

Series involving $\zeta(n)$

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These notes reproduce my article *Math. Gazette* **98** (2014), 58–66, with some extra material (Theorem 3 and its applications).

Recall that for integers $n \geq 2$, $\zeta(n)$ is defined by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots.$$

Of course, $\zeta(1)$ is not defined, since $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. A well-known particular value is $\zeta(2) = \pi^2/6$: numerous alternative proofs of this fact have been presented in the *Gazette*, e.g. the recent notes [1], [2].

Here we will consider infinite series of the form $\sum_{n=2}^{\infty} a_n \zeta(n)$ or $\sum_{n=2}^{\infty} a_n [\zeta(n) - 1]$. There are cases where the second type is convergent, while the first one is divergent. In fact, this is already seen in the case $a_n = 1$: clearly, $\zeta(n) > 1$ for all n , so $\sum_{n=2}^{\infty} \zeta(n)$ is divergent. However,

$$\zeta(n) - 1 = \frac{1}{2^n} + \frac{1}{3^n} + \cdots,$$

and from this it is quite easy to show (for example by integral estimation) that $\zeta(n) - 1 < \frac{1}{2^{n-1}}$ for all $n \geq 3$, so that $\sum_{n=2}^{\infty} [\zeta(n) - 1]$ is convergent. We will not bother with the details here, because this fact will emerge with no extra effort from the reasoning below.

Our investigation will take us on a round trip of several of the most basic infinite series (consequently, this topic can serve as a rather good revision course on series for students).

By definition,

$$\sum_{n=2}^{\infty} a_n \zeta(n) = \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{a_n}{k^n}.$$

To make further progress, the next step is inevitably to reverse the order of summation in this double series. Since everything will depend on it, we pause here to consider the validity of this reversal. In general terms, consider the repeated series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k}.$$

In this, $\sum_{k=1}^{\infty} a_{n,k}$ is the sum (assumed convergent) of row n of the infinite matrix $(a_{n,k})$: denoting this by R_n , the stated repeated series is then $\sum_{n=1}^{\infty} R_n$ (if it converges). Meanwhile, the reversed repeated series

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k}$$

is the sum of all the column sums, again assuming convergence at both stages. Are the two repeated sums always equal? Without further conditions, the answer is no, as the following rather startling example shows

Example. Let $a_{n,n} = 1$ and $a_{n,n+1} = -1$ for all n , and let $a_{n,k} = 0$ for all other n and k : this is an infinite matrix with 1 along the main diagonal and -1 along the next diagonal (it might help the reader to write it out). Clearly, for each n we have $\sum_{k=1}^{\infty} a_{n,k} = a_{n,n} + a_{n,n+1} = 1 - 1 = 0$, hence $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = 0$. However, the first column sum is $\sum_{n=1}^{\infty} a_{n,1} = a_{1,1} = 1$, while for all $k \geq 2$, we have $\sum_{n=1}^{\infty} a_{n,k} = a_{k-1,k} + a_{k,k} = -1 + 1 = 0$, so $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = 1$.

So we are not being over-fussy in giving a careful statement of the conditions under which reversal is valid. Here it is:

THEOREM A. *Suppose that $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{n,k}|$ is convergent. Then the two repeated series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k}$ and $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k}$ both converge, and their sums are equal.*

Of course, the example above fails this condition: $\sum_{k=1}^{\infty} |a_{n,k}| = 2$ for each n , so the summation over n is divergent.

In particular, if all the terms $a_{n,k}$ are non-negative and one of the repeated sums exists, then so does the other, with the same value. Furthermore, the same then applies to the repeated sums formed from $(-1)^n a_{n,k}$, since these have the same absolute value as $a_{n,k}$.

With this matter clarified, we can now state our basic result on series of the type we are considering. In the interests of simplicity, we will not state it in the most general form possible, but in a more user-friendly form that is suitable for the particular cases to follow.

