LINNIK'S PROOF OF THE WARING-HILBERT THEOREM FROM HUA'S BOOK

(with a correction)

Notes by Tim Jameson

For integers $s \geq 1$, $k \geq 2$ and $n \geq 0$, let $r_s^{(k)}(n)$ denote the number of solutions $(n_1, \ldots, n_s)$ of the equation $n_1^k + \cdots + n_s^k = n$ with $n_1, \ldots n_s \geq 0$.

The fact that $r_4^{(2)}(n) > 0$ for all n was probably already suspected by Diophantus (c.250). It was stated explicitly by Bachet in 1621, and later Fermat claimed to have a proof. The first accepted proof was by Lagrange in 1770, building on work of Euler.

Also in 1770, Waring wrote a letter to Euler in which he asserted that every natural number is a sum of 4 squares, 9 cubes, 19 biquadrates "and so on". By this it is usually assumed that he meant that for each k there exists an s such that $r_s^{(k)}(n) > 0$ for all n. We let $g(k)$ denote the least such value of s. The problem of showing that $g(k) < \infty$ for all k has become known as Waring's problem. It was first solved by Hilbert in 1909, by a complicated method [Hil]. Hardy and Littlewood [HL] gave a more elegant proof by the "circle method" in 1919, which was then refined and simplified by Vinogradov [Vin].

An "elementary" proof, using only number-theoretic methods, was given by Linnik in 1943 [Lin]. This method is presented in [Hua], but with one serious mistake. Here I give a version with this mistake corrected. A more general version, with x^k replaced by a polynomial, is given in [Nath].

Linnik's method depends on the notion of *Shnirelman density*, defined as follows. Let A be a set of non-negative integers (possibly including 0). For each $n \geq 1$, let $A(n)$ be the number of $a \in A$ with $1 \leq a \leq n$. The Shnirelman density is

$$\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n}.$$

The application to Waring's problem is via the following Lemma.

LEMMA 1. *If A contains 0 and has positive Shnirelman density, then there exists h such that every positive integer is expressible as the sum of h members of A.*

We omit the proof of this Lemma, which is quite straightforward. See [Nath, Theorem 11.4] or [Hua, Theorem 19.2.3].

So it will be enough to prove that for given k, there exists s such that $\{n : r_s^{(k)}(n) > 0\}$ has positive Shnirelman density.

LEMMA 2. *Let*

$$q(n; h_1, h_2) = \sum_{\substack{m_1, m_2 = -M \\ h_1 m_1 + h_2 m_2 = n}}^{M} 1.$$

For $(h_1, h_2) \neq (0, 0)$ *we have* $q(n; h_1, h_2) = 0$ *unless* $g = (h_1, h_2) | n$, *say* $h_1 = g a_1$, $h_2 = g a_2$ *(so* $(a_1, a_2) = 1$*). Then*

$$q(n; h_1, h_2) \leq \frac{2M}{\max(|a_1|, |a_2|)} + 1.$$

Proof. Say $n = gf$. Then the equation becomes

$$a_1 m_1 + a_2 m_2 = f.$$

On writing $a_1 \bar{a}_1 + a_2 \bar{a}_2 = 1$, we may write this as

$$a_1(m_1 - f\bar{a}_1) + a_2(m_2 - f\bar{a}_2) = 0.$$

Thus the general solution has the form

$$m_1 = f\bar{a}_1 + k a_2,$$

$$m_2 = f\bar{a}_2 - k a_1.$$

Wlog we may suppose $a_1 \geq a_2 \geq 0$. This implies $a_1 > 0$ and

$$q(n; h_1, h_2) = \sum_{\substack{k: \\ -M - f\bar{a}_1 \leq k a_2 \leq M - f\bar{a}_1 \\ -M + f\bar{a}_2 \leq k a_1 \leq M + f\bar{a}_2}} 1.$$

By the second condition, k is constrained to lie in an interval of length $2M/a_1$. Such an interval contains at most $2M/a_1 + 1$ integers, hence the statement. Note this all works fine if $n = 0$. □

