Some double series related to $\zeta(3)$

Tim Jameson

Recall that for integers $k \geq 2$, $\zeta(k)$ is defined by

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots.$$

The value $\zeta(2) = \pi^2/6$ is very well known, and for all even k, $\zeta(k)$ can be expressed in terms of the Bernoulli numbers. However, no closed expressions are known for odd k, and in particular for $\zeta(3)$, though of course it can be calculated numerically to any desired degree of accuracy: $\zeta(3) = 1.202057$ to six d.p. It was shown by Apéry [1] that $\zeta(3)$ is irrational, and it is sometimes known as Apéry's constant. Apéry's proof is quite difficult; a simpler one was given by Beukers [2].

Here we describe a number of double series whose sums can be expressed in terms of $\zeta(3)$, demonstrating that it really is of some interest. First, let

$$S_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m(m+n)^2}.$$

$$S_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn(m+n)},$$

$$S_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn\max(m,n)},$$

We start with S_1. Substituting $m+n=r$, and then reversing the order of summation, we can rewrite it in two ways as follows:

$$S_1 = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{r=m+1}^{\infty} \frac{1}{r^2} = \sum_{r=2}^{\infty} \frac{1}{r^2} \sum_{m=1}^{r-1} \frac{1}{m}. \tag{1}$$

Also, by writing S_1 with m and n interchanged and adding the original expression, we have

$$\begin{aligned} 2S_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^2}\left(\frac{1}{m} + \frac{1}{n}\right) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn(m+n)} \\ &= S_2. \end{aligned} \tag{2}$$

Now

$$\frac{1}{mn(m+n)} = \frac{1}{m^2}\,\frac{m}{n(m+n)} = \frac{1}{m^2}\left(\frac{1}{n} - \frac{1}{m+n}\right)$$

and by cancellation it is easily seen that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{m+n}\right)=1+\frac{1}{2}+\cdots+\frac{1}{m}.$$

So

$$\begin{aligned} S_2 &= \sum_{m=1}^{\infty}\frac{1}{m^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) \\ &= \zeta(3)+\sum_{m=2}^{\infty}\frac{1}{m^2}\sum_{n=1}^{m-1}\frac{1}{n} \\ &= \zeta(3)+S_1, \end{aligned} \tag{3}$$

by (1). By (2) and (3), we arrive at the rather pleasing conclusion

$$S_1=\zeta(3), \qquad S_2=2\zeta(3). \tag{4}$$

Now consider S_3. The terms with $m=n$ contribute $\zeta(3)$. The terms with $n>m$ contribute the same as those with $m>n$, so we have

$$\begin{aligned} S_3 &= \zeta(3)+2\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m+1}^{\infty}\frac{1}{n^2} \\ &= \zeta(3)+2S_1, \end{aligned}$$

by (1) again. So, by (4), we have

$$S_3=3\zeta(3). \tag{5}$$

The identity $S_1=\zeta(3)$ was already known to Euler. It is discussed extensively in [3] (thanks to Nick Lord for providing this reference), where the quick proof given above is attributed to Steinberg [4]. Numerous other proofs are given in [3]. For example, one can equate both S_1 and $\zeta(3)$ to the integral

$$\int_0^1 \frac{\ln x\ln(1-x)}{x}\,dx$$

or to the double integral

$$-\int_0^1\int_0^1 \frac{\ln(1-xy)}{1-xy}\,dx\,dy.$$

The proof using first of these integrals was given in [5]. However, it must surely be conceded that the method above is simpler and more direct.

The article [3] also describes a number of generalisations and related results. In what is now the established notation for "Euler sums", one defines

$$\zeta(j,k)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^k(m+n)^j}=\sum_{m=1}^{\infty}\frac{1}{m^k}\sum_{r=m+1}^{\infty}\frac{1}{r^j},$$

so that our S_1 is $\zeta(2,1)$ (warning: some writers interchange the j and k). By a fairly straightforward extension of the method above (which readers might care to attempt for themselves), one can prove

$$\zeta(n) = \sum_{j=1}^{n-2} \zeta(n-j, j)$$

for $n \geq 3$. Euler's equality $S_1 = \zeta(2,1) = \zeta(3)$ is the case $n = 3$. Another variation, also known to Euler, introduces alternating signs:

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{r=m+1}^{\infty} \frac{(-1)^r}{r^2} = \tfrac{1}{8}\zeta(3);$$

see [2, p. 8–9].

