-structure on a dg vector space V is a $\mathsf{C}\mathcal{L}ie$-algebra structure on $V[1]$.*
- *An A_∞-structure on a dg vector space V is a $\mathsf{C}\mathcal{C}om$-algebra structure on $V[1]$.*

This definition is a special case of a more general concept of a *homotopy algebra*. Given an operad $\mathcal{O}$ what is a homotopy algebra over $\mathcal{O}$? This has to do with cobar-duality. Namely, a homotopic $\mathcal{O}$-algebra could be defined as an algebra over the canonical cofibrant resolution $\mathsf{CC}\mathcal{O}$ of $\mathcal{O}$. For operads $\mathcal{O}$ such as $\mathcal{C}om, \mathcal{L}ie$ and $\mathcal{A}ss$ their cobar-duals are particularly simple (see Lecture 6) and so the definition of a homotopy $\mathcal{O}$-algebra is considerably simplified. The same phenomenon happens for all *Koszul* operads, [17]. We will not discuss Koszul operads here but mention that most operads of interest are in fact Koszul.

Let us unravel the definition of an A_∞-algebra. Recall that $\mathsf{C}\mathcal{A}ss$ is freely generated (disregarding the differential) by the Σ-module $\mathcal{A}ss[-1]$. It is not hard to prove that $\mathcal{A}ss(h)$ can be identified with the regular representation of S_n. We will represent the generators of $\mathcal{A}ss(n)$, where $n > 1$ by the corollas

$c(n)$:=



The corollas are placed in degree 1; permutation group S_n acts on them by permuting the labels of their inputs. It follows that a map $f : \mathsf{C}\mathcal{A}ss \to \mathcal{E}(V[1])$ is determined by a collection of elements of degree 1: $\check{m}_n \in \mathrm{Hom}(V[1]^{\otimes n}, V[1])$, $n \geq 3$. Here $\check{m}(n) = f(c_n)$.

The collection of elements $\check{m}_n$ is equivalent to a collection of elements $m_n \in \mathrm{Hom}(V^{\otimes n}, V$ of degrees $2 - n$ called *higher multiplications*. The condition that f be a map of dg operads translates into the collection of compatibility conditions between higher multiplications:

$$m_n(x_1, \ldots, x_n) - (-1)^n \sum_{i=1}^{n} \pm m_n(x_1, \ldots, dx_i, \ldots, x_n)$$

$$\sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{i=0}^{k-1} \pm m_k(x_1, \ldots, x_i, m_l(x_{i+1}, \ldots, x_{i+l}), \ldots, x_n)$$

For $n = 2$ and 3 these relations mean, respectively, that d is a derivation for m_2 and that m_2 is chain homotopy associative, the chain homotopy being the map m_3.

Analogous constructions could also be made in the context of L_∞ and C_∞-algebras.

8.3. **Geometric definition of homotopy algebras.** We now give an alternative (but, of course, equivalent), definition of an ∞-algebra. The full treatment could be found in [20] and [17]. for a (graded) vector space V we denote by TV, SV and LV the tensor algebra, symmetric algebra and the free Lie algebra respectively. We will denote by $\hat{T}V, \hat{S}V$ and $\hat{L}V$ the corresponding completions. For example, $\hat{S}V$ is the algebra of formal power series in elements of V. We will call a *vector field* on the corresponding completed algebra a continuous derivation of it. With this we have the following definition.

Definition 8.8. *Let V be a free graded vector space:*

(a) *An L_∞-structure on V is a vector field*

$$m : \hat{S}V^*[-1] \to \hat{S}V^*[-1]$$

of degree one and vanishing at zero, such that $m^2 = 0$.

(b) *An A_∞-structure on V is a vector field*

$$m : \hat{T}V^*[-1] \to \hat{T}V^*[-1]$$

of degree one and vanishing at zero, such that $m^2 = 0$.

(c) *A C_∞-structure on V is a vector field*

$$m : \hat{L}V^*[-1] \to \hat{L}V^*[-1]$$

of degree one, such that $m^2 = 0$.

Let us explain how this definition correspond to the one given previously. Any continuous derivation of $\hat{T}V^*[-1]$ is determined by a linear map $V^*[-1] \to \hat{T}V^*[-1]$, moreover the last map could be arbitrary. Next, such a map is determined by a collection of maps $V^*[-1] \to (V^*[-1])^{\otimes n}$. These maps are dual to the maps $\check{m}_n$ introduced in the previous section.

The advantage of this definition is that it is geometric: one can say concisely that, e.g. an L_∞-structure is a vector field on a formal manifold. The other homotopy algebras are interpreted in terms of noncommutative geometry. The notions of *homotopy morphisms* are best interpreted in this language. Additionally this definition involves no signs to memorize.

Let us define the notion of a *morphism* between two ∞-algebras.

Definition 8.9. *Let V and U be vector spaces:*

(a) *Let m and m' be L_∞-structures on V and U respectively. An L_∞-morphism from V to U is a continuous algebra homomorphism*

$$\phi : \hat{S}V^*[-1] \to \hat{S}V^*[-1]$$

of degree zero such that $\phi \circ m' = m \circ \phi$.

(b) *Let m and m' be A_∞-structures on V and U respectively. An A_∞-morphism from V to U is a continuous algebra homomorphism*

$$\phi : \hat{T}V^*[-1] \to \hat{T}V^*[-1]$$

of degree zero such that $\phi \circ m' = m \circ \phi$.

(c) *Let m and m' be C_∞-structures on V and U respectively. A C_∞-morphism from V to U is a continuous algebra homomorphism*

$$\phi : \hat{L}V^*[-1] \to \hat{L}V^*[-1]$$

of degree zero such that $\phi \circ m' = m \circ \phi$.

If one views an ∞-structure as an odd vector field on a formal (noncommutative) manifold, then an ∞-morphism is a (smooth) map between the corresponding formal manifolds 'carrying' one vector field to the other. It is not so easy to describe what's going on in terms of maps into the endomorphism operad. If the ∞-morphisms under consideration are invertible, i.e. are (noncommutative) diffeomorphisms then one can show, [5] that the corresponding maps of operads are *homotopic* in a suitable sense.

9. Operads of moduli spaces in genus zero and their algebras

We now return to the the operads of moduli spaces; we consider genus zero in this lecture.

9.1. **Open TCFT.** Consider the moduli space of holomorphic discs with $n+1$ marked and labeled points on the boundary; there is one 'output' marked point and n 'inputs'. We identify such discs with the standard unit disc in the complex plane and fix the output point. Two such discs are identified if there is a biholomorphic map from one to the other taking the marked input points to marked input points in a label-preserving fashion. It is clear that the moduli space of such discs is the same as the disjoint union of configuration spaces of $n-2$ ordered points on an interval; it is thus homeomorphic to a disjoint union of $n!$ open cells I^{n-2}, the symmetric group S_n acts on it by permuting these cells. We compactify this moduli space by glueing holomorphic discs at marked points. More precisely, to any planar tree T with n leaves (with vertices of valence ≥ 3) we associate (moduli space of) Riemann surfaces with boundary of genus 0. Any vertex v of T corresponds to a holomorphic disc D; the input edges of T would correspond to the input marked points on D and the output edge of v would correspond to the output marked point on D. These discs are glued according to the tree T.

The obtained moduli space will be denoted by $\mathscr{M}O(n)$. We will only consider spaces $\mathscr{M}O(n)$ for $n > 1$. It is clear that the spaces $\mathscr{M}O(n)$ form a topological (non-unital) operad; one can make it unital by inserting the operadic unit by hand. It turns out that trees give a cellular decomposition of the operad $\mathscr{M}O$, with the corolla (tree without internal edges) corresponding to the cell of the highest dimension and the binary trees correspond to the vertices, cf. [6]. The corresponding cellular chain complex is essentially the cobar-complex of the associative operad (more precisely, the cobar-complex of the *suspension* of the associative operad. Therefore the corresponding TCFT is simply an A_∞-algebra.