LEMMA 3. *Let*

$$q(n) = \sum_{\substack{h_1, h_2 = -H \\ h_1, h_2 \neq 0}}^{H} \sum_{\substack{m_1, m_2 = -M \\ h_1 m_1 + h_2 m_2 = n}}^{M} 1.$$

Then

$$q(n) \leq \begin{cases} 20 H M \sigma_{-1}(n) & (n \neq 0,\ H \leq M), \\ 20 H^2 M & (n = 0). \end{cases}$$

Proof. We have

$$\begin{aligned} q(n) &= \sum_{\substack{h_1,h_2=-H \\ h_1,h_2\neq 0}}^{H} q(n;h_1,h_2) \\ &= 4\sum_{h_1,h_2=1}^{H} q(n;h_1,h_2) \\ &\leq 4H^2 + 8M \sum_{\substack{g|n \\ g\leq H}} \sum_{\substack{1\leq a_1\leq H/g \\ 0\leq a_2\leq H/g}} \frac{1}{\max(a_1,a_2)}. \end{aligned}$$

We could have said $a_2 \geq 1$ here, but have allowed $a_2 = 0$ also (since it naturally leads to no worse an estimate and) so that the following applies to the variant of this lemma required for Lemma 7b:

$$\begin{aligned} q(n) &\leq 4H^2 + 8M \sum_{\substack{g|n \\ g\leq H}} \left(\sum_{1\leq a_1\leq H/g} \sum_{0\leq a_2<a_1} \frac{1}{a_1} + \sum_{1\leq a_2\leq H/g} \sum_{1\leq a_1\leq a_2} \frac{1}{a_2} \right) \\ &= 4H^2 + 16M \sum_{\substack{g|n \\ g\leq H}} \sum_{1\leq a\leq H/g} 1 \\ &\leq 4H^2 + 16HM \sum_{\substack{g|n \\ g\leq H}} \frac{1}{g}, \end{aligned}$$

and the result follows. □

Although we have thrown away a lot in the case $n = 0$ (we could give the result as $4H^2 + 16HM\log(eH)$) this will have little effect in the application.

The following lemmas are of some interest in their own right.

LEMMA 4. *We have*

$$\sum_{\substack{a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{a^2b^2} = \frac{5}{2}.$$

Proof. Denote the sum by S. We have

$$\sum_{d,e=1}^{\infty} \frac{1}{d^2e^2} = \zeta(2)^2.$$

Now write $(d,e) = g$ and $d = ga$, $e = gb$, so that $(a,b) = 1$. Then

$$\sum_{d,e=1}^{\infty} \frac{1}{d^2e^2} = \sum_{\substack{g,a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{g^4a^2b^2} = \zeta(4)S.$$

Hence

$$S = \frac{\zeta(2)^2}{\zeta(4)} = \frac{\pi^4/36}{\pi^4/90} = \frac{5}{2}. \qquad \square$$

LEMMA 5. *We have*

$$\sum_{d,e=1}^{\infty} \frac{1}{de[d,e]} = \frac{5}{2}\zeta(3).$$

Proof. Denote the sum by C. With g, a, b as above, we have $[d,e] = gab$, hence

$$C = \sum_{\substack{g,a,b=1\\(a,b)=1}}^{\infty} \frac{1}{g^3a^2b^2} = \zeta(3)S. \qquad \square$$

LEMMA 6. *We have*

$$\sum_{n\le x} \sigma_{-1}(n)^2 \le \frac{5}{2}\zeta(3)x.$$

Proof. This sum is

$$\begin{aligned}\sum_{n\le x}\sum_{d,e|n}\frac{1}{de} &= \sum_{d,e\le x}\frac{1}{de}\sum_{\substack{n\le x\\ n\equiv 0 \bmod [d,e]}} 1\\ &\le \sum_{d,e\le x}\frac{x}{de[d,e]}\\ &\le Cx,\end{aligned}$$

where C is as in Lemma 5. $\square$

A few further estimates show that in fact

$$\sum_{n\le x}\sigma_{-1}(n)^2 = \frac{5}{2}\zeta(3)x + O(\log^2 x).$$

LEMMA 7 (for the inductive step in Theorem 1). *For $H \le M$ we have*

$$\sum_{\substack{h_1,\dots,h_4=-H\\ h_1,\dots,h_4\ne 0\\ h_1m_1+h_2m_2=h_3m_3+h_4m_4}}^{H} \sum_{m_1,\dots,m_4=-M}^{M} 1 \le 5250(HM)^3.$$