THEOREM 1. *Suppose that $a_n \geq 0$ for all $n \geq 2$. Let*

$$F(t) = \sum_{n=2}^{\infty} a_n t^n, \quad G(t) = \frac{1}{2}[F(t) + F(-t)] = \sum_{n=1}^{\infty} a_{2n} t^{2n}$$

for the values of t for which these series converge. If $\sum_{n=2}^{\infty} a_n$ is convergent, then

$$\sum_{n=2}^{\infty} a_n \zeta(n) = \sum_{k=1}^{\infty} F\left(\frac{1}{k}\right), \tag{1}$$

$$\sum_{n=2}^{\infty} (-1)^n a_n \zeta(n) = \sum_{k=1}^{\infty} F\left(-\frac{1}{k}\right), \tag{2}$$

$$\sum_{n=1}^{\infty} a_{2n} \zeta(2n) = \sum_{k=1}^{\infty} G\left(\frac{1}{k}\right). \tag{3}$$

Now suppose that either $\sum_{n=2}^{\infty} a_n[\zeta(n) - 1]$ or $\sum_{k=2}^{\infty} F(1/k)$ is convergent. Then

$$\sum_{n=2}^{\infty} a_n[\zeta(n) - 1] = \sum_{k=2}^{\infty} F\left(\frac{1}{k}\right), \quad (4)$$

$$\sum_{n=2}^{\infty} (-1)^n a_n[\zeta(n) - 1] = \sum_{k=2}^{\infty} F\left(-\frac{1}{k}\right), \quad (5)$$

$$\sum_{n=1}^{\infty} a_{2n}[\zeta(2n) - 1] = \sum_{k=2}^{\infty} G\left(\frac{1}{k}\right). \quad (6)$$

Proof: First, assume that $\sum_{n=2}^{\infty} a_n$ is convergent. Since $\zeta(n) < 2$ for all n , it follows that $\sum_{n=2}^{\infty} a_n \zeta(n)$ is convergent. Consequently, reversal of summation is valid in the following:

$$\begin{aligned} \sum_{n=2}^{\infty} a_n \zeta(n) &= \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{a_n}{k^n} \\ &= \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \frac{a_n}{k^n} \\ &= \sum_{k=1}^{\infty} F\left(\frac{1}{k}\right). \end{aligned}$$

For (2), insert the factor $(-1)^n$, with the effect that k is replaced by $-k$. As remarked above, reversal is still valid. Statement (3) is now obtained by adding (1) and (2).

Statements (4), (5) and (6) are proved in exactly the same way, with k starting at 2 instead of 1. By Theorem A, reversal is valid if either of the two sides of (4) is convergent.

□

Note: We will prove in the next result that $\sum_{n=2}^{\infty} [\zeta(n) - 1]$ is convergent. It follows that a sufficient (though not necessary) condition for convergence of the left-hand side of (4) is that (a_n) is bounded.

We will now bring Theorem 1 to life by applying it to a number of particular cases.

APPLICATION 1. *We have*

$$\sum_{n=2}^{\infty} [\zeta(n) - 1] = 1, \quad (7)$$

$$\sum_{n=2}^{\infty} (-1)^n [\zeta(n) - 1] = \frac{1}{2}, \quad (8)$$

$$\sum_{n=1}^{\infty} [\zeta(2n) - 1] = \frac{3}{4}. \quad (9)$$

Proof. Denote the sums by S_1 , S_2 and S_3 respectively. Take $a_n = 1$ for all n , so that by the geometric series, $F(t) = \sum_{n=2}^{\infty} t^n = t^2/(1-t)$ for $|t| < 1$. Hence

$$F\left(\frac{1}{k}\right) = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Now by cancellation

$$\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1,$$

so (4) applies to show that $S_1 = 1$, hence (7). Also, $F(-1/k) = 1/[k(k+1)]$, so by (5),

$$S_2 = \sum_{k=2}^{\infty} \frac{1}{k(k+1)} = \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{2},$$

which establishes (8). Statement (9) follows, since $S_3 = \frac{1}{2}(S_1 + S_2)$. □

APPLICATION 2. *We have*

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{2^n} = \log 2, \tag{10}$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{2^n} = 1 - \log 2. \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{2n}} = \frac{1}{2}, \tag{12}$$

Proof. Denote the sums by S_4 , S_5 , S_6 respectively. We now have $a_n = \frac{1}{2^n}$, so

$$F\left(\frac{1}{k}\right) = \sum_{n=2}^{\infty} \frac{1}{(2k)^n} = \frac{1}{2k(2k-1)} = \frac{1}{2k-1} - \frac{1}{2k},$$

$$F\left(-\frac{1}{k}\right) = \frac{1}{2k(2k+1)} = \frac{1}{2k} - \frac{1}{2k+1}.$$

