

THE FOURTH POWER MOMENT OF $\zeta(\frac{1}{2} + it)$

Notes by Tim Jameson

Introduction

The following result was obtained by Ingham in 1926 [Ing]:

THEOREM 1. *We have*

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = \frac{1}{2\pi^2} T \log^4 T + O(T \log^3 T). \quad (1)$$

Another proof is given in [Iv, chap. 5]. At the expense of a lot more effort, a more specific estimation was obtained by Heath-Brown in 1979 [HB].

Theorem 1 obviously incorporates the following weaker statement (cf. [Ti, p. 146–7]):

THEOREM 2. *We have*

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt \ll T \log^4 T. \quad (2)$$

A discrete analogue (eg. [Iv, sect. 8.3]) is:

THEOREM 3. *Suppose that*

$$0 \leq t_1 < t_2 < \dots < t_R \leq T \quad \text{and} \quad t_r - t_{r-1} \geq 1 \quad \text{for each } r. \quad (3)$$

Then

$$\sum_{r=1}^R |\zeta(\frac{1}{2} + it_r)|^4 \ll T \log^5 T. \quad (4)$$

Here I present another proof of these results. In common with Ingham, it uses the approximate functional equation for $\zeta(s)^2$. Where it differs is in the use of an averaging technique which opens the way to an application of the Montgomery-Vaughan mean-value theorem for Dirichlet polynomials, and circumvents an estimation for the divisor function. The method is particularly efficient for Theorems 2. A partial version of it is enough for Theorem 3; this is outlined in [Iv, p. 227], and we repeat it here.

For Theorems 1 and 2, we will actually show that

$$I =: \int_T^{2T} |\zeta(\frac{1}{2} + it)|^4 dt \quad (5)$$

satisfies an estimate similar to the one stated. The result for $[0, T]$ then follows by application to the intervals $[2^{-j}T, 2^{-j+1}T]$. Similarly, in Theorem 3, we will assume that the points t_r lie in $[T, 2T]$.

The approximate functional equation for $\zeta(s)^2$

As usual, write

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s).$$

The approximate functional equation for $\zeta(s)^2$, specialised to the case $s = \frac{1}{2} + it$, states:

Let $\tau(n)$ be the divisor function. Let $x \asymp t$, and define y by $4\pi^2 xy = t^2$. Then

$$\zeta\left(\frac{1}{2} + it\right)^2 = S(x, t) + \chi\left(\frac{1}{2} + it\right)^2 S(y, -t) + O(\log t), \quad (6)$$

where

$$S(x, t) = \sum_{n \leq x} \tau(n) n^{-\frac{1}{2} - it}. \quad (7)$$

A pleasantly simple proof, restricted to the case $y = x$, was given by Motohashi [Moto]. This method actually adapts readily to the general case: details are given in my notes [Jam].

Note that by the elementary estimation $\sum_{n \leq x} \tau(n) n^{-1/2} \ll x^{1/2} \log x$ (see below), we have $|S(x, t)| \ll x^{1/2} \log x \asymp t^{1/2} \log t$.

For Theorem 1 (but not Theorems 2 and 3), we will need following modified version of (6). We use the known fact that for $t > 0$,

$$\chi\left(\frac{1}{2} - it\right) = e^{i\phi(t) - i\pi/4} + O(1/t),$$

where

$$\phi(t) = t \log \frac{t}{2\pi e},$$

so that

$$\chi\left(\frac{1}{2} - it\right)^2 = \psi(t) + O(1/t),$$

where $\psi(t) = -ie^{2i\phi(t)}$. Clearly, we have

$$\zeta\left(\frac{1}{2} + it\right)^2 = S(x, t) + \overline{\psi(t)} S(y, -t) + O(\log t), \quad (8)$$

Summation of $\tau(n)^2$

The following estimation of $\sum_{n \leq x} \tau(n)^2$ is originally due to Ramanujan. It can be seen in many books, e.g. [Nath, Theorem 7.8] or [Hux, p. 9]; we include a proof here for

completeness. We assume familiarity with convolutions and Abel summation. We start with the trivial estimate $\sum_{n \leq x} \tau(n) = x \log x + O(x)$. By Abel summation, it follows that

$$\sum_{n \leq x} \frac{\tau(n)}{n} = \frac{1}{2} \log^2 x + O(\log x),$$

and

$$\sum_{n \leq x} \frac{\tau(n)}{n^{1/2}} \ll x^{1/2} \log x$$

(as already assumed for the estimation of $|S(x, t)|$). Of course, more accurate estimations are possible, but they are not needed for our purposes.

