Another proof that $\zeta(2) = \pi^2/6$ via double integration

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Over the years several proofs that

$$\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

using just double integrals and elementary calculus have appeared. Perhaps the following version seems more transparent than some (in particular avoiding a two-dimensional substitution). Commonly proofs are given using Fourier series or (as Euler originally used) the partial fractions expansion of the cotangent.

The substitution $t = \frac{\sqrt{1-x^2}}{x} u$ shows that

$$\begin{aligned} x \int_0^1 \frac{dt}{1 - x^2 + x^2 t^2} &= \frac{1}{\sqrt{1-x^2}} \int_0^{\frac{x}{\sqrt{1-x^2}}} \frac{du}{1+u^2} \\ &= \frac{1}{\sqrt{1-x^2}} \arctan \frac{x}{\sqrt{1-x^2}} \\ &= \frac{\arcsin x}{\sqrt{1-x^2}}. \end{aligned}$$

Since also

$$\frac{d}{dx} (\arcsin x)^2 = \frac{2 \arcsin x}{\sqrt{1-x^2}},$$

we now have

$$\begin{aligned} \left(\frac{\pi}{2}\right)^2 &= (\arcsin 1)^2 \\ &= \int_0^1 \frac{2 \arcsin x}{\sqrt{1-x^2}}\, dx \\ &= \int_0^1 \int_0^1 \frac{2x\; dx}{1 - x^2 + x^2 t^2}\, dt \\ &= \int_0^1 \left[-\frac{\log(1 - x^2 + x^2 t^2)}{1 - t^2} \right]_{x=0}^{x=1} dt \\ &= -\int_0^1 \frac{2 \log t}{1 - t^2}\, dt \\ &= -2 \sum_{n=0}^{\infty} \int_0^1 t^{2n} \log t\; dt \\ &= -2 \sum_{n=0}^{\infty} \left(\left[\frac{t^{2n+1}}{2n+1} \log t \right]_0^1 - \int_0^1 \frac{t^{2n}}{2n+1}\, dt \right) \end{aligned}$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{t^{2n+1}}{(2n+1)^2}\right]_0^1$$
$$= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

Hence

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Finally, noting that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \left(1-\frac{1}{4}\right)\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{3}{4}\zeta(2),$$

gives

$$\zeta(2) = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

as claimed.

I discovered this proof while searching for a neat derivation of the power series for $(\arcsin x)/\sqrt{1-x^2}$. Indeed, on substituting $t^2 = 1-w$ our integral representation for this becomes a standard integral (due to Euler) for its hypergeometric series.

A variant, avoiding double integrals, is as follows. Start from

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} C_n x^{2n},$$

where

$$C_n = \frac{1.3.\ldots(2n-1)}{2.4.\ldots(2n)}.$$

Integration gives

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \sum_{n=0}^{\infty}\frac{C_n}{2n+1}x^{2n+1}.$$

Hence

$$\frac{\pi^2}{4} = \int_0^1 \frac{2\arcsin x}{\sqrt{1-x^2}}\,dx = 2\sum_{n=0}^{\infty}\frac{C_n}{2n+1}J_n,$$

where

$$J_n = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx.$$

Substituting $x = \sin\theta$, we have $J_n = \int_0^{\pi/2}\sin^{2n+1}\theta\,d\theta$, which (as is well known) equates to $1/[(2n+1)C_n]$. Very satisfyingly, C_n cancels, giving (again)

$$\frac{\pi^2}{4} = 2\sum_{n=0}^{\infty}\frac{1}{2n+1)^2}.$$

Given the multitude of proofs already known, I initially resisted publication – especially after I realised that my proof is in essence the same as Nick Lord's in [1], the substitutions $t = e^{-u}$ and $x = 1/\sqrt{1+y^2}$ transforming my double integral expression into his (writing y for his x).

There are also close similarities to Apostol's proof [2], which is the first appearing in Robin Chapman's collection of fourteen proofs [3].

I thank Nick Lord and my father (Graham Jameson) for pressing me to finally publish, and apologise to Robin Chapman for burdening him with yet more proofs.

References

1. Nick Lord, Yet another proof that $\sum \frac{1}{n^2} = \frac{1}{6}\pi^2$, *Math. Gaz.* **86** (2002) pp. 477-479.
2. Tom M. Apostol, A proof that Euler missed: evaluating $\zeta(2)$ the easy way, *Math. Intelligencer* **5** (1983), 59–60.
3. Robin Chapman, Evaluating $\zeta(2)$, http://empslocal.ex.ac.uk/people/staff/rjchapma/rjc.html