THE FOURTH POWER MOMENT OF $\zeta(\frac{1}{2} + it)$

Notes by Tim Jameson

Introduction

The following result was obtained by Ingham in 1926 [Ing]:

THEOREM 1. We have

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = \frac{1}{2\pi^2} T \log^4 T + O(T \log^3 T).$$

(1)

Another proof is given in [Iv, chap. 5]. At the expense of a lot more effort, a more specific estimation was obtained by Heath-Brown in 1979 [HB].

Theorem 1 obviously incorporates the following weaker statement (cf. [Ti, p. 146–7]):

THEOREM 2. We have

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T \log^4 T.$$

(2)

A discrete analogue (eg. [Iv, sect. 8.3]) is:

THEOREM 3. Suppose that

$$0 \leq t_1 < t_2 < \ldots < t_R \leq T \quad \text{and} \quad t_r - t_{r-1} \geq 1 \quad \text{for each } r.$$

Then

$$\sum_{r=1}^R |\zeta(\frac{1}{2} + it_r)|^4 \ll T \log^5 T.$$

(3)

Here I present another proof of these results. In common with Ingham, it uses the approximate functional equation for $\zeta(s)^2$. Where it differs is in the use of an averaging technique which opens the way to an application of the Montgomery-Vaughan mean-value theorem for Dirichlet polynomials, and circumvents an estimation for the divisor function. The method is particularly efficient for Theorems 2. A partial version of it is enough for Theorem 3; this is outlined in [Iv, p. 227], and we repeat it here.

For Theorems 1 and 2, we will actually show that

$$I =: \int_T^{2T} |\zeta(\frac{1}{2} + it)|^4 \, dt$$

(5)
satisfies an estimate similar to the one stated. The result for \([0, T]\) then follows by application to the intervals \([2^{-j}T, 2^{-j+1}T]\). Similarly, in Theorem 3, we will assume that the points \(t_r\) lie in \([T, 2T]\).

The approximate functional equation for \(\zeta(s)^2\)

As usual, write
\[
\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s).
\]
The approximate functional equation for \(\zeta(s)^2\), specialised to the case \(s = \frac{1}{2} + it\), states:

Let \(\tau(n)\) be the divisor function. Let \(x \asymp t\), and define \(y\) by
\[
4\pi^2 xy = t^2.
\]
Then
\[
\zeta\left(\frac{1}{2} + it\right)^2 = S(x, t) + \chi\left(\frac{1}{2} + it\right)^2 S(y, -t) + O(\log t),
\]
where
\[
S(x, t) = \sum_{n \leq x} \tau(n) n^{-\frac{1}{2}-it}.
\]

A pleasantly simple proof, restricted to the case \(y = x\), was given by Motohashi [Moto]. This method actually adapts readily to the general case: details are given in my notes [Jam].

Note that by the elementary estimation
\[
\sum_{n \leq x} \tau(n) n^{-1/2} \ll x^{1/2} \log x
\]
(see below), we have \(|S(x, t)| \ll x^{1/2} \log x \asymp t^{1/2} \log t\).

For Theorem 1 (but not Theorems 2 and 3), we will need following modified version of (6). We use the known fact that for \(t > 0\),
\[
\chi\left(\frac{1}{2} - it\right) = e^{i\phi(t) - i\pi/4} + O(1/t),
\]
where
\[
\phi(t) = t \log \frac{t}{2\pi e},
\]
so that
\[
\chi\left(\frac{1}{2} - it\right)^2 = \psi(t) + O(1/t),
\]
where \(\psi(t) = -ie^{2i\phi(t)}\). Clearly, we have
\[
\zeta\left(\frac{1}{2} + it\right)^2 = S(x, t) + \overline{\psi(t)} S(y, -t) + O(\log t),
\]
where
\[
\psi(t) = -ie^{2i\phi(t)}.
\]

Summation of \(\tau(n)^2\)

The following estimation of \(\sum_{n \leq x} \tau(n)^2\) is originally due to Ramanujan. It can be seen in many books, e.g. [Nath, Theorem 7.8] or [Hux, p. 9]; we include a proof here for
completeness. We assume familiarity with convolutions and Abel summation. We start with the trivial estimate
\[ \sum_{n \leq x} \tau(n) = x \log x + O(x). \]
By Abel summation, it follows that
\[ \sum_{n \leq x} \frac{\tau(n)}{n} = \frac{1}{2} \log^2 x + O(\log x), \]
and
\[ \sum_{n \leq x} \frac{\tau(n)}{n^{1/2}} \ll x^{1/2} \log x \]
(as already assumed for the estimation of \( |S(x, t)| \)). Of course, more accurate estimations are possible, but they are not needed for our purposes.