Denote the convolution of arithmetic functions a, b by $a * b$. Define u by $u(n) = 1$ for all n . As usual, denote the Möbius function by μ , and let $\tau_3 = \tau * u$, so that $\tau_3(n)$ is the number of triples (i, j, k) with $ijk = n$.

LEMMA 1. *We have $\tau^2 = \tau_3 * \mu^2$.*

Proof. Both sides are multiplicative, so it is sufficient to consider the values at a prime power p^k . We have

$$\tau_3(p^k) = (\tau * u)(p^k) = \sum_{r=0}^k \tau(p^r) = \sum_{r=0}^k (r+1) = \frac{1}{2}(k+1)(k+2),$$

hence

$$(\tau_3 * \mu^2)(p^k) = \tau_3(p^k) + \tau_3(p^{k-1}) = \frac{1}{2}(k+1)(k+2) + \frac{1}{2}k(k+1) = (k+1)^2 = \tau(p^k)^2.$$

LEMMA 2. *We have*

$$\sum_{n \leq x} \tau(n)^2 = \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x), \tag{9}$$

$$\sum_{n \leq x} \frac{\tau(n)^2}{n} = \frac{1}{4\pi^2} \log^4 x + O(\log^3 x). \tag{10}$$

Proof. By the formula for partial sums of convolutions,

$$\sum_{n \leq x} \tau_3(n) = \sum_{j \leq x} \tau(j) \left[\frac{x}{j} \right] = x \sum_{j \leq x} \frac{\tau(j)}{j} + O(x \log x) = \frac{1}{2} x \log^2 x + O(x \log x).$$

By Abel summation, it follows that

$$\sum_{n \leq x} \frac{\tau_3(n)}{n} = \frac{1}{6} \log^3 x + O(\log^2 x),$$

$$\sum_{n \leq x} \frac{\tau_3(n)}{n^{1/2}} = O(x^{1/2} \log^2 x).$$

Let $M_2(x) = \sum_{n \leq x} \mu(n)^2$. Assuming the well-known estimate $M_2(x) = (6/\pi^2)x + O(x^{1/2})$, we deduce from Lemma 1 that

$$\begin{aligned} \sum_{n \leq x} \tau(n)^2 &= \sum_{n \leq x} \tau_3(n) M_2\left(\frac{x}{n}\right) \\ &= \sum_{n \leq x} \tau_3(n) \left(\frac{6}{\pi^2} \frac{x}{n} + O\left(\frac{x^{1/2}}{n^{1/2}}\right) \right) \\ &= \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x). \end{aligned}$$

This establishes (9), and (10) follows by Abel summation. \square

Integrals and sums of $|S(x, t)|^2$

We use the following mean-value theorem for Dirichlet polynomials, an easy consequence of the Montgomery-Vaughan generalisation of Hilbert's inequality (see [Mont, p. 140] or [Iv, Theorem 5.2]):

Let $f(t) = \sum_{n=1}^N a_n n^{-it}$. Then

$$\int_{T_1}^{T_2} |f(t)|^2 dt = (T_2 - T_1) \sum_{n=1}^N |a_n|^2 + O\left(\sum_{n=1}^N n |a_n|^2\right). \quad (11)$$

A weaker version, proved rather more easily, has error term $N \sum_{n=1}^N |a_n|^2$.

LEMMA 3. Suppose that $x \asymp T_2 \asymp T$. Write $1/(4\pi^2) = K$ and $\log T = L$. Then

$$\int_{T_1}^{T_2} |S(x, t)|^2 dt = K(T_2 - T_1)L^4 + O(TL^3). \quad (12)$$

Proof. By (9), (10) and (11), with $a_n = \tau(n)/n^{1/2}$ and $N = [x]$, we have

$$\int_{T_1}^{T_2} |S(x, t)|^2 dt = K(T_2 - T_1) \log^4 x + (T_2 - T_1)O(\log^3 x) + O(x \log^3 x).$$

Since $x \asymp T$, we have $\log x = L + O(1)$, hence $\log^4 x = L^4 + O(L^3)$. \square

In particular, the integral is $O(TL^4)$; for this conclusion, the weaker version of (11) would have been sufficient.