9.2. **Deligne-Mumford field theory in genus 0.** We now consider the Deligne-Mumford analogue of the open TCFT (at the tree level). Consider the moduli space of Riemann spheres with $n+1$ marked and labeled points; there is one 'output' marked point and n 'inputs'. This moduli space in non-compact, to compactify it we consider the moduli space of nodal stable algebraic curves of genus zero. Each such curve is glued from Riemann spheres according to some tree; furthermore each irreducible component (sphere) minus its marked points and singularities has negative Euler characteristic. For a tree on n leaves we denote the space of such curves by $\overline{\mathscr{M}}(n)$, clearly the spaces $\overline{\mathscr{M}}(n)$ with $n > 1$ form a non-unital operad, as before, one can put in a unit by hand if one wishes. Each tree with n leaves determines a stratum in $\overline{\mathscr{M}}(n)$, the maximal stratum corresponds to a corolla while the binary trees correspond to zero-dimensional strata. The difference with the open case is that strata are not anymore contractible and thus do not give a cellular decomposition of $\overline{\mathscr{M}}(n)$. In fact, the open strata of $\overline{\mathscr{M}}(n)$ correspond to non-compact moduli spaces $\mathscr{M}(k)$ of smooth curves of genus zero with k marked points with $k \leq n$. The maximal stratum is $\mathscr{M}(n)$.

Let us give a description of $H^*(\mathscr{M}(n))$. For this we introduce the space $\mathbb{C}_n^0$ of configurations of n points in $\mathbb{C}$. If $1 \leq j \neq k \leq n$ let ω_{jk} be the differential 1-form on $\mathbb{C}_n^0$ defined by the formula

$$\omega_{jk} := \frac{d\log(z_j - z_k)}{2\pi i}.$$

Then we have the following result.

Theorem 9.1. *The cohomology ring $H^*(\mathbb{C}_n^0, \mathbb{Z})$ is the graded commutative ring with generators ω_{jk} and relations $\omega_{jk} = \omega_{kj}$ and $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$. The symmetric group S_n acts on the generators as $\sigma\omega_{ij} = \omega_{\sigma(i)\sigma(j)}$.*

This theorem could be deduced by induction from the fibration

$$\mathbb{C} \setminus (z_1, \dots, z_n) \to \mathbb{C}_{n+1}^0 \to \mathbb{C}_n^0$$

defined by projecting $(z_1, \dots, z_{n+1})$ onto $(z_1, \dots, z_n)$.

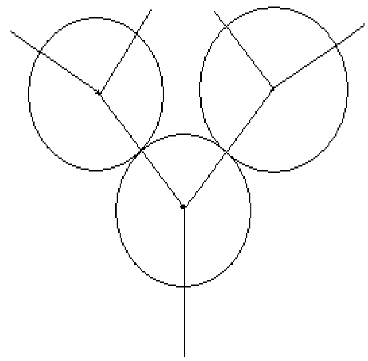
Corollary 9.2. *The cohomology ring $H^*(\mathscr{M}(n), \mathbb{C})$ is isomorphic to the kernel of the differential d on $H^*(\mathbb{C}_n^0)$ whose action on the generators is $d(\omega_{ij}) = 1$.*

Note that $\mathscr{M}(n)$ is the quotient of the configuration space of points on $\mathbb{C}P^1$ by the action of the group $PSL(2, \mathbb{C})$ of fractional linear transformations. The point ∞ is fixed by the subgroup $Aff(\mathbb{C})$ consisting of the matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. This subgroup is clearly homotopy equivalent to its circle subgroup consisting of diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $|a| = 1$. It follows that $\mathscr{M}(n)$ is homotopy equivalent to $C_n^0 \times S^1$. This allows one to reduce the problem to finding the S^1-equivariant cohomology of C_n^0 which is straightforward.

Let us return to the Deligne-Mumford operad $\overline{\mathscr{M}}$. We stress that the situation is similar to, but more interesting than, the corresponding open case. The reason for it is, that, firstly, the moduli space of marked points in $\mathbb{C}P^1$ is not contractible, in contrast with the space of holomorphic discs with marked points on the boundary and, secondly, that the corresponding compactification is a *closed manifold*.

Let us describe the space $\overline{\mathscr{M}}(n)$ in more detail. It will consist of nodal curves of arithmetic genus zero. Here 'nodal' means that a curve is smooth except at *nodes* or *double points* which are singularities locally isomorphic to the one given by the equation $xy = 0$ in $\mathbb{C}^2$. The arithmetic genus is the usual genus of a curve obtained by smoothing all nodes. We will require that these curves have $n+1$ marked points and that each irreducible component minus marked and singular points have negative Euler characteristic (stability condition). This moduli space is also denoted by $\overline{\mathscr{M}}_{n+1}$ for obvious reasons. For $n = 3$ $\overline{\mathscr{M}}_n$ consists of one point, for $n = 4$ it is isomorphic to $\mathbb{C}P^1$. In general, $\overline{\mathscr{M}}(n)$ is a smooth projective algebraic variety of complex dimension $n - 3$.

A stable curve Σ determines a graph Γ_Σ called the *dual graph* of Σ. Its vertices are the irreducible components of Σ and each edge corresponds to a double point. This graph will be a tree for stable curves of arithmetic genus zero. The following picture illustrates this definition.



The dual graphs of stable curves determine a filtration: of $\overline{\mathscr{M}}_n$ by closed subspaces

$$F_0 \subset F_1 \subset \ldots \subset \overline{\mathscr{M}}_n, \tag{9.1}$$

where F_p is formed by the stable curves of at least $n-3-p$ double points. Equivalently, one can say that $F_p = \bigcup \overline{\mathscr{M}}(T)$, the union of the closures of the moduli spaces corresponding to those trees which have $n-3-p$ internal edges. Note that F_0 corresponds to binary trees. The stratum $\overline{\mathscr{M}}(T)$, the closure of the subspace of stable curves corresponding to the tree T is a smooth algebraic variety; moreover the strata of dimension one less inside $\overline{\mathscr{M}}(T)$ form a divisor with normal crossing inside $\overline{\mathscr{M}}(T)$; that means that they are smooth algebraic subvarieties in $\overline{\mathscr{M}}(T)$ meeting transversally.

One should think of the stratification (9.1) as an analogue of the filtration of a cell complex by its skeleta (which is really the case for open TCFT). We have an associated spectral sequence:

$$E^1_{pq} = H_{p+q}(F_p, F_{p-1}) = H_{p+q}(F_p/F_{p-1}) \Rightarrow H_{p+q}(\overline{\mathscr{M}}_n).$$

The differential acts as follows:

$$d_1 : H_{p+q}(F_p, F_{p-1}) \to H_{p+q-1}(F_p, F_{p-1}).$$

Note that if (9.1) were indeed a filtration by skeleta of a cell complex then F_p/F_{p-1} would be a wedge of p-spheres corresponding to the p-cells, the differential d_1 would be the usual cellular differential and that our spectral sequence would collapse: $E_2 = E_\infty$ for dimensional reasons.

Let us now pretend that the space F_p is a smooth variety of real dimension $2p$; then the Poincaré-Lefschetz duality gives:

$$H_i(F_p, F_{p-1}) \cong H^{2p-i}(F_p \setminus F_{p-1})$$

and the E_1-term of our spectral sequence becomes

$$E^1_{pq} = H^{p-q}(F_p \setminus F_{p-1}).$$

Of course, F_p is not smooth but it is so away from F_{p-1} and this makes the above formula valid: we can blow up F_p along the divisor F_p and use the usual Poincaré-Lefschetz duality.