So by (1) and (2),

$$S_4 = \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2,$$

$$S_5 = \sum_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+1} \right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots = 1 - \log 2,$$

establishing (10) and (11). Statement (12) follows, since $S_6 = \frac{1}{2}(S_4 + S_5) = \frac{1}{2}$ (and for this there was no need to know about $\log 2$). □

APPLICATION 3. *We have*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{4^{2n}} = \frac{1}{2} - \frac{\pi}{8}. \quad (13)$$

Proof. Denote the sum by S_7 . We apply (3) with $a_{2n} = \frac{1}{4^{2n}}$. Using the geometric series in the form $\sum_{n=1}^{\infty} t^{2n} = t^2/(1-t^2)$, we have

$$G\left(\frac{1}{k}\right) = \sum_{n=1}^{\infty} \frac{1}{(4k)^{2n}} = \frac{1}{16k^2 - 1} = \frac{1}{2} \left(\frac{1}{4k-1} - \frac{1}{4k+1} \right).$$

Now consider the sum of K terms:

$$2 \sum_{k=1}^K G\left(\frac{1}{k}\right) = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots - \frac{1}{4K+1} \rightarrow 1 - \frac{\pi}{4} \quad \text{as } K \rightarrow \infty,$$

since $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$. This proves (13). \square

For the next group of examples, we need to recall that Euler's constant γ is defined to be $\lim_{k \rightarrow \infty} (H_k - \log k)$, where $H_k = \sum_{r=1}^k \frac{1}{r}$. Of course, γ also equals $\lim_{k \rightarrow \infty} H_{k-1} - \log k$, since $H_k - H_{k-1} = \frac{1}{k}$. The following identities are unashamedly repeated from [3, p. 421]. However, we are approaching them from the opposite direction: the emphasis in [3] was on finding interesting formulae for γ , while ours is on evaluation of series involving $\zeta(n)$.

APPLICATION 4. *We have*

$$\sum_{n=2}^{\infty} \frac{1}{n} [\zeta(n) - 1] = 1 - \gamma, \quad (14)$$

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) [\zeta(n) - 1] = \gamma, \quad (15)$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \gamma. \quad (16)$$

Proof. Denote the sums by S_8, S_9, S_{10} . By the log series, we now have (for $|t| < 1$)

$$F(t) = \sum_{n=2}^{\infty} \frac{t^n}{n} = -t - \log(1-t),$$

so

$$F\left(\frac{1}{k}\right) = -\frac{1}{k} - \log\left(1 - \frac{1}{k}\right) = -\frac{1}{k} - \log(k-1) + \log k,$$

hence

$$\sum_{k=2}^K F\left(\frac{1}{k}\right) = -\sum_{k=2}^K \frac{1}{k} + \log K = 1 - H_K + \log K \rightarrow 1 - \gamma \quad \text{as } K \rightarrow \infty,$$

so (14) follows from (4). Statement (15) follows from (14) and (7), since $S_9 = S_1 - S_8$. For (16), first consider

$$S_{11} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} [\zeta(n) - 1].$$

Now

$$F\left(-\frac{1}{k}\right) = \frac{1}{k} - \log\left(1 + \frac{1}{k}\right) = \frac{1}{k} + \log k - \log(k+1),$$

so

$$\sum_{k=2}^K F\left(-\frac{1}{k}\right) = H_K - 1 + \log 2 - \log(K+1),$$

from which we see that

$$S_{11} = \gamma - 1 + \log 2. \quad (17)$$

Identity (16) now follows, since

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n} = 1 - \log 2. \quad \square$$

It might appear that we have taken a rather roundabout route to (16) via (17). However, the tempting idea of deriving (16) directly from (2) would have been outside the rules for reversal of summation, because $\sum_{n=2}^{\infty} \frac{\zeta(n)}{n}$ is divergent.

By adding (14) and (17), we obtain the following further identity:

$$\sum_{n=1}^{\infty} \frac{1}{2n} [\zeta(2n) - 1] = \frac{1}{2} \log 2, \quad (18)$$

in which γ has disappeared by cancellation. Of course, (18) can be derived directly from (6): the reader may care to explore what happens if one does this.

The next identity is derived in the same way from the series $\sum_{n=1}^{\infty} nt^n = t/(1-t)^2$. Again we leave the details as an exercise for the reader.

