

Some double series related to $\zeta(3)$

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Recall that for integers $k \geq 2$, $\zeta(k)$ is defined by

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots.$$

The value $\zeta(2) = \pi^2/6$ is very well known, and for all even k , $\zeta(k)$ can be expressed in terms of the Bernoulli numbers. However, no closed expressions are known for odd k , and in particular for $\zeta(3)$, though of course it can be calculated numerically to any desired degree of accuracy: $\zeta(3) = 1.202057$ to six d.p. It was shown by Apéry [1] that $\zeta(3)$ is irrational, and it is sometimes known as Apéry's constant. Apéry's proof is quite difficult; a simpler one was given by Beukers [2].

Here we describe a number of double series whose sums can be expressed in terms of $\zeta(3)$, demonstrating that it really is of some interest. First, let

$$\begin{aligned} S_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m(m+n)^2}. \\ S_2 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)}, \\ S_3 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn \max(m, n)}, \end{aligned}$$

We start with S_1 . Substituting $m+n=r$, and then reversing the order of summation, we can rewrite it in two ways as follows:

$$S_1 = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{r=m+1}^{\infty} \frac{1}{r^2} = \sum_{r=2}^{\infty} \frac{1}{r^2} \sum_{m=1}^{r-1} \frac{1}{m}. \quad (1)$$

Also, by writing S_1 with m and n interchanged and adding the original expression, we have

$$\begin{aligned} 2S_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^2} \left(\frac{1}{m} + \frac{1}{n} \right) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} \\ &= S_2. \end{aligned} \quad (2)$$

Now

$$\frac{1}{mn(m+n)} = \frac{1}{m^2} \frac{m}{n(m+n)} = \frac{1}{m^2} \left(\frac{1}{n} - \frac{1}{m+n} \right)$$

and by cancellation it is easily seen that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{m+n} \right) = 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

So

$$\begin{aligned} S_2 &= \sum_{m=1}^{\infty} \frac{1}{m^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m} \right) \\ &= \zeta(3) + \sum_{m=2}^{\infty} \frac{1}{m^2} \sum_{n=1}^{m-1} \frac{1}{n} \\ &= \zeta(3) + S_1, \end{aligned} \tag{3}$$

by (1). By (2) and (3), we arrive at the rather pleasing conclusion

$$S_1 = \zeta(3), \quad S_2 = 2\zeta(3). \tag{4}$$

Now consider S_3 . The terms with $m = n$ contribute $\zeta(3)$. The terms with $n > m$ contribute the same as those with $m > n$, so we have

$$\begin{aligned} S_3 &= \zeta(3) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=m+1}^{\infty} \frac{1}{n^2} \\ &= \zeta(3) + 2S_1, \end{aligned}$$

by (1) again. So, by (4), we have

$$S_3 = 3\zeta(3). \tag{5}$$

The identity $S_1 = \zeta(3)$ was already known to Euler. It is discussed extensively in [3] (thanks to Nick Lord for providing this reference), where the quick proof given above is attributed to Steinberg [4]. Numerous other proofs are given in [3]. For example, one can equate both S_1 and $\zeta(3)$ to the integral

$$\int_0^1 \frac{\ln x \ln(1-x)}{x} dx$$

or to the double integral

$$- \int_0^1 \int_0^1 \frac{\ln(1-xy)}{1-xy} dx dy.$$

The proof using first of these integrals was given in [5]. However, it must surely be conceded that the method above is simpler and more direct.

The article [3] also describes a number of generalisations and related results. In what is now the established notation for ‘‘Euler sums’’, one defines

$$\zeta(j, k) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^k (m+n)^j} = \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{r=m+1}^{\infty} \frac{1}{r^j},$$

so that our S_1 is $\zeta(2, 1)$ (warning: some writers interchange the j and k). By a fairly straightforward extension of the method above (which readers might care to attempt for themselves), one can prove

$$\zeta(n) = \sum_{j=1}^{n-2} \zeta(n-j, j)$$

for $n \geq 3$. Euler's equality $S_1 = \zeta(2, 1) = \zeta(3)$ is the case $n = 3$. Another variation, also known to Euler, introduces alternating signs:

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{r=m+1}^{\infty} \frac{(-1)^r}{r^2} = \frac{1}{8} \zeta(3);$$

see [2, p. 8–9].

Several much more exotic series expressions that equate to $\zeta(3)$ are listed on p. 6 of [3]. One that converges very rapidly, so is good for calculation, is

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

This identity was used by Apéry in his proof, and has sometimes been attributed to him, but in fact it was established by Hjortnaes in 1953 [6]. A proof can be seen in my website notes [7].

Instead of repeating any of these results and proofs here, we will now go in a more number-theoretic direction and consider some double sums involving coprime pairs or lowest common multiples. Write (a, b) for the greatest common divisor of a and b and for $r, s \geq 2$, let

$$T_{r,s} = \sum_{\substack{a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{a^r b^s}.$$

Clearly,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} = \sum_{m=1}^{\infty} \frac{1}{m^r} \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(r) \zeta(s).$$

Now write $(m, n) = g$ and $m = ag, n = bg$, so that $(a, b) = 1$. Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} = \sum_{\substack{g,a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{g^{r+s} a^r b^s} = \sum_{g=1}^{\infty} \frac{1}{g^{r+s}} \sum_{\substack{a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{a^r b^s} = \zeta(r+s) T_{r,s},$$

hence

$$T_{r,s} = \frac{\zeta(r) \zeta(s)}{\zeta(r+s)}. \quad (6)$$

In particular, using the known values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we have

$$T_{2,2} = \frac{\zeta(2)^2}{\zeta(4)} = \frac{\pi^4/36}{\pi^4/90} = \frac{5}{2}. \quad (7)$$

Now write $[m, n]$ for the lowest common multiple of m and n . Let

$$U = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn[m, n]}.$$

With g, a, b as above, we have $[m, n] = abg$, hence

$$U = \sum_{\substack{g, a, b=1 \\ (a, b)=1}}^{\infty} \frac{1}{g^3 a^2 b^2} = \sum_{g=1}^{\infty} \frac{1}{g^3} \sum_{\substack{a, b=1 \\ (a, b)=1}}^{\infty} \frac{1}{a^2 b^2} = \zeta(3)T_{2,2},$$

so by (7), we have

$$U = \frac{5}{2}\zeta(3). \tag{8}$$

By obvious variations of this reasoning, one obtains, for example,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{[m, n]^2} = \zeta(2)T_{2,2} = \frac{5\pi^2}{12}$$

and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{[m, n]^3} = \zeta(3)T_{3,3} = \frac{\zeta(3)^3}{\zeta(6)}.$$

More generally, one can apply (6) to express

$$U_{r,s,t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s [m, n]^t}$$

in terms of zeta values: we leave this to the reader.

References

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Sadly, Tim Jameson died in September 2013. This note was submitted after his death by his father, Graham Jameson.