Denote the convolution of arithmetic functions \( a, b \) by \( a * b \). Define \( u \) by \( u(n) = 1 \) for all \( n \). As usual, denote the Möbius function by \( \mu \), and let \( \tau_3 = \tau * u \), so that \( \tau_3(n) \) is the number of triples \( (i, j, k) \) with \( ijk = n \).

**LEMMA 1.** We have \( \tau^2 = \tau_3 * \mu^2 \).

**Proof.** Both sides are multiplicative, so it is sufficient to consider the values at a prime power \( p^k \). We have
\[ \tau_3(p^k) = (\tau * u)(p^k) = \sum_{r=0}^{k} \tau(p^r) = \sum_{r=0}^{k} (r + 1) = \frac{1}{2}(k + 1)(k + 2), \]

hence
\[ (\tau_3 * \mu^2)(p^k) = \tau_3(p^k) + \tau_3(p^{k-1}) = \frac{1}{2}(k + 1)(k + 2) + \frac{1}{2}k(k + 1) = (k + 1)^2 = \tau(p^k)^2. \]

**LEMMA 2.** We have
\[ \sum_{n \leq x} \tau(n)^2 = \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x), \quad (9) \]
\[ \sum_{n \leq x} \frac{\tau(n)^2}{n} = \frac{1}{4\pi^2} \log^4 x + O(\log^3 x). \quad (10) \]

**Proof.** By the formula for partial sums of convolutions,
\[ \sum_{n \leq x} \tau_3(n) = \sum_{j \leq x} \tau(j) \left[ \frac{x}{j} \right] = x \sum_{j \leq x} \frac{\tau(j)}{j} + O(x \log x) = \frac{1}{2}x \log^2 x + O(x \log x). \]

By Abel summation, it follows that
\[ \sum_{n \leq x} \frac{\tau_3(n)}{n} = \frac{1}{6} \log^3 x + O(\log^2 x), \]
\[ \sum_{n \leq x} \frac{\tau_3(n)}{n^{1/2}} = O(x^{1/2} \log^2 x). \]

Let \( M_2(x) = \sum_{n \leq x} \mu(n)^2 \). Assuming the well-known estimate \( M_2(x) = (6/\pi^2)x + O(x^{1/2}) \), we deduce from Lemma 1 that
\[ \sum_{n \leq x} \tau(n)^2 = \sum_{n \leq x} \tau_3(n)M_2 \left( \frac{x}{n} \right) \]
\[ = \sum_{n \leq x} \tau_3(n) \left( \frac{6}{\pi^2} \frac{x}{n} + O \left( \frac{x^{1/2}}{n^{1/2}} \right) \right) \]
\[ = \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x). \]

This establishes (9), and (10) follows by Abel summation. \( \square \)

**Integrals and sums of \(|S(x,t)|^2\)**

We use the following mean-value theorem for Dirichlet polynomials, an easy consequence of the Montgomery-Vaughan generalisation of Hilbert’s inequality (see [Mont, p. 140] or [Iv, Theorem 5.2]):

Let \( f(t) = \sum_{n=1}^{N} a_n n^{-it} \). Then
\[ \int_{T_1}^{T_2} |f(t)|^2 \, dt = (T_2 - T_1) \sum_{n=1}^{N} |a_n|^2 + O \left( \sum_{n=1}^{N} n |a_n|^2 \right). \] (11)

A weaker version, proved rather more easily, has error term \( N \sum_{n=1}^{N} |a_n|^2 \).

**LEMMA 3.** Suppose that \( x \asymp T_2 \asymp T \). Write \( 1/(4\pi^2) = K \) and \( \log T = L \). Then
\[ \int_{T_1}^{T_2} |S(x,t)|^2 \, dt = K(T_2 - T_1)L^4 + O(TL^3). \] (12)

**Proof.** By (9), (10) and (11), with \( a_n = \tau(n)/n^{1/2} \) and \( N = \lfloor x \rfloor \), we have
\[ \int_{T_1}^{T_2} |S(x,t)|^2 \, dt = K(T_2 - T_1) \log^4 x + (T_2 - T_1)O(\log^3 x) + O(x \log^3 x). \]

Since \( x \asymp T \), we have \( \log x = L + O(1) \), hence \( \log^4 x = L^4 + O(L^3) \). \( \square \)

In particular, the integral is \( O(TL^4) \); for this conclusion, the weaker version of (11) would have been sufficient.