Proof. The LHS is

$$\begin{aligned}\sum_{n=-2HM}^{2HM} q(n)^2 &\le 20^2\left(H^4M^2 + 2H^2M^2\sum_{n=1}^{2HM}\sigma_{-1}(n)^2\right)\\ &\le 20^2\left(H^4M^2 + 2H^2M^2\cdot\frac{5}{2}\zeta(3)\cdot 2HM\right)\\ &= 20^2\left(H^4M^2 + 10\zeta(3)H^3M^3\right)\\ &\le 20^2(1+10\zeta(3))(HM)^3.\end{aligned}$$

Calculation shows that $20^2(1+10\zeta(3)) \approx 5208$.

LEMMA 7b (irritating variant needed for Lemma 8). *For $2H \le M$ we have*

$$\sum_{\substack{h_1,\ldots,h_4=-H \\ h_1m_1+h_2m_2=h_3m_3+h_4m_4}}^{H} \sum_{m_1,\ldots,m_4=-M}^{M} 1 \le 162M^4 + 5250(HM)^3.$$

Proof. Let

$$\begin{aligned} Q(n) &= \sum_{h_1,h_2=-H}^{H} \sum_{\substack{m_1,m_2=-M \\ h_1m_1+h_2m_2=n}}^{M} 1 \\ &= \sum_{h_1,h_2=-H}^{H} q(n;h_1,h_2) \\ &= q(n;0,0) + 4\sum_{\substack{1\le h_1\le H \\ 0\le h_2\le H}} q(n;h_1,h_2). \end{aligned}$$

For $n \ne 0$ we have $q(n;0,0)=0$ and obtain the same bound for $Q(n)$ as that for $q(n)$ given by Lemma 2: In the working of Lemma 2 the $8H^2$ is replaced by $8H(H+1)$, but since $H < M$ we can still say $8H(H+1) \le 8HM$.

However, in the case $n=0$ we have an additional term $q(n;0,0) = (2M+1)^2 \le 9M^2$. Thus we have

$$Q(0) \le 9M^2 + 20H^2M,$$

and so

$$\begin{aligned} Q(0)^2 &\le 2(9M^2)^2 + 2(20H^2M)^2 \\ &= 162M^4 + 20^2H^3M^2 \cdot 2H \\ &\le 162M^4 + 20^2(HM)^3. \end{aligned}$$

The working of Lemma 7 now gives

$$\sum_{\substack{h_1,\ldots,h_4=-H \\ h_1m_1+h_2m_2=h_3m_3+h_4m_4}}^{H} \sum_{m_1,\ldots,m_4=-M}^{M} 1 = \sum_{n=-2HM}^{2HM} Q(n)^2 \le 162M^4 + 5250(HM)^3,$$

as required. □

LEMMA 8 (case $k=2$ of Theorem 1). *Let*

$$f(n) = a_2n^2 + a_1n$$

where a_2, a_1 are integers with

$$0 < |a_2| \le c_2, \quad |a_1| \le c_1 N.$$

Then for $N \ge 1$ we have

$$\int_0^1 \left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^8 d\alpha \le C N^6$$

where

$$C = 162(2c_2 + c_1)^4 + 5250(2c_2 + c_1)^3.$$

In particular $C = 44592$ when $f(n) = n^2$, $c_2 = 1$, $c_1 = 0$.