Note that $F_p \setminus F_{p-1}$ is a cartesian product of *open* moduli spaces; in fact our spectral sequence looks formally as a cobar-construction of the operad (called the *gravity operad*): $\mathrm{Grav}(k) := H_*(\mathscr{M}_{k+1})[-1]$ where $\mathscr{M}_{k+1}$ is the open moduli space of Riemann spheres with $k+1$ points. The cobar-differential is d_1 and the shift $[-1]$ is there because d_1 has degree -1. The operad structure on $\mathrm{Grav}(k)$ is deduced from the definition of d_1 and associativity conditions follow from $d_1^2 = 0$. It is not so easy to interpret geometrically the structure maps in the operad $\mathrm{Grav}(k)$, i.e. the maps

$$\circ_i : H_l(\mathscr{M}_k) \otimes H_m(\mathscr{M}_n) \to H_{m+l+1}(\mathscr{M}_{k+l-2}).$$

It turns out (cf. [13]) that the last map is still related with the sort of glueing of Riemann surfaces, but this time in a subtler fashion. Consider a map

$$\mathscr{M}_k \times \mathscr{M}_n \hookrightarrow \overline{\mathscr{M}}_{n+k-2}$$

which is induced by the usual glueing of marked points. Note that this map, of course, does not give an operad structure on $\mathscr{M}_*$ and if it did, this would not be one that we are after (we need a shift in grading). The required map is called the *Poincaré residue map* and could be roughly described as follows. Take a tubular neighborhood of $\mathscr{M}_k \times \mathscr{M}_n$ inside $\overline{\mathscr{M}}_{n+k-2}$. An l-dimensional cycle in $\mathscr{M}_k \times \mathscr{M}_n$ gives rise to an $l+1$-dimensional cycle in the tubular neighborhood via the Gysin map; considering it as a cycle in $\overline{\mathscr{M}}_{n+k-2} \setminus (\mathscr{M}_k \times \mathscr{M}_n)$ and thus, a cycle in $\mathscr{M}_{n+k-2}$. This is the desired structure map.

The gravity operad is actually a dg operad; it could be expressed in terms of a real version of the DM compactification of moduli spaces with marked points known as the *Kimura-Stasheff-Voronov compactificaion* [26]. It also exist in higher genera.

Finally, we describe the algebras over the operads $\overline{\mathscr{M}}$ and Grav.

Theorem 9.3.

- *An algebra over the operad* $\overline{\mathscr{M}}$ *(also called a* hypercommutative algebra *is a graded vector space* V *together with a sequence of graded symmetric products*

$$(x_1, \ldots, x_n) : V^{\otimes n} \to V$$

of degree $2(n-2)$ *satisfying the following generalized associativity condition: if* $a, b, c, x_1, \ldots, x_n \in V$ *then*

$$\sum_{S_1 \bigsqcup S_2 = \{1,\ldots,n\}} \pm((a, b, x_{S_1}, c, x_{S_2} = \sum_{S_1 \bigsqcup S_2 = \{1,\ldots,n\}} \pm(a, (b, x_{S_1}), c, x_{S_2})$$

where $S = s_1, \ldots, x_{s_k}$ *is a finite set and* x_S *stands for* $x_{s_1}, \ldots, x_{s_k}$.

- *An algebra over the operad* Grav *(also called a* gravity algebra*) is a graded vector space* V *together with a sequence of graded antisymmetric products*

$$[x_1, \ldots, x_n] : V^{\otimes n} \to V$$

of degree $n-2$ *satisfying the following generalized Jacobi relation: if* $k > 2, l \geq 0$ *and* $a_1, \ldots, a_k, b_1, \ldots, b_l \in V$ *then*

$$\sum_{1 \leq i < j \leq k} \pm[[a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots a_k, b_1, \ldots, b_l] = \begin{cases} [[a_1, \ldots, a_k], b_1, \ldots, b_l], & l > 0 \\ 0, & l = 0. \end{cases}$$

Remark 9.4.

- *The proof of the first part of the theorem is an elegant reformulation of the theorem of Keel* [34] *which states that the fundamental cycles of the closed strata* $\overline{\mathscr{M}}(T)$ *span the homology of* $\overline{\mathscr{M}}$ *together with the relations between these strata. Hypercommutative algebras with a compatible scalar product is equivalent to the notion of a* formal Frobenius manifold *(roughly, a formal manifold with a structure of a Frobenius algebra on its tangent sheaf). This structure arise in the quantum cohomology of compact Kähler manifolds, the spaces of universal unfoldings of isolated singularities and many other situations. It is easy to see that the first nontrivial product* (a, b) *is associative and graded commutative.*
- *The gravity algebras are dual to the hypercommutative algebras in the same way as the Lie algebras are dual to the commutative algebras; indeed, it is easy to see that the product* $[a, b]$ *determines the structure of a graded Lie algebra. Gravity algebras have not been studied as extensively as the dual notion, in particular it is not clear what is*

(if any) the geometric object behind them (the one 'dual' to Frobenius manifolds). It would also be quite interesting to study ∞-versions of Frobenius manifolds and gravity algebras.

9.3. **BV-algebras.** We now consider noncompact TCFT's in genus zero. The reference for this material is [14].

Definition 9.5. *Let $\mathcal{P} := \{\mathcal{P}(n)\}$ be the moduli space of Riemann spheres with $n+1$ holomorphically embedded disks. The permutation group S_n acts on $\mathcal{P}(n)$ by permuting the first n (input) disks, leaving the last one (output) fixed. The composition*

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(m+n-1)$$

taking (C_1, C_2) to $C_1 \circ_i C_2$ is obtained by cutting the ith disk out of C_1 and $m+1$th disk out of C_2 and glueing along their boundaries according to the rule $z \mapsto 1/z$.

A (closed) TCFT in genus zero is an algebra over the singular chain operad of $\mathcal{P}$. We will be considering algebras over the homology operad of $\mathcal{P}$; these turn out to be the Batalin-Vilkovisky (BV) algebras:

Definition 9.6. *A BV-algebra is a graded-commutative algebra A^* together with an operator $\Delta : A^* \to A^{*-1}$ such that $\Delta^2 = 0$ and*

$$\begin{aligned}\Delta(abc) &= \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ &\quad - (\Delta a)bc - (-1)|a|a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).\end{aligned}$$

To explain this result let us introduce another operad called the operad of *framed disks.*

Definition 9.7. *Let D denote the unit disk about the origin in $\mathbb{C}$ and $F(n)$ consist of configuration of n disks embedded into D via a composition of a translation, rotation and dilation and intersecting at most along their boundaries. We can represent an element of $F(n)$ as n little disks inside D together with one marked point on the boundary of each disc. It is understood that D has 1 as its marked point. The spaces $F(n)$ form an operad under 'inserting one configuration into the other'.*

It is easy to see that the operad F is homotopy equivalent to $\mathcal{P}$ and so it is sufficient to study algebras over F.

To proceed further, we need to introduce another operad called the *little disks operad.*

Definition 9.8. *The operad Ω is a suboperad in F consisting of little disks inside D embedded via a composition of a translation and dilation only. Pictorially that means that these disks has their marked points located on the rightmost point of their boundary.*

It is clear that $F(n) \cong \Omega(n) \times (S^1)^{\times n}$. It turns out that the algebras over $H_*(\Omega)$ are *Gerstenhaber algebras.*

Definition 9.9. *A Gertstenhaber algebra is a graded commutative and associative algebra A with a graded Lie bracket $[,]$ such that the operator $[a, ?]$ is a (graded) derivation of A for all homogeneous elements $a \in A$.*

The proof of this result follows from the simple observation that $\Omega(n)$ is equivalent to $\mathbb{C}^0_n$. The space $\Omega(2)$ is homotopically equivalent to S^1. The commutative product corresponds to the generator of $H_0(S^1)$ whereas the Lie bracket corresponds to the generator of $H^1(S^1)$. The relations between these operations are not hard to prove and to show that there are no other generators and relations one uses the presentation of the cohomology of configuration spaces described previously.

