

Four methods for a trigonometric integral

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It can be very illuminating to compare alternative methods for the same problem. Here we consider the evaluation of

$$I(a) = \int_{-\pi}^{\pi} \frac{1}{1 - a \cos \theta} d\theta,$$

where $|a| < 1$. We describe four completely different methods; undoubtedly, many more are possible. There is no need to try to choose the “best” one. Instead, the reader is invited to appreciate the contrasting merits of all of them, and to enjoy the round trip of mathematical ideas encountered. Methods 1 and 3 are very standard exercises of their type, while the other two are a little less routine.

Preliminary remark: The substitution $\theta = \pi - \phi$ shows easily that $I(-a) = I(a)$. Actually, only Method 3 will require this observation; the others deliver it automatically.

Method 1: integration by substitution. This is perhaps the most “elementary” method. First, substitute $\theta = 2\phi$ to obtain

$$I(a) = \int_{-\pi/2}^{\pi/2} \frac{2}{1 + a - 2a \cos^2 \phi} d\phi.$$

Now substitute $\tan \phi = t$, so that $\sec^2 \phi = 1 + t^2$ and $\sec^2 \phi d\phi = dt$. We obtain

$$I(a) = \int_{-\pi/2}^{\pi/2} \frac{2 \sec^2 \phi}{(1 + a) \sec^2 \phi - 2a} d\phi = \int_{-\infty}^{\infty} \frac{2}{(1 + a)t^2 + (1 - a)} dt.$$

Write $b^2 = (1 - a)/(1 + a)$. Then

$$\begin{aligned} I(a) &= \frac{2}{1 + a} \int_{-\infty}^{\infty} \frac{1}{t^2 + b^2} dt \\ &= \frac{2}{1 + a} \left[\frac{1}{b} \tan^{-1} \frac{t}{b} \right]_{-\infty}^{\infty} \\ &= \frac{2}{1 + a} \frac{\pi}{b} \\ &= \frac{2\pi}{(1 - a^2)^{1/2}}, \end{aligned}$$

since $(1 + a)b = (1 + a)^{1/2}(1 - a)^{1/2} = (1 - a^2)^{1/2}$.

At the cost of less transparency, the two substitutions can be combined by putting $t = \tan \frac{1}{2}\theta$, but we will refrain from claiming this as a fifth method!

Method 2: the real geometric series. By the geometric series,

$$\frac{1}{1 - a \cos \theta} = \sum_{n=0}^{\infty} a^n \cos^n \theta.$$

Let $I_n = \int_{-\pi}^{\pi} \cos^n \theta d\theta$. Writing $\cos^n \theta = \cos^{n-1} \theta \cos \theta$ and integrating by parts, we have for $n \geq 2$

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \cos^{n-1} \theta \cos \theta d\theta \\ &= \left[\cos^{n-1} \theta \sin \theta \right]_{-\pi}^{\pi} - (n-1) \int_{-\pi}^{\pi} \cos^{n-2} \theta (-\sin \theta) \sin \theta d\theta \\ &= 0 + (n-1) \int_{-\pi}^{\pi} \cos^{n-2} \theta (1 - \cos^2 \theta) d\theta \\ &= (n-1)(I_{n-2} - I_n), \end{aligned}$$

hence $I_n = \frac{n-1}{n} I_{n-2}$. Now $I_0 = \int_{-\pi}^{\pi} 1 d\theta = 2\pi$ and $I_1 = \int_{-\pi}^{\pi} \cos \theta d\theta = 0$, hence $I_{2n-1} = 0$ for all n and

$$I_{2n} = 2\pi \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)} = \frac{2\pi}{n!} \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \dots \left(n - \frac{1}{2}\right) = 2\pi (-1)^n \binom{-\frac{1}{2}}{n}.$$

So by the binomial series,

$$I(a) = 2\pi \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} a^{2n} = 2\pi (1 - a^2)^{-1/2}.$$

Method 3: Cauchy integral calculus. Let C be the circle $C(0, 1)$. One circuit of C is described by $z = e^{i\theta} = \cos \theta + i \sin \theta$ for $-\pi \leq \theta \leq \pi$. Then $z + \frac{1}{z} = 2 \cos \theta$ and $\frac{dz}{d\theta} = ie^{i\theta} = iz$, so by the definition of a contour integral, we have

$$\begin{aligned} I(a) &= \int_C \frac{1}{1 - \frac{a}{2}(z + \frac{1}{z})} \frac{1}{iz} dz \\ &= \frac{1}{i} \int_C \frac{2}{2z - az^2 - a} dz. \end{aligned}$$