LEMMA 4. If $x \asymp T_2 \asymp T$, then

$$\int_{T_1}^{T_2} t^2 |S(x, t)|^2 dt = KL^4 \int_{T_1}^{T_2} t^2 dt + O(T^3 L^3).$$

Proof. Write $|S(x, t)|^2 - KL^4 = g(t)$ and $G(t) = \int_{T_1}^t g(u) du$. By (12), $G(t) \ll TL^3$ for $T_1 \leq t \leq T_2$. Integrating by parts, we have

$$\int_{T_1}^{T_2} t^2 (|S(x, t)|^2 - KL^4) dt = \int_{T_1}^{T_2} t^2 g(t) dt = \left[t^2 G(t) \right]_{T_1}^{T_2} - \int_{T_1}^{T_2} 2tG(t) dt \ll T^3 L^3. \quad \square$$

For Theorem 3, we will apply instead the following “large values estimate”, which can be deduced easily from (11) by Gallagher’s method [Iv, Theorem 5.3]:

Let $f(t) = \sum_{n=1}^N a_n^{-it}$, and let t_r ($1 \leq r \leq R$) satisfy (3). Then

$$\sum_{r=1}^R |f(t_r)|^2 \ll (T + N) \log N \sum_{n=1}^N |a_n|^2.$$

For our case, we deduce:

LEMMA 5. Suppose that $x \asymp T$ and t_r ($1 \leq r \leq R$) satisfy (3). Then

$$\sum_{r=1}^R |S(x, t_r)|^2 \ll TL^5. \quad \square$$

Proofs of Theorems 2 and 3

We consider both x and t in the interval $[T, 2T]$. Write $K = 1/4\pi^2$ and $y = Kt^2/x$, so that $\frac{1}{2}KT \leq y \leq 4KT$ and $x \asymp y \asymp T$. Also, write $L = \log T$.

Since $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and $|\chi(\frac{1}{2} + it)| = 1$, (6) gives

$$|\zeta(\frac{1}{2} + it)|^4 \ll |S(x, t)|^2 + |S(y, t)|^2 + L^2. \quad (13)$$

Proof of Theorem 2. Let $I_0(x) = I_1(x) + I_2(x)$, where

$$I_1(x) = \int_T^{2T} |S(x, t)|^2 dt,$$

$$I_2(x) = \int_T^{2T} |S(y, t)|^2 dt = \int_T^{2T} \left| S\left(\frac{Kt^2}{x}, t\right) \right|^2 dt.$$

Then for any $x \in [T, 2T]$,

$$I \ll I_0(x) + TL^2.$$

By Lemma 3, $I_1(x) \ll TL^4$ for each x . However, since y varies with t , $I_2(x)$ is not of the same form. This is where we introduce our averaging technique. Instead of estimating $I_0(x)$ for one fixed x , we consider its average over x in $[T, 2T]$, as follows. For $j = 0, 1, 2$, let

$$A_j = \frac{1}{T} \int_T^{2T} I_j(x) dx, \quad (14)$$

Clearly, $A_0 = A_1 + A_2$ and $I \ll A_0 + TL^2$.

Clearly, $A_1 \ll TL^4$; of course, the averaging step was not needed for this term. Now consider A_2 . Reversing the order of integration, substituting $x = Kt^2/y$ and reversing again, we have

$$\begin{aligned}
A_2 &= \frac{1}{T} \int_T^{2T} \int_T^{2T} |S(y, t)|^2 dx dt \\
&= \frac{1}{T} \int_T^{2T} \int_{\frac{Kt^2}{2T}}^{\frac{Kt^2}{T}} |S(y, t)|^2 \frac{Kt^2}{y^2} dy dt \\
&= \frac{1}{T} \int_{\frac{1}{2}KT}^{4KT} \frac{K}{y^2} \int_{T_1}^{T_2} t^2 |S(y, t)|^2 dt dy,
\end{aligned} \tag{15}$$