These arguments help to prove the following result:

Theorem 9.10. *An algebra over the operad F is a BV-algebra.*

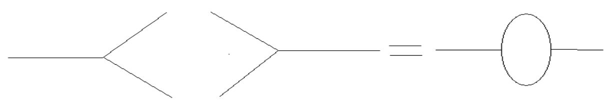
Note that $F(1)$ is homotopically equivalent to S^1; the fundamental class of S^1 corresponds to the operator Δ.

To conclude, note that there is a subtle interplay between compactified and uncompactified moduli spaces operads, particularly Barannikov and Kontsevich showed how to construct a formal Frobenius manifold out of a BV-algebra of a special sort. It seems that there is still something to understand about this construction. Related to this is the question of describing the cobar-construction of the operad $H_*(F)$, their algebras and 'homotopy BV-algebras'.

10. Modular operads and surfaces of higher genus

In this lecture we consider TCFT's of higher genus. They are best expressed as algebras over *modular operads*, the notion introduced by Getzler and Kapranov, [16] which is the main source for the material in this lecture. Another one is [4].

The notion of a modular operad is technically complicated, and it will be useful to start with some motivation. We saw that operads were based on trees and algebras over them were variations on TFT's in genus zero. This is because operads encode algebraic operations of type n to 1; iterated compositions of such result in a tree. To get more general TFT-like constructions we need operations of type 1 to n; taking compositions with operations of the first kind will result in general graphs with non-trivial topology.



So the first idea of a modular operad is to include self-glueings among its structure maps. This idea could also be implemented by considering PROP's, however the latter are much less economical constructions and even though they are easier to define but harder to work with.

The second idea is that since we really want to devise an apparatus suitable for working with Riemann surfaces there is no reason to specify separately inputs and outputs; the latter are being thought of as marked points or holes in Riemann surfaces. In other words, we want to impose an additional symmetry, mixing the inputs and the outputs; note that imposing this condition alone (without introducing self-glueings leads to the notion of a *cyclic operad*, cf. [15]).

The third idea stems from the observation that there are infinitely many graphs with a fixed number of external edges while only finitely many trees with a fixed number of leaves. To overcome this unpleasant circumstance note that if we consider only (connected) graphs whose internal vertices have valences three or higher then for a fixed number of external edges and *first Betti number* there is only finitely many such graphs. This suggests to introduce an additional grading by genus (=1st Betti number) to ensure some kind of finite dimensionality *and* disallow certain kind of graphs, called *unstable*. Somewhat imprecisely, one can say that these unstable graphs correspond to unstable surfaces, i.e. those with an infinite automorphism group; these are spheres with less than 3 marked points and tori with no marked points. In particular, modular operads will not have a unit. The last step is not obligatory but useful, however in some situations it is sensible to re-introduce a unit or consider other types of prohibited graphs.

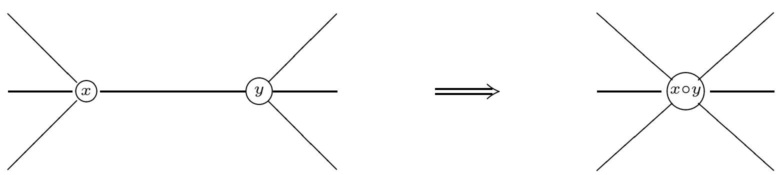
One encounters one additional difficulty when working with moduli spaces of Riemann surfaces of higher genus: they have non-trivial automorphism groups, although the stability condition ensures their finiteness. The outcome is that the resulting moduli spaces are *orbifolds* and *orbi-simplicial* spaces rather than smooth manifolds or simplicial complexes. We will not elaborate on these notions and pretend that this difficulty is not present. This is possible, roughly speaking, because we are working over a field of characteristic zero – $\mathbb{C}$, to be precise – and is related to the fact that the cohomology of finite groups is zero in characteristic zero. On the combinatorial side the automorphism groups are also present (as automorphism groups of graphs); note that trees have no non-trivial automorphisms.

10.1. **Definitions.** We are now coming to precise definitions. Recall that for a Σ-module E we previously defined a free (non-unital) operad ΦE; the functor $E \mapsto \Phi E$ is a *triple*; i.e. there is a natural morphism $\eta : E \to \Phi E$ and $\mu : \Phi^2 E \to \Phi E$ satisfying the natural associativity and two-sided unit axioms, cf. [31]. The map η is the inclusion of E into ΦE as he space of decorated corollas; for the map μ we take an element of $\Phi^2 E$ which is a decorated tree of trees and consider it as a big tree – every element of the original tree is replaced by another tree (its decoration). An operad can then be defined as an algebra over the triple Φ i.e. a Σ-module E together with a map $m : \Phi E \to E$ such that the following diagrams commute:

$$\begin{array}{ccc} \Phi^2 E & \xrightarrow{\mu} & \Phi E \\ \downarrow{\scriptstyle \Phi m} & & \downarrow{\scriptstyle m} \\ \Phi E & \xrightarrow{m} & E \end{array} \qquad\qquad \begin{array}{ccc} E & \xrightarrow{\eta} & \Phi E \\ & \searrow^{=} & \downarrow{\scriptstyle m} \\ & & E \end{array}$$

The map m can be pictorially represented as

(10.1)



It is technically convenient to give a definition of a modular operad in a similar fashion, as an algebra over a certain triple. The category of Σ-modules gets replaced with the category of *stable Σ-modules*:

Definition 10.1. *An ordered pair (g, n) of nonnegative integers is* stable *if $2(g-1)+n > 0$. The only pairs which are* unstable*, i.e. not stable, are $(0,0)$, $(0,1)$, $(0,2)$ and $(1,0)$. A* cyclic *$\mathbb{S}$-module is a collection of dg vector spaces $\{\mathcal{V}((n)) \mid n \geq 0\}$ with an action of S_n on $V((n))$. A* stable *$\mathbb{S}$-module is a collection of dg vector spaces $\{\mathcal{V}((g,n)) \mid g,n \geq 0\}$ with an action of S_n on each $\mathcal{V}((g,n))$, such that $\mathcal{V}((g,n)) = 0$ for unstable (g,n). Furthermore, if $\mathcal{V}$ is a stable $\mathbb{S}$-module and I is a finite set then we set*

$$\mathcal{V}((g,I)) := \Big[\bigoplus \mathcal{V}((g,n))\Big]_{S_n}$$

where the direct sum is extended over all bijections $\{1,2,\dots,n\} \to I$. We also set $\mathcal{V}((n)) := \bigoplus_{g \geq 0} \mathcal{V}((g,n))$ and $\mathcal{V}((I)) := \bigoplus_{g\geq 0} \mathcal{V}((g,I))$.

A *graph* is a one-dimensional cell complex; we will only consider connected graphs. It is worthwhile to give a more formal definition of a graph.

Definition 10.2. *A graph consists of the following data:*

(1) *A finite set, also denoted by G, consisting of the* half-edges *of G.*

(2) *A partition* Vert(G) *of* Flag(G). *The elements of* Vert(G) *are called the* vertices *of* G. *The half-edges belonging to the same vertex* v *will be denoted by* Flag(v)*; the cardinality of* Flag(v) *will be called the* valence *of* v *and denoted by* $n(v)$.
(3) *An involution* σ *acting on* Flag(G). *The* edges *of* G *are pairs of half-edges of* G *forming a two-cycle of* σ*; the set of edges will be denoted by* Edge(G). *The* legs *of* G *are fixed points of* σ*; the set of legs will be denoted by* Leg(G).

A *stable* graph is a graph G having each vertex $v \in \mathrm{Vert}(G)$ decorated by a non-negative integer $g(v)$, the *genus* of v; it is required that $(g(v), n(v))$ is stable for each vertex v of G. The *genus* $g(G)$ of stable graph G is defined by the formula

$$g(G) = \dim(H_1(G)) + \sum_{v \in \mathrm{Vert}(G)} g(v). \tag{10.2}$$

their sets of half-edges preserving the involutions σ and Contracting an edge e in a stable graph G yields a new stable graph G_e; the decorations $g(v)$ of the vertices of G_e are defined in the obvious way, so that $g(G_e) = g(G)$. The set of stable graphs forms a category whose morphisms are generated by isomorphisms and edge-contractions $G \to G_e$.