APPLICATION 5. *We have*

$$\sum_{n=2}^{\infty} n[\zeta(n) - 1] = \zeta(2) + 1. \quad (19)$$

Are these results special cases of something more general? Yes, several of them are indeed cases of certain power series. For the first one, we assume the well-known cotangent series: for non-integer values of x ,

$$\pi x \cot \pi x = 1 - 2x^2 \sum_{k=1}^{\infty} \frac{1}{k^2 - x^2}. \quad (20)$$

(A good way to prove this identity is by the Fourier series for $\cos ax$, where a is not an integer, e.g. [4, p. 700].)

THEOREM 2. For $|x| < 1$, we have

$$2 \sum_{n=1}^{\infty} \zeta(2n) x^{2n} = 1 - \pi x \cot \pi x. \quad (21)$$

Proof. Regarding x as fixed, we apply (3) with $a_{2n} = x^{2n}$ (note that $\sum_{n=1}^{\infty} x^{2n}$ is convergent). By the geometric series again,

$$G\left(\frac{1}{k}\right) = \sum_{n=1}^{\infty} \frac{x^{2n}}{k^{2n}} = \frac{x^2/k^2}{1 - x^2/k^2} = \frac{x^2}{k^2 - x^2}.$$

The statement follows at once from (3) and (20). \square

(12) and (13) are special cases. For a further example, take $x = \frac{1}{3}$. Since $\cot \frac{\pi}{3} = 1/\sqrt{3}$, we obtain

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{3^{2n}} = \frac{1}{2} - \frac{\pi}{6\sqrt{3}}.$$

Meanwhile, the power series for $\pi x \cot \pi x$ can also be derived by multiplication of series. We can then apply (21) to determine values of $\zeta(2n)$. Let the power series for $x \cot x$ be $\sum_{n=0}^{\infty} a_{2n} x^{2n}$. Now $(x \cot x)[(\sin x)/x] = \cos x$, so, inserting the power series for $\sin x$ and $\cos x$, we have

$$(a_0 + a_2 x^2 + a_4 x^4 + \dots) \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

from which one works out that $a_0 = 1$, $a_2 = -\frac{1}{3}$, $a_4 = -\frac{1}{45}$. Hence

$$\pi x \cot \pi x = 1 - \frac{\pi^2}{3} x^2 - \frac{\pi^4}{45} x^4 - \dots.$$

Equating coefficients with those in (21), we obtain

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}.$$

In principle, this process can be continued; however, the general expression for $\zeta(2n)$ involves the Bernoulli numbers, and we will not embark on this topic here.

We can deduce a second power series by integration of (21):

THEOREM 3. For $|x| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^{2n} = \log \pi x - \log \sin \pi x. \quad (22)$$

Proof. Rewrite (21) as

$$2 \sum_{n=1}^{\infty} \zeta(2n) t^{2n-1} = \frac{1}{t} - \pi t \cot \pi t.$$

Integrate both sides from 0 to x . The integral of the series is exactly the left-hand side of (22) (for those who care, termwise integration is valid for power series). Now

$$\begin{aligned} \int_{\delta}^x \left(\frac{1}{t} - \cot \pi t \right) dt &= \log x - \log \sin \pi x + \log \frac{\sin \pi \delta}{\delta} \\ &\rightarrow \log x - \log \sin \pi x + \log \pi \quad \text{as } x \rightarrow 0^+, \end{aligned}$$

since $(\sin \pi \delta)/\delta \rightarrow \pi$ as $\delta \rightarrow 0^+$. This gives (22). \square

The case $x = \frac{1}{2}$ gives

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n2^{2n}} = \log \frac{\pi}{2}. \quad (23)$$

Further applications are obtained by integrating again. We assume the logsine integral [5, p. 246]:

$$\int_0^{\pi} \log \sin x \, dx = -\pi \log 2, \quad \int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2,$$

hence

$$\int_0^1 \log \sin \pi x \, dx = -\log 2, \quad \int_0^{1/2} \log \sin \pi x \, dx = -\frac{1}{2} \log 2.$$

Note that $\int_0^x \log \pi t \, dt = x \log \pi x - x$. Integrating (22) on $[0, 1]$, we obtain

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)} = \log \pi - 1 + \log 2 = \log 2\pi - 1. \quad (24)$$

Integrating on $[0, \frac{1}{2}]$ (and doubling), we obtain

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)2^{2n}} = \log \frac{\pi}{2} - 1 + \log 2 = \log \pi - 1. \quad (25)$$

Yet further identities can be found by multiplying (22) by x and then integrating, using the integral $\int_0^1 x \log \sin \pi x \, dx = -\frac{1}{2} \log 2$. We leave it to the reader to explore this.