**LEMMA 4.** If \( x \asymp T_2 \asymp T \), then
\[ \int_{T_1}^{T_2} t^2 |S(x,t)|^2 \, dt = KL^4 \int_{T_1}^{T_2} t^2 \, dt + O(T^3 L^3). \]
Proof. Write \(|S(x,t)|^2 - KL^4 = g(t)\) and \(G(t) = \int_{T_1}^{T_2} g(u) \, du\). By (12), \(G(t) \ll TL^3\) for \(T_1 \leq t \leq T_2\). Integrating by parts, we have
\[
\int_{T_1}^{T_2} t^2(|S(x,t)|^2 - KL^4) \, dt = \int_{T_1}^{T_2} t^2 g(t) \, dt = \left[t^2 G(t)\right]_{T_1}^{T_2} - \int_{T_1}^{T_2} 2t G(t) \, dt \ll T^3 L^3.
\]
\(\square\)

For Theorem 3, we will apply instead the following “large values estimate”, which can be deduced easily from (11) by Gallagher’s method [Iv, Theorem 5.3]:

Let \(f(t) = \sum_{n=1}^{N} a_n e^{-itn}\), and let \(t_r (1 \leq r \leq R)\) satisfy (3). Then
\[
\sum_{r=1}^{R} |f(t_r)|^2 \ll (T + N) \log N \sum_{n=1}^{N} |a_n|^2.
\]
For our case, we deduce:

LEMMA 5. Suppose that \(x \asymp T\) and \(t_r (1 \leq r \leq R)\) satisfy (3). Then
\[
\sum_{r=1}^{R} |S(x,t_r)|^2 \ll TL^5.
\]
\(\square\)

Proofs of Theorems 2 and 3

We consider both \(x\) and \(t\) in the interval \([T, 2T]\). Write \(K = 1/4\pi^2\) and \(y = Kt^2/x\), so that \(\frac{1}{2}KT \leq y \leq 4KT\) and \(x \asymp y \asymp T\). Also, write \(L = \log T\).

Since \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) and \(|\chi(\frac{1}{2} + it)| = 1\), (6) gives
\[
|\zeta(\frac{1}{2} + it)|^4 \ll |S(x,t)|^2 + |S(y,t)|^2 + L^2.
\]
(13)

Proof of Theorem 2. Let \(I_0(x) = I_1(x) + I_2(x)\), where
\[
I_1(x) = \int_{T}^{2T} |S(x,t)|^2 \, dt,
\]
\[
I_2(x) = \int_{T}^{2T} |S(y,t)|^2 \, dt = \int_{T}^{2T} \left|S\left(\frac{Kt^2}{x}, t\right)\right|^2 \, dt.
\]
Then for any \(x \in [T, 2T]\),
\[
I \ll I_0(x) + TL^2.
\]
By Lemma 3, \(I_1(x) \ll TL^4\) for each \(x\). However, since \(y\) varies with \(t\), \(I_2(x)\) is not of the same form. This is where we introduce our averaging technique. Instead of estimating \(I_0(x)\) for one fixed \(x\), we consider its average over \(x\) in \([T, 2T]\), as follows. For \(j = 0, 1, 2\), let
\[
A_j = \frac{1}{T} \int_{T}^{2T} I_j(x) \, dx,
\]
(14)
Clearly, $A_0 = A_1 + A_2$ and $I \ll A_0 + TL^2$.