where for present purposes we only need to know that $[T_1, T_2]$ is contained in $[T, 2T]$, so that by Lemma 3 again,

$$\int_{T_1}^{T_2} |S(y, t)|^2 dt \ll TL^4,$$

Since $t^2/y^2 = O(1)$, it follows that

$$A_2 \ll \frac{1}{T} T \cdot TL^4 = TL^4,$$

so $I \ll TL^4$. □

Proof of Theorem 3. This time, we average the expression for $|\zeta(\frac{1}{2} + it)|^4$ itself, not its integral. Let

$$\begin{aligned}
A_1(t) &= \frac{1}{T} \int_T^{2T} |S(x, t)|^2 dx, \\
A_2(t) &= \frac{1}{T} \int_T^{2T} |S(y, t)|^2 dx.
\end{aligned}$$

By (13), for each t ,

$$|\zeta(\frac{1}{2} + it)|^4 \ll A_1(t) + A_2(t) + L^2.$$

Since $R \leq T + 1$,

$$\sum_{r=1}^R |\zeta(\frac{1}{2} + it_r)|^4 \ll \sum_{r=1}^R [A_1(t_r) + A_2(t_r)] + TL^2.$$

By Lemma 5, $\sum_{r=1}^R A_1(t_r) \ll TL^5$. As in the proof of Theorem 2 (without the integration with respect to t), we have

$$A_2(t) \ll \frac{1}{T} \int_{\frac{1}{2}KT}^{4KT} |S(y, t)|^2 dy,$$

for each t in $[T, 2T]$, hence also $\sum_{r=1}^R A_2(t_r) \ll TL^5$. □

Note. For Theorems 2 and 3 (but not Theorem 1), we can work with the approximate functional equation for $\zeta(s)$ itself instead of $\zeta(s)^2$:

$$\zeta\left(\frac{1}{2} + it\right) = Z(x, t) + \chi\left(\frac{1}{2} + it\right)Z(y, -t) + O(1).$$

where $Z(x, t) = \sum_{n \leq x} n^{-\frac{1}{2} - it}$ and $2\pi xy = t$, so that

$$|\zeta\left(\frac{1}{2} + it\right)|^4 \ll |Z(x, t)|^4 + |Z(y, t)|^4 + 1.$$

Now

$$|Z(x, t)|^2 = \sum_{n \leq x^2} \tau(n, x) n^{-\frac{1}{2} - it},$$

where $\tau(n, x)$ counts factorisations jk of n with $j, k \leq x$, so that $\tau(n, x) \leq \tau(n)$, and hence

$$\int_T^{2T} |Z(x, t)|^4 dt \ll (T + x^2) \sum_{n \leq x^2} \frac{\tau(n)^2}{n} \ll TL^4.$$

The steps are now as before, but with the averaging applied over $T^{1/2} \leq x \leq 2T^{1/2}$.

Proof of Theorem 1

We must replace the upper estimate (13) used in Theorem 2 by an asymptotic estimate. Recall from (8) that $|\zeta\left(\frac{1}{2} + it\right)|^2 = W(x, t) + O(L)$, where

$$W(x, t) = S(x, t) + \overline{\psi(t)}S(y, -t). \tag{16}$$

Just writing W and ζ , we have $|W| = |\zeta|^2 + \rho$, where $\rho = O(L)$. Hence

$$|W|^2 = |\zeta|^4 + 2\rho|\zeta|^2 + \rho^2.$$

By Theorem 2, $\int_T^{2T} |\zeta|^4 \ll TL^4$. So by the Cauchy-Schwarz inequality,

$$\int_T^{2T} |\zeta|^2 \ll T^{1/2}(T^{1/2}L^2) = TL^2.$$

(In fact, it is well known that $\int_T^{2T} |\zeta|^2 \ll TL$: see [Ti, p. 141].) So if

$$I(x) = \int_T^{2T} |W(x, t)|^2 dt,$$

then

$$I = I(x) + O(TL^3),$$

for each x in $[T, 2T]$.