Another power series is in terms of the gamma function. For this, we assume Weierstrass's product formula for the gamma function, which can be stated as follows (e.g. [6, p. 236] or [7, formula 6.1.3]): for all x except negative integers,

$$\Gamma(1+x) = e^{-\gamma x} \prod_{k=1}^{\infty} e^{x/k} \left(1 + \frac{x}{k}\right)^{-1}. \quad (26)$$

(In [7], this identity is called “Euler’s product formula”, but this label has not been adopted generally, and would appear to be historically inaccurate, although Euler did state another closely related product formula; see [6] for more on this.)

For present purposes, we actually want the following logarithmic version of (26): for all $x > -1$,

$$\log \Gamma(1+x) = -\gamma x + \sum_{k=1}^{\infty} \left[\frac{x}{k} - \log \left(1 + \frac{x}{k} \right) \right]. \quad (27)$$

THEOREM 4. For $|x| < 1$,

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} x^n = \log \Gamma(1+x) + \gamma x. \quad (28)$$

Proof. For $x > 0$, we apply (2) with $a_n = x^n/n$. Then

$$F(t) = \sum_{n=2}^{\infty} \frac{x^n t^n}{n} = -xt - \log(1-xt)$$

for $|xt| < 1$, so

$$F\left(-\frac{1}{k}\right) = \frac{x}{k} - \log\left(1 + \frac{x}{k}\right),$$

for all $k \geq 1$, and the statement follows from (27). For $x < 0$, we apply (1) with $a_n = (-x)^n/n$: the conclusion is the same. \square

Using Euler’s reflection formula $\Gamma(1+x)\Gamma(1-x) = \pi x/(\sin \pi x)$ and combining (28) for x and $-x$, we obtain (22) again.

Our (16) shows that (28) is also valid when $x = 1$, since $\Gamma(2) = 1$. Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the case $x = -\frac{1}{2}$ gives

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{n2^n} = \frac{1}{2} \log \pi - \frac{1}{2} \gamma. \quad (29)$$

Replacing $\zeta(n)$ by $\zeta(n) - 1$, we obtain the following variant of (28), valid for a wider range of x (cf. [6, formula 6.1.33]):

THEOREM 5. For $-1 < x < 2$,

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) - 1}{n} x^n = \log \Gamma(1+x) + \log(1+x) - (1-\gamma)x. \quad (30)$$

Proof. Modify the previous proof by applying (5) instead of (2). We now assume the inequality $\zeta(n) - 1 < \frac{1}{2^{n-1}}$ for $n \geq 3$, which implies that $\sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n} x^n$ converges for

$|x| < 2$. If $-1 < x < 2$, the expression for $F(-1/k)$ is valid for $k \geq 2$. Removing the term $F(-1) = x - \log(1+x)$, we obtain (30). \square

The case $x = 1$ reproduces (17). As stated, the series is actually convergent for $|x| < 2$, but when $-2 < x < -1$, both $1+x$ and $\Gamma(1+x)$ are negative, so the two logarithmic terms in (30) are undefined in real numbers. However, $(1+x)\Gamma(1+x) = \Gamma(2+x)$, and with the substitution of $\log \Gamma(2+x)$ for the two previous terms, (30) is still valid; the case $x = -1$ is (14).

Finally, by differentiation of (28), we obtain:

THEOREM 6. For $|x| < 1$,

$$\sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} = \frac{\Gamma'(1+x)}{\Gamma(1+x)} + \gamma. \quad \square \quad (31)$$

One can easily derive an expression for $\sum_{n=2}^{\infty} \zeta(n)x^n$ itself, if desired. In a sense, (10) and (11) are special cases, but we found them without any reference to the gamma function. In fact, we can use (10), together with the case $x = -\frac{1}{2}$ of (31), to deduce the value $\Gamma'(\frac{1}{2}) = -(\gamma + 2 \log 2)\sqrt{\pi}$.

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