# Evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by a double integral

Graham Jameson and Nick Lord (*Math. Gazette* **97** (2013), 504–505

Let $\sum_{n=1}^{\infty} \frac{1}{n^2} = S$. Numerous proofs are known of the well-known identity $S = \pi^2/6$. Robin Chapman's website [1] contains a very useful collation of 14 such proofs. Here we give a short one based on a certain double integral. The first two proofs in [1] give alternative double integral proofs from [2, 3] and another can be seen in [4]. Note first that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \tfrac{3}{4}S.$$

We consider the double integral $I = \int_0^1 J(x)\,dx$, where

$$J(x) = \int_0^{\pi/2} \frac{\sin\theta}{(1 - x^2\sin^2\theta)^{1/2}}\,d\theta.$$

First we evaluate $J(x)$. Let $x^* = (1-x^2)^{1/2}$, so that $x^2 + x^{*2} = 1$. Then $1 - x^2\sin^2\theta = x^{*2} + x^2\cos^2\theta$. Substitute $t = x\cos\theta$ to obtain

$$J(x) = \frac{1}{x}\int_0^x \frac{1}{(x^{*2}+t^2)^{1/2}}\,dt = \frac{1}{x}\sinh^{-1}\frac{x}{x^*}.$$

Now $\sinh^{-1} u = \log[u + (1+u^2)^{1/2}]$ and $1 + x^2/x^{*2} = 1/x^{*2}$, so

$$J(x) = \frac{1}{x}\log\frac{1+x}{x^*}.$$

Also,

$$\frac{1+x}{x^*} = \frac{1+x}{(1-x^2)^{1/2}} = \frac{(1+x)^{1/2}}{(1-x)^{1/2}}.$$

By the well-known power series for $\log(1+x)$, we deduce that

$$J(x) = \frac{1}{2x}\log\frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1}.$$

Termwise integration now gives

$$I = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \tfrac{3}{4}S.$$

Now reverse the double integral. The substitution $u = x\sin\theta$ gives

$$\int_0^1 \frac{\sin\theta}{(1-x^2\sin^2\theta)^{1/2}}\,dx = \int_0^{\sin\theta} \frac{1}{(1-u^2)^{1/2}}\,du = \sin^{-1}(\sin\theta) = \theta,$$

so that

$$I = \int_0^{\pi/2} \theta \, d\theta = \frac{\pi^2}{8},$$

hence $S = \pi^2/6$.

*References*


1. R. J. Chapman, Evaluating $\zeta(2)$, http://empslocal.ex.ac.uk/people/staff/rjchapma/rjc.html
2. T. M. Apostol, A proof that Euler missed: evaluating $\zeta(2)$ the easy way, *Math. Intelligencer* **5** (1983), 59–60.
3. F. Beukers, J. A. C. Kolk and E. Calabi, Sums of generalised harmonic series and volumes, *Nieuw Archief voor Wiskunde* **4** (1993), 217–224.
4. N. Lord, Yet another proof that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, *Math. Gazette* **86** (2002), 477–479.