Proof. We have

$$\begin{aligned}\int_0^1 \left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^8 d\alpha &= \sum_{n_1,\dots,n_8=0}^{N} \int_0^1 e(\alpha(f(n_1) + \cdots + f(n_4) - f(n_5) - \cdots - f(n_8)))\, d\alpha \\ &= \sum_{\substack{n_1,\dots,n_8=0 \\ f(n_1)+\cdots+f(n_4)=f(n_5)+\cdots+f(n_8)}}^{N} 1.\end{aligned}$$

We may write the equation here as

$$\begin{aligned}\sum_{i=1}^{4}(f(n_i) - f(n_{i+4})) &= \sum_{i=1}^{4}(a_2(n_i^2 - n_{i+4}^2) + a_1(n_i - n_{i+4})) \\ &= \sum_{i=1}^{4} h_i m_i \\ &= 0,\end{aligned}$$

where

$$\begin{aligned} h_i &= n_i - n_{i+4} \\ m_i &= a_2(n_i + n_{i+4}) + a_1.\end{aligned}$$

Note that (h_i, m_i) uniquely determines (n_i, n_{i+4}) since

$$\begin{pmatrix} 1 & -1 \\ a_2 & a_2 \end{pmatrix} \begin{pmatrix} n_i \\ n_{i+4} \end{pmatrix} = \begin{pmatrix} h_i \\ m_i - a_1 \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} n_i \\ n_{i+4} \end{pmatrix} = \frac{1}{2a_2} \begin{pmatrix} a_2 & 1 \\ -a_2 & 1 \end{pmatrix} \begin{pmatrix} h_i \\ m_i - a_1 \end{pmatrix}$$

for $a_2 \ne 0$. Clearly we have

$$|h_i| \le N$$

and

$$|m_i| \leq M, \text{ where } M = (2c_2 + c_1)N.$$

Noting that $M \geq 2N$ since $c_2 \geq |a_2| \geq 1$, the result follows from Lemma 6b.

THEOREM 1. *Let $k \geq 2$ and*

$$f(n) = a_k n^k + \cdots + a_1 n$$

where $a_1, \ldots a_k$ are integers with $a_k \neq 0$ and

$$|a_j| \leq c_{j,k} N^{k-j}.$$

Then for $N \geq 1$ we have

$$\int_0^1 \left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^{8^{k-1}} d\alpha \ll_{k, c_{1,k}, \ldots, c_{k,k}} N^{8^{k-1}-k}.$$

Proof. The statement is more than we need for the application. It has been elaborated to make its proof by induction on k work. The case $k = 2$ is given by Lemma 8. We will omit the suffices in the $\ll$ notation. Suppose the statement is true with $k-1$ in place of k. We have

$$\left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^2 = \sum_{m,n=0}^{N} e(\alpha f(m) - \alpha f(n)) = N + 1 + \sum_{\substack{h=-N \\ h \neq 0}}^{N} b_h, \tag{1}$$

where

$$b_h = \sum_{\substack{m,n=0 \\ m-n=h}}^{N} e(\alpha f(m) - \alpha f(n)) = \sum_{n=\max(0,-h)}^{\min(N,N-h)} e(\alpha h \phi(n,h)),$$

where

$$\begin{aligned} \phi(n,h) &= \frac{1}{h}(f(n+h) - f(n)) \\ &= \frac{1}{h} \sum_{j=1}^{k} a_j((n+h)^j - n^j) \\ &= \sum_{j=1}^{k} a_j \sum_{r=0}^{j-1} \binom{j}{r} h^{j-r-1} n^r \\ &= \sum_{r=0}^{k-1} \left(\sum_{j=r+1}^{k} \binom{j}{r} a_j h^{j-r-1} \right) n^r \end{aligned}$$

is a degree $k-1$ polynomial in n. From the definition we see that $\phi(n,h) \ll N^{k-1}$. The coefficient of n^{k-1} in $\phi(n,h)$ is

$$\binom{k}{k-1} a_k h^{k-(k-1)-1} = k a_k \neq 0,$$

and the coefficient of n^r is

$$\begin{aligned}\sum_{j=r+1}^{k}\binom{j}{r}a_j h^{j-r-1} &\ll \sum_{j=r+1}^{k}\binom{j}{r}N^{k-j}N^{j-r-1}\\ &\ll N^{k-r-1}.\end{aligned}$$