Now apply the averaging process: let

$$A = \frac{1}{T} \int_T^{2T} I(x) dx.$$

Then $I = A + O(TL^3)$, so Theorem 1 will follow if we can show that

$$A = 2KTL^4 + O(TL^3). \quad (17)$$

Now

$$|W(x, t)|^2 = W(x, t)\overline{W(x, t)} = |S(x, t)|^2 + |S(y, t)|^2 + 2\operatorname{Re} (\psi(t)S(x, t)S(y, t)),$$

so

$$I(x) = I_1(x) + I_2(x) + 2I_3(x),$$

where $I_1(x)$ and $I_2(x)$ are as before, and

$$I_3(x) = \int_T^{2T} \operatorname{Re} (\psi(t)S(x, t)S(y, t)) dt.$$

With A_j defined by (14) for $j = 1, 2, 3$, we have

$$A = A_1 + A_2 + 2A_3.$$

By Lemma 3, $I_1(x) = KTL^4 + O(TL^3)$ for each x , hence

$$A_1 = KTL^4 + O(TL^3). \quad (18)$$

Now consider A_2 . In the expression (15) for A_2 , we now need to specify

$$T_1 = \max \left(T, \sqrt{\frac{Ty}{K}} \right), \quad T_2 = \min \left(2T, \sqrt{\frac{2Ty}{K}} \right).$$

By Lemma 4,

$$\int_{T_1}^{T_2} t^2 |S(y, t)|^2 dt = KL^4 \int_{T_1}^{T_2} t^2 dt + O(T^3 L^3).$$

Now reversing the steps that led to (15), with $|S(y, t)|^2$ replaced by 1, we have

$$\frac{1}{T} \int_{\frac{1}{2}KT}^{4KT} \frac{K}{y^2} \int_{T_1}^{T_2} t^2 dt dy = \frac{1}{T} \int_T^{2T} \int_T^{2T} 1 dx dt = T,$$

also

$$\frac{1}{T} \int_{\frac{1}{2}KT}^{4KT} \frac{K}{y^2} dy < \frac{2}{T^2},$$

hence

$$A_2 = KTL^4 + O(TL^3). \quad (19)$$

The required estimation (17) will follow if we can establish that $A_3 \ll TL^3$. Since $\psi(t) = -ie^{2i\phi(t)}$,

$$\operatorname{Re}(\psi(t)S(x,t)S(y,t)) = \operatorname{Im}(e^{2i\phi(t)}S(x,t)S(y,t)).$$

Now

$$e^{2i\phi(t)}S(x,t)S(y,t) = e^{2i\phi(t)} \sum_{m \leq x} \sum_{n \leq \frac{Kt^2}{x}} \frac{\tau(m)\tau(n)}{(mn)^{1/2+it}} = \sum_{m \leq x} \sum_{n \leq \frac{Kt^2}{x}} \frac{\tau(m)\tau(n)}{(mn)^{1/2}} e^{if_{m,n}(t)},$$

where

$$f_{m,n}(t) = 2\phi(t) - t \log mn = t(2 \log t - 2 - 2 \log 2\pi - \log mn).$$

For a given n , integration is on the interval $[T_0, 2T]$, where

$$T_0 = \max\left(T, \sqrt{\frac{nx}{K}}\right).$$

So

$$I_3(x) = \sum_{m \leq x} \sum_{n \leq \frac{4KT^2}{x}} \frac{\tau(m)\tau(n)}{(mn)^{1/2}} \operatorname{Im} J_{m,n}, \quad (20)$$

where

$$J_{m,n} = \int_{T_0}^{2T} e^{if_{m,n}(t)} dt.$$

Denote by $I_{3,1}(x)$ the contribution to $I_3(x)$ of the terms with $m \leq x-1$, and by $I_{3,2}(x)$ the contribution of the single term with $x-1 < m \leq x$, with corresponding averages $A_{3,1}$ and $A_{3,2}$.