Definition 10.3. *Let* $\Gamma((g, n))$ *be the category whose objects are stable graphs* G *of genus* g *with* n *labeled legs (i.e., equipped with a bijection between* $\{1, \ldots, n\}$ *and* Leg(G)*). Note that* $\Gamma((g, n))$ *is empty whenever* (g, n) *is unstable.*

For a stable graph G and a stable $\mathbb{S}$-module $\mathcal{V}$ we define the dg vector space

$$\mathcal{V}((G)) := \bigotimes_{v \in Vert(G)} \mathcal{V}((g(v), \mathrm{Flag}(v))$$

of $\mathcal{V}$-decorations on G. We are now ready to define the functor 'free modular operad'.

Definition 10.4. *Let the* $\mathbb{M}$ *be the endofunctor on the category of stable* $\mathbb{S}$*-modules by the formula*

$$\mathbb{M}\mathcal{V}((g, n)) = \operatorname*{colim}_{\mathrm{Iso}\,\Gamma((g,n))} \mathcal{V}((G))$$

where $\mathrm{Iso}\,\Gamma((g, n))$ *is the subcategory of isomorphisms in* $\Gamma((g, n))$. *The functor* $\mathbb{M}$ *is a triple; a* modular operad *is an algebra over this triple.*

Remark 10.5. *Note that there is an isomorphism*

$$\mathbb{M}\mathcal{V}((g, n)) = \bigoplus_{[\Gamma((g,n))]} \mathcal{V}((G))_{\mathrm{Aut}(G)};$$

here the summation is extended over the isomorphism classes of stable graphs in $\Gamma((g, n))$ *and the subscript* Aut(G) *means the space of coinvariants with respect to the group of automorphism of the graph* G. *It is clear that* $\mathbb{M}$ *is a direct analogue of the functor* Φ*: it associates to a stable* $\mathbb{S}$*-module* $\mathcal{V}$ *the space of all* $\mathcal{V}$*-decorated stable graphs. The structure of a triple on* $\mathbb{M}$ *is represented by the same picture (10.1) as in the operad case.*

Remark 10.6. *Note that the structure of a modular operad on a stable* $\mathbb{S}$*-module* $\mathcal{V}$ *is determined by a collection of 'operadic' glueings*

$$\circ_i : \mathcal{V}((g, n)) \otimes \mathcal{V}((g', n')) \to \mathcal{V}((g + g', n + n' - 2)$$

plus a collection of 'self-glueings' for any $i, j \leq n, i \neq j$*:*

$$\xi_{ij} : \mathcal{V}((g, n)) \to \mathcal{V}((g + 1, n - 2)).$$

The maps ξ_{ij} *should be visualized as the closing up of the* i*th and* j*th legs of a stable graph, clearly this operation entails the increasing of the genus.*

10.2. **Algebras over modular operads; examples.** Moduli spaces of Riemann surfaces in higher genera furnish examples of topological modular operads (we have not defined these but the definition carries over almost verbatim from the linear situation; in effect we can define a modular operad in any symmetric monoidal category).

Example 10.7.
- *The Deligne-Mumford modular operad forms perhaps the most natural example of a (topological) modular operad:* $\overline{\mathscr{M}}((g,n)) := \overline{\mathscr{M}}_{g,n}$*. The structure maps are simply the glueing at marked points.*
- *The closed CFT operad:* $CCFT((g,n))$ *is defined as the moduli space of Riemann surfaces of genus* g *with* n *holes with parametrized boundaries; the glueings or self-glueings are along the boundaries.*
- *The open CFT operad:* $OCFT((g,n))$ *is defined as the moduli space of Riemann surfaces of genus* g *with* n *parametrized intervals embedded into their boundary components; the structure maps are the glueings along these intervals.*
- *The open-closed CFT operad* $OCCFT((g,n))$ *is defined as the moduli space of Riemann surfaces of genus* g *with* n_1 *holes (closed boundaries and* n_2 *parametrized intervals (open boundaries) embedded into their other boundary components with* $n = n_1 + n_2$*; the structure maps are the glueings along both closed and open boundaries.*

Applying the singular complex functor to these operads one obtains modular operads in dg vector spaces.

Let V be a chain complex with symmetric inner product (x,y). The *endomorphism operad* $\mathcal{E}(V)$ of V is defined as $\mathcal{E}(V)((g,n)) := V^{\otimes n}$. Note that the inner product gives an isomorphism $V^{\otimes n} \cong \mathrm{Hom}(V^{\otimes n-1}, V$; the operadic compositions are thus defined as in the usual endomorphism operad. The self-glueing maps $\xi_{ij} : V^{\otimes n} \to V^{\otimes n-2}$ are defined via contracting the i and jth tensor factors with the help of the inner product.

Definition 10.8. *A structure of an algebra over a modular operad* $\mathcal{O}$ *on a dg vector space* V *is a map of modular operads* $\mathcal{O} \to \mathcal{E}(V)$

Algebras over the homology of the operad $\overline{\mathscr{M}}$ are called *cohomological field theories*. Algebras over singular (or cellular) chain complexes of the modular operads $CCSFT, OTFT, OCSFT$ are called *closed TCFT*, *open TCFT* and *open-closed TCFT* respectively. The algebras over H_0 of these operads are called *TFT* (with an appropriate adjective). It is clear that these definitions agree with the notions discussed in Lectures 4 and 5.

10.3. **Cyclic operads and their modular closures.** Recall that an operad was defined as a collection of S_n-modules $\mathcal{O}(n)$together with maps $\circ_i : \mathcal{O}(k)\otimes \to \mathcal{O}(k+l-1)$ which are S_n-equivariant.

Definition 10.9. *Suppose that for an operad* $\mathcal{O}$ *the* S_n*-action on* $\mathcal{O}(n)nis$ *extended to an action of* S_{n+1}*. For* $a \in \mathcal{O}(n)$ *we denote the additional action of the cycle* $(1, 2, ldots, n+1$ *by* a^**. Then the operad* $\mathcal{O}$ *is called* cyclic *if for any* $a \in \mathcal{O}(m), b \in \mathcal{O}(n)$ *we have*

$$(a \circ_m b)^* = b^* \circ_1 a^*.$$

Remark 10.10. *Informally speaking, the additional action of the cyclic group is thought of as permuting the inputs with the output of* $\mathcal{O}$*. the graphs governing cyclic operads are still trees but without explicit separating inputs from the output. In other words, these are simply contractible graphs.*

Example 10.11.
- *The commutative operad* $\mathcal{C}om(n) = \mathbb{C}$ *has a trivial cyclic structure.*
- *Let us represent the basis elements of the associative operad* $\mathcal{A}ss(n)$ *as* $n+1$*-corollas (graphs with one vertex and* $n+1$ *legs) whose legs are cyclically ordered. This makes clear that* $\mathcal{A}ss$ *is a cyclic operad.*

- *The Lie operad $\mathcal{L}ie$ has the property that $\mathcal{L}ie(n)$ is isomorphic to the subspace in the free Lie algebra on n generators spanned by all Lie monomials containing each generator precisely once. This gives $\mathcal{L}ie$ a cyclic structure.*

If $\mathcal{O}$ is a cyclic operad we will denote its component $\mathcal{O}(n)$ as $\mathcal{O}((n=1))$; this stresses the equal rights of the inputs and the output.

The notion of a cyclic operad can, of course, be defined as an algebra over a suitable triple. Going through the definitions we obtain the following result.

Proposition 10.12. *Let $\mathcal{O}$ be a modular operad. Then its genus zero part $\mathcal{O}((0,n))$ is a cyclic operad.*

Definition 10.13. *Let $\mathcal{O}$ be a cyclic operad. Its* modular closure $\overline{\mathcal{O}}$ *is the left adjoint functor to the functor associating to $\mathcal{O}$ its genus zero part. The* naive closure *of $\mathcal{O}$ is the right adjoint to the genus zero functor.*

The existence of left and right adjoint functors is easy. Informally, $\overline{\mathcal{O}}$ is obtained from $\mathcal{O}$ by adjoining freely the self-glueing maps whereas $\underline{\mathcal{O}}$ is obtained by setting all these self-glueings to zero.