Raising (1) to the power 8^{k-2} using Hölder's inequality gives

$$\begin{aligned}\left|\sum_{n=0}^{N}e(\alpha f(n))\right|^{2\cdot 8^{k-2}} &\ll N^{8^{k-2}}+\left|\sum_{\substack{h=-N\\h\neq 0}}^{N}b_h\right|^{8^{k-2}}\\ &\ll N^{8^{k-2}}+\left(\sum_{\substack{h=-N\\h\neq 0}}^{N}1\right)^{8^{k-2}-1}\sum_{\substack{h=-N\\h\neq 0}}^{N}|b_h|^{8^{k-2}}\\ &\ll N^{8^{k-2}}+N^{8^{k-2}-1}\sum_{\substack{h=-N\\h\neq 0}}^{N}|b_h|^{8^{k-2}}.\end{aligned}$$

Raising this to a further fourth power and integrating over α then gives

$$\int_0^1\left|\sum_{n=0}^{N}e(\alpha f(n))\right|^{8^{k-1}}d\alpha \ll N^{4\cdot 8^{k-2}}+N^{4\cdot 8^{k-2}-4}\int_0^1\left(\sum_{\substack{h=-N\\h\neq 0}}^{N}|b_h|^{8^{k-2}}\right)^4 d\alpha. \qquad (2)$$

As a function of α, b_h has period $1/|h|$. Let $|b_h|^{8^{k-2}}$ have the Fourier series

$$|b_h|^{8^{k-2}} = \sum_{m=-\infty}^{\infty}A(m,h)e(\alpha hm).$$

This is finite really because

$$A(m,h)\neq 0 \Rightarrow m \ll \max_{0\le n\le N}|\phi(n,h)| \ll N^{k-1},$$

so we may write the range for m as $|m|\le CN^{k-1}$ (where C is independent of h). The coefficients are given by

$$\begin{aligned}A(m,h) &= \int_0^1 |b_h|^{8^{k-2}}e(-\alpha hm)\,d\alpha\\ &= \int_0^{|h|}\left|\sum_{n=\max(0,-h)}^{\min(N,N-h)}e((\mathrm{sgn}\ h)\beta\phi(n,h))\right|^{8^{k-2}}e(-(\mathrm{sgn}\ h)\beta m)\,\frac{d\beta}{|h|}\\ &= \int_0^1\left|\sum_{n=\max(0,-h)}^{\min(N,N-h)}e(\beta\phi(n,h))\right|^{8^{k-2}}e(-(\mathrm{sgn}\ h)\beta m)\,d\beta.\end{aligned}$$

Thus

$$\begin{aligned} |A(m,h)| &\le \int_0^1 \left| \sum_{n=\max(0,-h)}^{\min(N,N-h)} e(\beta\phi(n,h)) \right|^{8^{k-2}} d\beta \\ &\ll N^{8^{k-2}-(k-1)}, \end{aligned}$$

by the inductive hypothesis. (We have trivially translated the region of summation over n. This should be written in as part of the official statement or perhaps done away with by restricting to $h > 0$ using a $2\Re$.) Now we have