Since $\sum_{n \leq x} \tau(n)n^{-1/2} \ll x^{1/2} \log x$, we have

$$\sum_{n \leq \frac{4KT^2}{x}} \frac{\tau(n)}{n^{1/2}} \ll T^{1/2}L. \quad (21)$$

Consider $A_{3,2}$ first: for this, we have $m = [x]$. By (21) and the trivial bound $|J_{m,n}| \leq T$,

$$I_{3,2}(x) \ll \frac{\tau(m)}{m^{1/2}} T^{3/2}L \asymp \tau(m)TL.$$

Hence

$$A_{3,2} \ll L \int_T^{2T} \tau([x]) dx \leq L \sum_{m \leq 2T} \tau(m) \ll TL^2.$$

(Note that we cannot deduce such an estimate for $I_{3,2}(x)$ for a fixed x , since $\tau(m)$ is not bounded by a power of L .)

Now consider $A_{3,1}$. We use the following Lemma [Ti, Lemma 4.3] to estimate $J_{m,n}$:

LEMMA 6. *If f is a real-valued function with $f'(t)$ increasing on $[a, b]$ and $f'(a) = \mu > 0$, then*

$$\left| \int_a^b e^{if(t)} dt \right| \leq \frac{2}{\mu}.$$

Proof. Let $h(t) = 1/f'(t)$. Integrating by parts, we have

$$\int_a^b e^{if(t)} dt = \int_a^b h(t) f'(t) e^{if(t)} dt = J_1 + J_2,$$

where

$$J_1 = \left[\frac{1}{i} h(t) e^{-if(t)} \right]_a^b, \quad J_2 = \int_a^b h'(t) e^{if(t)} dt.$$

Clearly, $|J_1| \leq h(a) + h(b)$, and since $|h'(t)| = -h'(t)$,

$$|J_2| \leq - \int_a^b h'(t) dt = h(a) - h(b). \quad \square$$

We have

$$f'_{m,n}(t) = 2 \log t - 2 \log 2\pi - \log mn = \log \frac{Kt^2}{mn},$$

which is clearly increasing. Also, $KT_0^2 \geq nx$, so for $m < x$, $f'_{m,n}(T_0) \geq \log x/m$. Now

$$\log \frac{x}{m} = \int_m^x \frac{1}{t} dt > \frac{x-m}{x},$$

so

$$\frac{1}{\log x/m} < \frac{x}{x-m} = 1 + \frac{m}{x-m}.$$

Hence for each x , we have

$$\begin{aligned} I_{3,1}(x) &\ll T^{1/2} L \sum_{m \leq x-1} \frac{\tau(m)}{m^{1/2}} \left(1 + \frac{m}{x-m} \right) \\ &\ll T^{1/2} L \cdot T^{1/2} L + T^{1/2} L \sum_{m \leq x-1} \frac{m^{1/2} \tau(m)}{x-m} \\ &\ll TL^2 + TL \sum_{m \leq x-1} \frac{\tau(m)}{x-m}. \end{aligned}$$

Now integrate with respect to x on $[T, 2T]$. For a fixed m , we integrate $1/(x-m)$ on an interval contained in $[m+1, 2T]$: the contribution is less than $\log 2T$. So

$$A_{3,1} \ll TL^2 + L^2 \sum_{m < 2T} \tau(m) \ll TL^3.$$

So $A_3 \ll TL^3$, as required. □

Comparison with Ingham's method

Ingham's method takes $x = y = t/(2\pi)$, requiring (in our notation) the estimation of $\int_0^T |S(t/2\pi, t)|^2 dt$. This in turn requires the estimation

$$\sum_{n=1}^N \sum_{m < n} \frac{\tau(m)\tau(n)}{(mn)^{1/2}(\log n - \log m)} \ll N \log^3 N, \quad (22)$$

which is achieved by calculations specific to the divisor function. For Theorem 2, this estimation would still be needed, with bound $N \log^4 N$; this is essentially the method implied in [Ti, p. 146–7]. We remark that there is no short cut to (22) by the Montgomery-Vaughan theorems, since it really involves $|\log n - \log m|$, not $\log n - \log m$. Indeed, it is not hard to show that the best constant C in the bilinear form estimate

$$\sum_{n=1}^N \sum_{m < n} \frac{x_m \bar{x}_n}{\log n - \log m} \leq C \sum_{n=1}^N |x_n|^2$$

satisfies $C \sim N \log N$. This would only lead to the bound $N \log^5 N$ in (22).

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