Example 10.14. *Consider the cyclic operad $\mathcal{C}om$ whose algebras are simply commutative algebras: $\mathcal{C}om((n)) = \mathbb{C}$. Then $\overline{\mathcal{C}om}((g,n)) = \mathbb{C}$ and the operad $\mathcal{C}om$ is just the closed TFT operad whose algebras are commutative Frobenius algebras. The operad $\underline{\mathcal{C}om}$ has $\mathcal{C}om((g,n)) = \begin{cases}\mathbb{C}, g=0\\ 0, g\neq 0\end{cases}$. The algebras over this modular operad are Frobenius algebras of a special nature: they vanish on surfaces whose genus is greater than zero. Let A be such an algebra and $\sum_i a_i \otimes b_i$ be the inverse to its scalar product. The condition could then be written as $\sum_i a_i b_i = 0$.*

For operads $\mathcal{A}ss$ and Lie the situation is similar. For example, an algebra over $\overline{\mathcal{A}ss}$ is a non-commutative Frobenius algebra.

The following result describes algebras over the modular operad $\overline{\mathsf{C}\mathcal{A}ss[1]}$; recall that algebras over $\mathsf{C}\mathcal{A}ss[1]$ are A_∞-algebras.

Proposition 10.15. *A vector space V with an inner product $\langle , \rangle$ has the structure of an algebra over the modular operad $\overline{\mathsf{C}\mathcal{A}ss[1]}$ if and only if V has the structure of an A_∞-algebra specified by a collection of maps $m_i : V^{\otimes i} \to V$ with $i \geq 2$ such that*

$$\langle m_n(x_1,\ldots,x_n), x_0\rangle = (-1)^{n+|x_0|(|x_1|+\ldots+|x_n|)}\langle m_n(x_0,\ldots,x_{n-1}), x_n\rangle$$

The proof of this result is a simple check, note that the condition on the structure maps are a kind of ∞-version of the Frobenius condition. Algebras over $\overline{\mathsf{C}\mathcal{A}ss[1]}$ are called *symplectic* or *cyclic* A_∞-algebras. The significance of the operad $\overline{\mathsf{C}\mathcal{A}ss[1]}$ stems from the following result, cf. [6]

Theorem 10.16. *The operad $\overline{\mathsf{C}\mathcal{A}ss[1]}$ is quasi-isomorphic to the operad $OTCFT$.*

In other words, the cohomology of the operad $\overline{\mathsf{C}\mathcal{A}ss[1]}$ is the cohomology of the corresponding moduli space. Of particular importance is the complex $\overline{\mathsf{C}\mathcal{A}ss[1]}(g,0)$ of 'vacuum' graphs (i.e. graphs without legs). Since the endomorphism modular operad $\mathcal{E}(V)$ of a dg space V with an inner product has $\mathcal{E}(V)(g,0) = \mathbb{C}$ a symplectic A_∞-algebra gives rise to an inhomogeneous cocycle on $\mathscr{M}_{g,n}$ for all g and n. It is known that so-called *tautological classes* could be obtained in this way.

11. Graph complexes and Gelfand-Fuks type cohomology; characteristic classes of ∞-algebras

In this lecture we introduce graph-complexes and describe Kontsevich's theorem [28, 29] expressing their homology through the homology of certain infinite-dimensional Lie algebras of vector fields. This type of cohomology is often named after Gelfand and Fuks who made extensive computations of them in the 1970's.

Kontsevich's original papers, while sketchy, are still highly recommended to get a quick introduction into this circle of ideas. There are several detailed expositions, e.g. [18, 21].

The most general formulation of a graph complex is as a Feynman transform of a modular operad, cf. [16]. The Feynman transform is the modular analogue of the cobar-construction. Unfortunately, in its general formulation it is unavoidable to use *twisted* modular operads; a rather technical notion which we opted to omit. We have seen the shadow of it when discussing cobar-constructions, e.g. the cobar-construction of $\mathcal{C}om$ is quasi-isomorphic *not* to the operad $\mathcal{L}ie$ as one should like but to its shift. This issue becomes rather more serious in the modular context.

It needs to be mentioned that the most general formulation of Kontsevich's theorem expressing the cohomology of a Feynman transform through certain Lie algebra cohomology has not yet been written (but see, however, [19] where the corresponding result was proved for the modular closure of the operad $\mathcal{A}ss$.)

11.1. **Graph complexes.** As mentioned above there is a notion of a graph complex corresponding to any modular operad but we restrict ourselves to giving precise details only for the *ribbon graph* case. The corresponding modular operad here is $\underline{\mathcal{A}ss}$.

Definition 11.1. *A ribbon graph is a graph* Γ *such that the set of half-edges around each vertex* $v \in \mathrm{Vert}(\Gamma)$ *has a cyclic order. We consider only graphs without legs (vacuum graphs) such that each vertex has valence* ≥ 3.

An orientation *on a ribbon graph is an ordering of its vertices and orienting its edges. Switching the ordering of two vertices or the orientation of one edge reverses the orientation; thus there are precisely two orientations on a given graph, opposite to each other.*

Two oriented ribbon graphs are called isomorphic *if there is there is an map between them which is an isomorphism of abstract graphs and which preserves the ribbon structure and the orientations. The group of automorphisms of an oriented ribbon graph* Γ *will be denoted by* $\mathrm{Aut}(\Gamma)$.

Remark 11.2. *Note that a ribbon graph is nothing but an* $\underline{\mathcal{A}ss}$*-decorated graph; indeed we noted in Lecture 9 that* $\mathcal{A}ss(n)$ *is naturally identified with the set of cyclic (i.e. considered up to a cyclic permutation) orderings on the set of* $n+1$ *elements.*

The cyclic ordering around each vertex of a ribbon graph allows one to embed its neighborhood into a plane and fatten up the half-edges into ribbons. Glueing the obtained fat corollas in an orientation-preserving fashion we get a 2-dimensional surface with boundary together with the embedding of the original graph into it. This explains the name 'ribbon graph'.

Given a ribbon graph Γ and its edge e denote by Γ/e the graph obtained from Γ by collapsing the edge e to a vertex. Note that the Γ/e clearly has an induced ribbon structure. Furthermore, if Γ was oriented, then Γ/e has an induced orientation. The only problem is how to order the vertices of Γ/e. If e was an oriented edge pointing from the first vertex of Γ to the second vertex, then the new vertex obtained from coalescing the endpoints of e is set to be the first in the ordering of $\mathrm{Vert}(\Gamma/e)$. The general case could always be reduced to this one by renumbering vertices of Γ.

Definition 11.3. *There is a complex, denoted by* $\mathcal{G}_\bullet$*, called the graph complex. The underlying space of* $\mathcal{G}_\bullet$ *is the vector space spanned by isomorphism classes of graphs modulo the relation*

$$\Gamma^\dagger = -\Gamma,$$

where $\Gamma^\dagger$ *denotes the graph* Γ *with the* opposite *orientation.*

The differential ∂ *on* $\mathcal{G}_\bullet$ *is given by the formula:*

$$\partial(\Gamma) := \sum_{e \in E(\Gamma)} \Gamma/e$$

where Γ is any graph and the sum is taken over all edges of Γ. Graph homology is defined to be the homology of this complex and is denoted by $H\mathcal{G}_\bullet$. Graph cohomology, *denoted by $H\mathcal{G}^\bullet$, is defined by taking the $\mathbb{C}$-linear dual of the above definition.*

Remark 11.4. *The graph complex splits up as a direct sum of subcomplexes which are indexed by the Euler characteristic of the corresponding graph,*

$$\mathcal{G}_\bullet = \bigoplus_{\chi=-1}^{-\infty} \mathcal{G}_\bullet(\chi) := \bigoplus_{\chi=-1}^{-\infty} \left[\bigoplus_{i-j=\chi} \mathcal{G}_{ij} \right].$$

Moreover, since all the vertices of any graph are at least trivalent, each of these subcomplexes has finite dimension.