$$\begin{aligned} \int_0^1 \left(\sum_{\substack{h=-N \\ h\neq 0}}^{N} |b_h|^{8^{k-2}} \right)^4 d\alpha &= \int_0^1 \left(\sum_{\substack{h=-N \\ h\neq 0}}^{N} \sum_{|m|\le CN^{k-1}} A(m,h)e(\alpha hm) \right)^4 d\alpha \\ &= \sum_{\substack{h_1,\ldots,h_4=-N \\ h_1,\ldots,h_4\neq 0}}^{N} \sum_{|m_1|,\ldots,|m_4|\le CN^{k-1}} \left(\prod_{i=1}^4 A(m_i,h_i) \right) \int_0^1 e\left(\alpha \sum_{i=1}^4 h_i m_i \right) d\alpha \\ &= \sum_{\substack{h_1,\ldots,h_4=-N \\ h_1,\ldots,h_4\neq 0 \\ h_1m_1+h_2m_2+h_3m_3+h_4m_4=0}}^{N} \sum_{|m_1|,\ldots,|m_4|\le CN^{k-1}} \prod_{i=1}^4 A(m_i,h_i) \\ &\ll N^{4(8^{k-2}-(k-1))} \sum_{\substack{h_1,\ldots,h_4=-N \\ h_1,\ldots,h_4\neq 0 \\ h_1m_1+h_2m_2+h_3m_3+h_4m_4=0}}^{N} \sum_{|m_1|,\ldots,|m_4|\le CN^{k-1}} 1 \\ &\ll N^{4(8^{k-2}-(k-1))} N^{3k} \\ &= N^{4\cdot 8^{k-2}-k+4}, \end{aligned}$$

by Lemma 7 (note that we may assume $C \ge 1$). So finally (2) gives

$$\begin{aligned} \int_0^1 \left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^{8^{k-1}} d\alpha &\ll N^{4\cdot 8^{k-2}} + N^{4\cdot 8^{k-2}-4} N^{4\cdot 8^{k-2}-k+4} \\ &\ll N^{4\cdot 8^{k-2}} + N^{8^{k-1}-k}. \end{aligned}$$

The fact that the second term dominates is equivalent to $8^k \ge 16k$, which is certainly true for $k \ge 3$ so completing the proof. □

THEOREM 2. *Let $k \ge 2$ and $s = 8^{k-1}$. Then the set $A = \{n \ge 1 : r_s^{(k)}(n) > 0\}$ has positive Schnirelmann density.*

Proof. Write $r(n)$ for $r_s^{(k)}(n)$. We have

$$\sum_{n=0}^{N} r(n) = \sum_{\substack{m_1,\ldots,m_s\ge 0 \\ m_1^k+\cdots+m_s^k\le N}} 1$$

$$
\begin{aligned}
&\geq \sum_{0\leq m_1,\ldots,m_s\leq (N/s)^{1/k}} 1 \\
&\geq (N/s)^{s/k} \\
&\gg_k N^{s/k},
\end{aligned}
$$

But on the other hand, for $n \geq 1$,

$$
\begin{aligned}
r(n) &= \sum_{0\leq m_1,\ldots,m_s\leq n^{1/k}} \int_0^1 e(\alpha(m_1^k+\cdots+m_s^k-n))\,d\alpha \\
&= \int_0^1 \left(\sum_{0\leq m\leq n^{1/k}} e(\alpha m^k) \right)^s e(-\alpha n)\,d\alpha \\
&\leq \int_0^1 \left| \sum_{0\leq m\leq n^{1/k}} e(\alpha m^k) \right|^s d\alpha \\
&\ll_k (n^{1/k})^{s-k} \quad \text{(by Theorem 1)} \\
&= n^{s/k-1},
\end{aligned}
$$

so that

$$
\sum_{n=0}^{N} r(n) \ll_k 1 + N^{s/k-1} A(N).
$$

The statement clearly follows. □

By Lemma 1, we can deduce at once:

THEOREM 3 (the Waring-Hilbert theorem). *For each $k \geq 2$, there exists $s(k)$ such that $r_s(n) > 0$ for all $n \geq 0$.*

The mistake in [Hua] is the claim that $Q(0) \ll \min(H^2M, M^2H) \ll (HM)^{3/2}$, which is wrong because there are $(2M+1)^2$ solutions with $h_1 = h_2 = 0$. This leads to the incorrect bound $\ll (HM)^3$ (regardless of the relative sizes of H and M) for the quantity in Lemma 6b. This is wrong if M is much bigger than H (as it is in the application to the inductive step in Theorem 1) since $Q(0)^2$ then dominates. The way I've corrected it is simply to note that $h_1, \ldots, h_4 = 0$ does not occur in Theorem 1 so we can use Lemma 7 instead.

References *(added by Graham Jameson)*

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