Clearly, $A_1 \ll TL^4$; of course, the averaging step was not needed for this term. Now consider $A_2$. Reversing the order of integration, substituting $x = Kt^2/y$ and reversing again, we have

$$A_2 = \frac{1}{T} \int_T^{2T} \int_T^{2T} |S(y,t)|^2 dx \, dt$$
$$= \frac{1}{T} \int_T^{2T} \int_{Kt^2/y}^{Kt^2} |S(y,t)|^2 \frac{Kt^2}{y^2} dy \, dt$$
$$= \frac{1}{T} \int_{\frac{1}{4} KT}^{KT} \int_{\frac{1}{4} KT}^{Kt^2} t^2 |S(y,t)|^2 dy \, dt,$$

where for present purposes we only need to know that $[T_1, T_2]$ is contained in $[T, 2T]$, so that by Lemma 3 again,

$$\int_{T_1}^{T_2} |S(y,t)|^2 dt \ll TL^4,$$

Since $t^2/y^2 = O(1)$, it follows that

$$A_2 \ll \frac{1}{T} TTL^4 = TL^4,$$

so $I \ll TL^4$.  \[\square\]

**Proof of Theorem 3.** This time, we average the expression for $|\zeta(\frac{1}{2} + it)|^4$ itself, not its integral. Let

$$A_1(t) = \frac{1}{T} \int_T^{2T} |S(x,t)|^2 dx,$$

$$A_2(t) = \frac{1}{T} \int_T^{2T} |S(y,t)|^2 dx.$$

By (13), for each $t$,

$$|\zeta(\frac{1}{2} + it)|^4 \ll A_1(t) + A_2(t) + L^2.$$

Since $R \leq T + 1$,

$$\sum_{r=1}^{R} |\zeta(\frac{1}{2} + it_r)|^4 \ll \sum_{r=1}^{R} [A_1(t_r) + A_2(t_r)] + TL^2.$$

By Lemma 5, $\sum_{r=1}^{R} A_1(t_r) \ll TL^5$. As in the proof of Theorem 2 (without the integration with respect to $t$), we have

$$A_2(t) \ll \frac{1}{T} \int_{\frac{1}{4} KT}^{KT} |S(y,t)|^2 dy,$$

for each $t$ in $[T, 2T]$, hence also $\sum_{r=1}^{R} A_2(t_r) \ll TL^5$.  \[\square\]
Note. For Theorems 2 and 3 (but not Theorem 1), we can work with the approximate functional equation for $\zeta(s)$ itself instead of $\zeta(s)^2$:

$$\zeta\left(\frac{1}{2} + it\right) = Z(x, t) + \chi\left(\frac{1}{2} + it\right)Z(y, -t) + O(1),$$

where $Z(x, t) = \sum_{n \leq x} n^{-\frac{1}{2} - it}$ and $2\pi xy = t$, so that

$$|\zeta\left(\frac{1}{2} + it\right)|^4 \ll |Z(x, t)|^4 + |Z(y, t)|^4 + 1.$$

Now

$$|Z(x, t)|^2 = \sum_{n \leq x^2} \tau(n, x) n^{-\frac{1}{2} - it},$$

where $\tau(n, x)$ counts factorisations $jk$ of $n$ with $j, k \leq x$, so that $\tau(n, x) \leq \tau(n)$, and hence

$$\int_{T}^{2T} |Z(x, t)|^4 dt \ll (T + x^2) \sum_{n \leq x^2} \frac{\tau(n)^2}{n} \ll TL^4.$$

The steps are now as before, but with the averaging applied over $T^{1/2} \leq x \leq 2T^{1/2}$.

**Proof of Theorem 1**

We must replace the upper estimate (13) used in Theorem 2 by an asymptotic estimate. Recall from (8) that $|\zeta\left(\frac{1}{2} + it\right)|^2 = W(x, t) + O(L)$, where

$$W(x, t) = S(x, t) + \overline{\psi(t)}S(y, -t). \quad (16)$$

Just writing $W$ and $\zeta$, we have $|W| = |\zeta|^2 + \rho$, where $\rho = O(L)$. Hence

$$|W|^2 = |\zeta|^4 + 2\rho|\zeta|^2 + \rho^2.$$

By Theorem 2, $\int_{T}^{2T} |\zeta|^4 \ll TL^4$. So by the Cauchy-Schwarz inequality,

$$\int_{T}^{2T} |\zeta|^2 \ll T^{1/2}(T^{1/2}L^2) = TL^2.$$

(In fact, it is well known that $\int_{T}^{2T} |\zeta|^2 \ll TL$: see [Ti, p. 141].) So if

$$I(x) = \int_{T}^{2T} |W(x, t)|^2 dt,$$

then

$$I = I(x) + O(TL^3),$$

for each $x$ in $[T, 2T]$. 7
Now apply the averaging process: let
\[
A = \frac{1}{T} \int_{T}^{2T} I(x) \, dx.
\]
Then \(I = A + O(TL^3)\), so Theorem 1 will follow if we can show that
\[
A = 2KTL^4 + O(TL^3).
\] (17)