The following result follows directly from definitions.

Proposition 11.5. *There is an isomorphism of complexes*

$$H\mathcal{G}^\bullet \cong \bigoplus_{g=0}^{\infty} \overline{\mathsf{C}\mathcal{A}ss}((g,0)). \tag{11.1}$$

The additional grading of the right-hand side of (11.1) by genus corresponds to the grading of the left-hand side by the Euler characteristic χ; the comparison is given by the formula $\chi = 1 - g$ (note that the genus g is understood in the sense of modular operad, i.e. as the first Betti number of the ribbon graph, not as the genus of the corresponding surface).

Recall that the complex $\overline{\mathsf{C}\mathcal{A}ss}((g,0))$ computes the homology of the complex $\bigoplus_{g=0}^{\infty} OTCFT((g,0))$ and the latter is, essentially the direct sum of the homologies of the spaces $\mathscr{M}_{g,0}$. After sorting out the grading we get the following corollary.

Corollary 11.6. *Let $H^*(\chi)$ denote the cohomology of the ribbon graph complex $\mathcal{G}^*(\chi)$, There is an isomorphism*

$$H^i(\chi) \cong \bigoplus_n H_{i+\chi}(\mathscr{M}_{g,n}),$$

where $\mathscr{M}_{g,n}$ is the moduli space of Riemann surfaces of genus g with n marked points; the genus g is related to χ by the usual formula $\chi = 2 - 2g + n$; moreover the summation is over such n for which χ is negative.

11.2. **Lie algebra homology.** In this section we briefly recall the definition of Lie algebra homology and *relative* Lie algebra homology in the $\mathbb{Z}/2\mathbb{Z}$-graded setting. A reference for this material is [12].

Definition 11.7. *Let $\mathfrak{l}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded (or super-)Lie algebra: the underlying space of the Chevalley-Eilenberg complex of $\mathfrak{l}$ is the exterior algebra $\Lambda_\bullet(\mathfrak{l})$ which is defined to be the quotient of $T(\mathfrak{l})$ by the ideal generated by the relation*

$$g \otimes h = -(-1)^{|g||h|} h \otimes g; \quad g, h \in \mathfrak{l}.$$

There is a natural grading on $\Lambda_\bullet(\mathfrak{l})$ where an element $g \in \mathfrak{l}$ has bidegree $(1, |g|)$ in $\Lambda_{\bullet\bullet}(\mathfrak{l})$ and total degree $|g| + 1$; this implicitly defines a bigrading and total grading on the whole of $\Lambda_\bullet(\mathfrak{l})$.

The differential $d : \Lambda_i(\mathfrak{l}) \to \Lambda_{i-1}(\mathfrak{l})$ is defined by the following formula:

$$d(g_1 \wedge \ldots \wedge g_m) := \sum_{1 \le i < j \le m} (-1)^{p(g)} [g_i, g_j] \wedge g_1 \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge \hat{g}_j \wedge \ldots \wedge g_m,$$

for $g_1, \ldots, g_m \in \mathfrak{l}$; where

$$p(g) := |g_i|(|g_1| + \ldots + |g_{i-1}|) + |g_j|(|g_1| + \ldots + |g_{j-1}|) + |g_i||g_j| + i + j - 1.$$

The differential d has bidegree $(-1, 0)$ in the above bigrading. We will denote the Chevalley-Eilenberg complex of $\mathfrak{l}$ by $C_\bullet(\mathfrak{l})$; the homology of $C_\bullet(\mathfrak{l})$ will be called the Lie algebra homology of $\mathfrak{l}$ and will be denoted by $H_\bullet(\mathfrak{l})$.

Remark 11.8. *Let $\mathfrak{l}$ be a Lie algebra; $\mathfrak{l}$ acts on $\Lambda(\mathfrak{l})$ via the adjoint action. This action commutes with the Chevalley-Eilenberg differential d; in fact it is nullhomotopic..*

Let $\mathfrak{h}$ be a Lie ideal of $\mathfrak{l}$ (that means that $[\mathfrak{l}, \mathfrak{h}] \subset \mathfrak{h}$). Then $\mathfrak{l}/\mathfrak{h}$ is itself a Lie algebra and the space $C_\bullet(\mathfrak{l}/\mathfrak{h})_\mathfrak{h}$ of $\mathfrak{h}$-coinvariants of the Chevalley-Eilenberg complex of $\mathfrak{l}/\mathfrak{h}$ forms a complex when equipped with the Chevalley-Eilenberg differential d. This is the *relative* Chevalley-Eilenberg complex of $\mathfrak{l}$ modulo $\mathfrak{h}$ (cf. [12]) and is denoted by $C_\bullet(\mathfrak{l}, \mathfrak{h})$. The homology of this complex is called the *relative* homology of $\mathfrak{l}$ modulo $\mathfrak{h}$ and is denoted by $H_\bullet(\mathfrak{l}, \mathfrak{h})$.

11.3. **Kontsevich's theorem.** As was mentioned above Kontsevich's theorem expresses graph homology as the homology of a certain Lie algebra. There are several flavors of graph homology depending on the kind of cyclic (or modular) operad on which it is based. The version described in this lecture is based on the associative operad the reason being is that it is closely related to the moduli of Riemann surfaces. However the formulation of Kontsevich's theorem is simpler for a *commutative* graph complex (the general formulation requires a fair bit of non-commutative geometry). Therefore we will stick to the commutative graph complex; its definition is very similar to that of a ribbon graph complex except the graphs under considerations are not required to possess ribbon structure. Additionally, it will be convenient for us to consider *not necessarily connected* graphs. We will denote the corresponding complex of (possible disconnected) graphs by $\tilde{G}_\bullet$. In the ribbon graph context the homology of the complex of disconnected ribbon graphs correspond to the homology of the moduli of possibly disconnected Riemann surfaces.

Let us consider the algebra of polynomial function on the space $\mathbb{C}^{2n}$ whose degree is at least two. This algebra possesses a Poisson structure defined for two functions f, g by the formula

$$\{f, g\} = \sum_{i=1}^{n} \partial_{p_i} f \partial_{q_i} g - \partial_{q_i} f \partial_{p_i} g,$$

where p_i, q_i are standard flat coordinates on $\mathbb{C}$. We denote by $\mathfrak{g}_{2n}$ the Lie algebra of polynomial functions on $\mathbb{C}$ with respect to this Poisson bracket. The quadratic polynomials form a Lie subalgebra in $\mathfrak{g}_{2n}$ isomorphic to the symplectic Lie algebra $sp_{2n}(C)$. It is clear that there is a system of inclusions $\mathfrak{g}_{2n} \to \mathfrak{g}_{2n+2} \to \ldots$ and the corresponding inclusions of the subalgebras $sp_{2n}(C) \to sp_{2n+2}(C) \to \ldots$. We denote the direct limits of these systems by $\mathfrak{g}_\infty$ and sp_∞ respectively; thus sp_∞ is a Lie subalgebra in $\mathfrak{g}_\infty$.