Several much more exotic series expressions that equate to $\zeta(3)$ are listed on p. 6 of [3]. One that converges very rapidly, so is good for calculation, is

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

This identity was used by Apéry in his proof, and has sometimes been attributed to him, but in fact it was established by Hjortnaes in 1953 [6]. A proof can be seen in my website notes [7].

Instead of repeating any of these results and proofs here, we will now go in a more number-theoretic direction and consider some double sums involving coprime pairs or lowest common multiples. Write (a, b) for the greatest common divisor of a and b and for r, $s \geq 2$, let

$$T_{r,s} = \sum_{\substack{a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{a^r b^s}.$$

Clearly,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} = \sum_{m=1}^{\infty} \frac{1}{m^r} \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(r)\zeta(s).$$

Now write $(m, n) = g$ and $m = ag$, $n = bg$, so that $(a, b) = 1$. Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} = \sum_{\substack{g,a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{g^{r+s} a^r b^s} = \sum_{g=1}^{\infty} \frac{1}{g^{r+s}} \sum_{\substack{a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{a^r b^s} = \zeta(r+s) T_{r,s},$$

hence

$$T_{r,s} = \frac{\zeta(r)\zeta(s)}{\zeta(r+s)}. \tag{6}$$

In particular, using the known values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we have

$$T_{2,2} = \frac{\zeta(2)^2}{\zeta(4)} = \frac{\pi^4/36}{\pi^4/90} = \frac{5}{2}. \tag{7}$$

Now write $[m,n]$ for the lowest common multiple of m and n. Let

$$U=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn[m,n]}.$$

With g, a, b as above, we have $[m,n]=abg$, hence

$$U=\sum_{\substack{g,a,b=1\\(a,b)=1}}^{\infty}\frac{1}{g^3a^2b^2}=\sum_{g=1}^{\infty}\frac{1}{g^3}\sum_{\substack{a,b=1\\(a,b)=1}}^{\infty}\frac{1}{a^2b^2}=\zeta(3)T_{2,2},$$

so by (7), we have

$$U=\frac{5}{2}\zeta(3). \tag{8}$$

By obvious variations of this reasoning, one obtains, for example,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{[m,n]^2}=\zeta(2)T_{2,2}=\frac{5\pi^2}{12}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{[m,n]^3}=\zeta(3)T_{3,3}=\frac{\zeta(3)^3}{\zeta(6)}.$$

More generally, one can apply (6) to express

$$U_{r,s,t}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^rn^s[m,n]^t}$$

in terms of zeta values: we leave this to the reader.

References

1. R. Apéry, Interpolation de fractions continues et irrationalité de certaines constantes, *Math. CTHS Bull. Sec. Sci. III*, Bibl. Nat. Paris (1981), 37–53.
2. F. Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* **11** (1979), 268–272.
3. Jonathan M. Borwein and David M. Bradley, Thirty two Goldbach variations, *Int. J. Number Theory* **2** (2006), 65–103; also at: arxiv.org/abs/math/0502034, pp. 1–41.
4. R. Steinberg, Solution to problem 4431, *Amer. Math. Monthly* **59** (1952), 471–472.
5. Solution to Problem 89D, *Math. Gazette* **89** (2005), p. 541.
6. M. M. Hjortnaes, Overføring av rekken $\sum_{k=1}^{\infty}1/k^3$ til et bestemt integral, *Proc. 12th Cong. Scand. Math. (Lund, 1953)*, Lund (1954).
7. Tim Jameson, Polylogarithms, multiple zeta values and the series of Hjortnaes and Comtet, at: www.maths.lancs.ac.uk/˜jameson.

Sadly, Tim Jameson died in September 2013. This note was submitted after his death by his father, Graham Jameson.