Now
\[
|W(x,t)|^2 = W(x,t)\overline{W(x,t)} = |S(x,t)|^2 + |S(y,t)|^2 + 2\text{Re} \, (\psi(t)S(x,t)S(y,t)),
\]
so
\[
I(x) = I_1(x) + I_2(x) + 2I_3(x),
\]
where \(I_1(x)\) and \(I_2(x)\) are as before, and
\[
I_3(x) = \int_{T}^{2T} \text{Re} \, (\psi(t)S(x,t)S(y,t)) \, dt.
\]
With \(A_j\) defined by (14) for \(j = 1, 2, 3\), we have
\[
A = A_1 + A_2 + 2A_3.
\]

By Lemma 3, \(I_1(x) = KTL^4 + O(TL^3)\) for each \(x\), hence
\[
A_1 = KTL^4 + O(TL^3).
\] (18)

Now consider \(A_2\). In the expression (15) for \(A_2\), we now need to specify
\[
T_1 = \max \left( T, \sqrt{\frac{Ty}{K} } \right), \quad T_2 = \min \left( 2T, \sqrt{\frac{2Ty}{K} } \right).
\]
By Lemma 4,
\[
\int_{T_1}^{T_2} t^2 |S(y,t)|^2 \, dt = KL^4 \int_{T_1}^{T_2} t^2 \, dt + O(T^3L^3).
\]
Now reversing the steps that led to (15), with \(|S(y,t)|^2\) replaced by 1, we have
\[
\frac{1}{T} \int_{\frac{T}{2}}^{4K} \int_{T_1}^{T_2} t^2 \, dt \, dy = \frac{1}{T} \int_{T}^{2T} \int_{T}^{2T} 1 \, dx \, dt = T,
\]
also
\[
\frac{1}{T} \int_{\frac{T}{2}}^{4K} \frac{K}{y^2} \, dy < \frac{2}{T^2},
\]
hence

$$A_2 = KTL^4 + O(TL^3).$$ (19)

The required estimation (17) will follow if we can establish that

$$A_3 \ll TL^3.$$ Since

$$\psi(t) = -ie^{2i\phi(t)},$$

$$\Re(\psi(t)S(x,t)S(y,t)) = \Im(e^{2i\phi(t)}S(x,t)S(y,t)).$$

Now

$$e^{2i\phi(t)}S(x,t)S(y,t) = e^{2i\phi(t)}\sum_{m \leq x} \sum_{n \leq \frac{x^k}{T}} \tau(m)\tau(n) \frac{(mn)^{1/2}}{(mn)^{1/2}} e^{if_{m,n}(t)},$$

where

$$f_{m,n}(t) = 2\phi(t) - t \log mn = t(2 \log t - 2 - 2 \log 2\pi - \log mn).$$

For a given \(n\), integration is on the interval \([T_0, 2T]\), where

$$T_0 = \max\left(T, \sqrt{\frac{nx}{K}}\right).$$

So

$$I_3(x) = \sum_{m \leq x} \sum_{n \leq \frac{x^k}{T}} \tau(m)\tau(n) \frac{(mn)^{1/2}}{(mn)^{1/2}} \Im J_{m,n},$$ (20)

where

$$J_{m,n} = \int_{T_0}^{2T} e^{if_{m,n}(t)} dt.$$

Denote by \(I_{3,1}(x)\) the contribution to \(I_3(x)\) of the terms with \(m \leq x - 1\), and by \(I_{3,2}(x)\) the contribution of the single term with \(x - 1 < m \leq x\), with corresponding averages \(A_{3,1}\) and \(A_{3,2}\).