Theorem 11.9. *There is an isomorphism of complexes*

$$C_\bullet(\mathfrak{g}_\infty, sp_\infty) \cong \tilde{\mathcal{G}}_{\bullet+1}.$$

The proof of this theorem is remarkably simple and based on classical invariant theory. A good exposition of the latter is contained in e.g. [30]. According to the Fundamental Theorem of Invariants theory the space of (co)invariants of the group $sp(V)$ in the tensor module $V^{\otimes n}$ has a basis consisting of *chord diagrams* on n vertices as long as the dimension of V is sufficiently big (not less than n to be precise). A chord diagram is a partition of the set of n elements into pairs (chords). Note that this implies, in particular, that this space of (co)invariants is non-empty if and only if n is even. Take the sp-coinvariants of the complex $C_\bullet(\mathfrak{g}_\infty)/sp_\infty$. The latter complex is a quotient by the actions of various symmetric groups of the direct sum of

$$V^{*\otimes n_1} \otimes V^{*\otimes n_2} \otimes \ldots V^{*\otimes n_k}.$$

Here V is the infinite-dimensional symplectic space $\mathbb{C}^{2\infty}$, n_i corresponds to the degree of a polynomial function on V and k corresponds to the homological degree. The space of sp-coinvariants in this space is a bunch of chord diagrams; we will organize them into a graph with k vertices and whose vertex number i has i half-edges issuing from it. The actions of symmetric groups lead to considering oriented graphs and the Chevalley-Eilenberg differential corresponds to edge-contractions. This concludes our sketch proof of Kontsevich's theorem.

Note that there is a striking resemblance of this construction to the proof of the Feynman diagram expansion of integrals discussed in Lecture 1; this resemblance could be made even more obvious by considering the Chevalley-Eilenberg *cohomological* complex; there is a *pairing* between it and the graph complex given by taking Feynman amplitudes. This approach is described in [21].

11.4. **Characteristic classes of ∞-algebras.** In this section we switch from $\mathbb{Z}$-graded to $\mathbb{Z}/\mathbb{Z}$-graded context.

We aim to describe how ∞-algebras give rise to classes in (appropriate versions of) graph complexes. To understand the material in this section a certain amount of familiarity with supergeometry will be useful. There are two equivalent ways to define these classes. For example, a symplectic A_∞-algebra introduced in Lecture 9 give rise to a homology class in the ribbon graph complex because the latter is isomorphic to $\overline{\mathsf{C}\mathcal{A}ss}((*,0))$ and a symplectic A_∞-algebra is an algebra over the modular operad $\overline{\mathsf{C}\mathcal{A}ss}$. On the other hand, there is another, equivalent way to describe the same class using Kontsevich's theorem. This alternative, geometric, description is complementary to the operadic, combinatorial one; it is more convenient to use in some situations. For example the proof of the invariance of the characteristic class under ∞-equivalences is much simpler with its second description.

We will describe this construction in the L_∞-case because the relevant version of graph complex is the complex of commutative graphs for which we formulated Kontsevich's theorem. The general formulation of the characteristic class construction involves symplectic non-commutative geometry and will not be discussed here.

First of all, we mention that Kontsevich's theorem generalizes to the $\mathbb{Z}/2\mathbb{Z}$-graded case. Let us consider the $\mathbb{Z}/2\mathbb{Z}$-graded space $V = \mathbb{C}_{2n|m}$; the space of (polynomial) superfunctions on it is by definition the symmetric algebra on V^* which is by definition $S(\mathbb{C}^{2n^*}) \otimes \Lambda(\mathbb{C}^m)$. Let us choose the even coordinates p_i, q_i and the odd coordinates x_i in V. Note that ∂_{x_i} make perfect sense as odd derivations of our algebra of superfunctions. Now for two homogeneous superfunctions $f, g \in S(V^*)$ we can define their Poisson bracket:

$$\{f,g\} = \sum_{i=1}^{n}(\partial_{p_i} f \partial_{q_i} g - \partial q_i f \partial_{p_i} g) + \sum_{i=1}^{m}(-1)^{|f|}\partial_{x_i} f \partial_{x_i} g.$$

The space SV^* becomes a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra; we denote it by $\mathfrak{g}_{2n|m}$. Just as before we denote by $\mathfrak{g}_{\infty|\infty}$ the direct limit of these Lie algebras as n, m tend to ∞; clearly $\mathfrak{g}_\infty$ is a Lie subalgebra in $\mathfrak{g}_{\infty|\infty}$. We can define similarly the symplectic Lie (super)algebra $sp_{2n|m}$ and its infinite version $sp_{\infty|\infty}$

Remark 11.10. *We can also consider* formal *superfunctions by replacing polynomials in even an odd variables by formal power series. Formal superfunctions also form a Poisson Lie algebra and by abuse of notation we will still denote it by the same symbol $\mathfrak{g}_{2n|m}$ and by $\mathfrak{g}_{\infty|\infty}$ in the infinite-dimensional situation. We need to use formal superfunctions to treat symplectic L_∞-algebras; it is easy to show that the Chevalley-Eilenberg cohomology is the same in the formal and polynomial setting.*

We have the following result:

Theorem 11.11. *The inclusion $\mathfrak{g}_\infty \subset \mathfrak{g}_{\infty|\infty}$ induces an isomorphism on the homology of these Lie algebras.*

The reason this theorem holds is that the invariant theory works in the same way in the ungraded case as in the $\mathbb{Z}/2\mathbb{Z}$-graded case.

We will now define the notion of an L_∞ and symplectic (a.k.a. cyclic) L_∞-algebra.

Definition 11.12.

- *An L_∞-algebra is an algebra over the operad $\mathsf{C}\mathcal{C}om[1]$.*
- *A symplectic L_∞-algebra is an algebra over the modular operad $\overline{\mathsf{C}\mathcal{C}om[1]}$.*

Remark 11.13. *The notion of a symplectic L_∞-algebra is a homotopy invariant analogue of Lie algebras with an invariant scalar product. As in the A_∞-case a symplectic L_∞-structure on a dg vector space V is determined by a collection of multilinear maps $m_i : V^{\otimes i} \to V$ for*

$i \geq 2$; these will be graded symmetric and cyclically invariant in an appropriate sense. If V has vanishing differential then m_2 is a (graded) Lie bracket.

We have the following alternative definition of the notion of an L_∞-algebra:

Definition 11.14. *An L_∞-structure on a vector space $\mathbb{C}^{2n|m}$ is a formal odd superfunction $m \in \mathfrak{g}_{2n|m}$: $f = f_3 + f_4 + \ldots$ where m_i is a polynomial of degree i such that $\{m, m\} = 0$.*

Remark 11.15. *Note that the homogeneous polynomial f_i is obtained from the product $m_i : (\mathbb{C}^{2n|m})^{i-1} \to \mathbb{C}^{2n|m}$ by 'lowering an index' using the standard linear (super)symplectic form on $\mathbb{C}^{2n|m}$.*

We can now give the alternative construction of the characteristic class. Let m be a superfunction on $\mathbb{C}^{2n|m}$ corresponding to a symplectic L_∞-algebra. It is an odd element in $\mathfrak{g}_{2n|m}$ and we will regard it as an element in $\mathfrak{g}_{\infty|\infty}$ via the embedding $\mathfrak{g}_{2n|m} \subset \mathfrak{g}_{\infty|\infty}$. Consider the Chevalley-Eilenberg complex of $\mathfrak{g}_{\infty|\infty}$:

$$\mathbb{C} \leftarrow \mathfrak{g}_{\infty|\infty} \leftarrow \ldots \leftarrow \Lambda^l \mathfrak{g}_{\infty|\infty} \leftarrow \ldots$$

and the inhomogeneous chain

$$e^m = 1 + m + \frac{1}{2!} m \wedge m + \ldots + \frac{1}{m!} l^\wedge + \ldots.$$

We have the following result:

Proposition 11.16. *The element e^m is a cycle in $C_\bullet(\mathfrak{g}_{\infty|\infty}, sp_{\infty|\infty})$.*

One can show that the graph homology class in $\mathcal{G}_\bullet$ corresponding to the Chevalley-Eilenberg class $[e^m]$ essentially coincides with the class obtained by considering the given symplectic L_∞-algebra as an algebra over the modular operad $\overline{\mathbb{C}\mathcal{C}om}$ (more precisely, the latter is the *logarithm* of the former). Also, this class is invariant under a suitable notion of equivalence of L_∞-algebras called *L_∞-equivalence*. We will not make this relationship precise here.

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