Since \(\sum_{n \leq x} \tau(n)n^{-1/2} \ll x^{1/2} \log x\), we have

$$\sum_{n \leq \frac{x^k}{T}} \frac{\tau(n)}{n^{1/2}} \ll T^{1/2} L.$$ (21)

Consider \(A_{3,2}\) first: for this, we have \(m = [x]\). By (21) and the trivial bound \(|J_{m,n}| \leq T\),

$$I_{3,2}(x) \ll \frac{\tau(m)}{m^{1/2}} T^{3/2} L \asymp \tau(m)TL.$$ Hence

$$A_{3,2} \ll L \int_T^{2T} \tau([x]) \, dx \leq L \sum_{m \leq 2T} \tau(m) \ll TL^2.$$
(Note that we cannot deduce such an estimate for \( I_{3,2}(x) \) for a fixed \( x \), since \( \tau(m) \) is not bounded by a power of \( L \).)

Now consider \( A_{3,1} \). We use the following Lemma [Ti, Lemma 4.3] to estimate \( J_{m,n} \):

**LEMMA 6.** If \( f \) is a real-valued function with \( f'(t) \) increasing on \([a, b]\) and \( f'(a) = \mu > 0 \), then

\[
\left| \int_a^b e^{if(t)} \, dt \right| \leq \frac{2}{\mu}.
\]

**Proof.** Let \( h(t) = 1/f'(t) \). Integrating by parts, we have

\[
\int_a^b e^{if(t)} \, dt = \int_a^b h(t)f'(t)e^{if(t)} \, dt = J_1 + J_2,
\]

where

\[
J_1 = \left[ \frac{1}{t} h(t)e^{-f(t)} \right]_a^b, \quad J_2 = \int_a^b h'(t)e^{if(t)} \, dt.
\]

Clearly, \( |J_1| \leq h(a) + h(b) \), and since \( |h'(t)| = -h'(t) \),

\[
|J_2| \leq - \int_a^b h'(t) \, dt = h(a) - h(b). \quad \square
\]

We have

\[
f'_{m,n}(t) = 2 \log t - 2 \log 2 \pi - \log mn = \log \frac{Kt^2}{mn},
\]

which is clearly increasing. Also, \( KT_0^2 \geq nx \), so for \( m < x \), \( f'_{m,n}(T_0) \geq \log x/m \). Now

\[
\log \frac{x}{m} = \int_m^x \frac{1}{t} \, dt > \frac{x - m}{x},
\]

so

\[
\frac{1}{\log x/m} < \frac{x}{x - m} = 1 + \frac{m}{x - m}.
\]

Hence for each \( x \), we have

\[
I_{3,1}(x) \ll T^{1/2}L \sum_{m \leq x-1} \frac{\tau(m)}{m^{1/2}} \left( 1 + \frac{m}{x - m} \right)
\ll T^{1/2}L T^{1/2}L + T^{1/2}L \sum_{m \leq x-1} \frac{m^{1/2}\tau(m)}{x - m}
\ll TL^2 + TL \sum_{m \leq x-1} \frac{\tau(m)}{x - m}.
\]

Now integrate with respect to \( x \) on \([T, 2T]\). For a fixed \( m \), we integrate \( 1/(x - m) \) on an interval contained in \([m + 1, 2T]\): the contribution is less than \( \log 2T \). So

\[
A_{3,1} \ll TL^2 + L^2 \sum_{m < 2T} \tau(m) \ll TL^3.
\]
So $A_3 \ll TL^3$, as required.

Comparison with Ingham’s method

Ingham’s method takes $x = y = t/(2\pi)$, requiring (in our notation) the estimation of
$$\int_0^T |S(t/2\pi, t)|^2 \, dt.$$ This in turn requires the estimation
$$\sum_{n=1}^N \sum_{m<n} \frac{\tau(m)\tau(n)}{(mn)^{1/2}(\log n - \log m)} \ll N \log^3 N,$$
which is achieved by calculations specific to the divisor function. For Theorem 2, this estimation would still be needed, with bound $N \log^4 N$; this is essentially the method implied in [Ti, p. 146–7]. We remark that there is no short cut to (22) by the Montgomery-Vaughan theorems, since it really involves $|\log n - \log m|$, not $\log n - \log m$. Indeed, it is not hard to show that the best constant $C$ in the bilinear form estimate
$$\sum_{n=1}^N \sum_{m<n} \frac{x_m x_n}{\log n - \log m} \leq C \sum_{n=1}^N |x_n|^2$$
satisfies $C \sim N \log N$. This would only lead to the bound $N \log^5 N$ in (22).

References


[Jam] Tim Jameson, The approximate functional equation for $\zeta(s)^2$, at www.maths.lancs.ac.uk/~jameson