Now

$$az^2 - 2z + a = a \left(z^2 - \frac{2}{a}z + 1 \right) = a(z - z_1)(z - z_2),$$

where

$$z_1 = \frac{1}{a} + \left(\frac{1}{a^2} - 1 \right)^{1/2}, \quad z_2 = \frac{1}{a} - \left(\frac{1}{a^2} - 1 \right)^{1/2}.$$

As mentioned in the preliminary remark, $I(-a) = I(a)$, so it is enough to consider $a > 0$. Then $z_1 > 1$. Also, $z_1 z_2 = 1$, so $0 < z_2 < 1$, hence z_2 is inside C . So

$$I(a) = \frac{1}{i} \int_C \frac{f(z)}{z - z_2} dz,$$

where $f(z) = 2/a(z_1 - z)$, which is holomorphic on a disc containing C . By Cauchy's integral formula,

$$I(a) = 2\pi f(z_2) = \frac{4\pi}{a(z_1 - z_2)}.$$

Now

$$a(z_1 - z_2) = 2a \left(\frac{1}{a^2} - 1 \right)^{1/2} = 2(1 - a^2)^{1/2},$$

hence $I(a) = 2\pi/(1 - a^2)^{1/2}$.

Method 4: the complex geometric series. For complex z with $|z| < 1$, we have from the geometric series

$$\frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n.$$

With $z = re^{i\theta}$, this becomes

$$D_r(\theta) = 1 + 2 \sum_{n=1}^{\infty} r^n e^{in\theta},$$

for $|r| < 1$, where

$$D_r(\theta) = \frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{(1 + re^{i\theta})(1 - re^{-i\theta})}{F_r(\theta)} = \frac{(1 - r^2) + 2ir \sin \theta}{F_r(\theta)},$$

in which

$$F_r(\theta) = (1 - re^{i\theta})(1 - re^{-i\theta}) = 1 - 2r \cos \theta + r^2.$$

Taking the real part, we have

$$\frac{1 - r^2}{F_r(\theta)} = 1 + 2 \sum_{n=1}^{\infty} r^n \cos n\theta. \quad (1)$$

(This function is called the *Poisson kernel*; it has further applications in Fourier analysis).

Since $\int_{-\pi}^{\pi} \cos n\theta \, d\theta = 0$ for integers $n \neq 0$, we now have at once

$$\int_{-\pi}^{\pi} \frac{1}{F_r(\theta)} \, d\theta = \frac{2\pi}{1 - r^2}. \quad (2)$$

No substitutions, reduction formulae or Cauchy theorems!

It just remains to translate from $F_r(\theta)$ to $1 - a \cos \theta$. Now $F_r(\theta) = (1 + r^2)(1 - a \cos \theta)$, where $a(1 + r^2) = 2r$. If a (with $|a| < 1$) given, we can solve this quadratic for r as in Method 3. But the following is rather more elegant: write $g(r) = 2r - ar^2$. Then $g(1) = 2 - a > a$ and $g(-1) = -2 - a < a$, so by the intermediate value theorem, there exists r in $(-1, 1)$ with $g(r) = a$. So, with this r ,

$$\int_{-\pi}^{\pi} \frac{1}{1 - a \cos \theta} \, d\theta = \int_{-\pi}^{\pi} \frac{1 + r^2}{F_r(\theta)} \, d\theta = 2\pi \frac{1 + r^2}{1 - r^2}.$$

Since $a = 2r/(1 + r^2)$, we have

$$1 - a^2 = \frac{(1 + r^2)^2 - 4r^2}{(1 + r^2)^2} = \frac{(1 - r^2)^2}{(1 + r^2)^2},$$

so $I(a) = 2\pi/(1 - a^2)^{1/2}$.

A minor variant is as follows. By considering

$$C_r(\theta) = \frac{1}{1 - re^{i\theta}} = \sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{(1 - r \cos \theta) + ir \sin \theta}{F_r(\theta)},$$

(the ‘‘Cauchy kernel’’), we obtain similarly

$$\int_{-\pi}^{\pi} \frac{1 - r \cos \theta}{F_r(\theta)} d\theta = 2\pi.$$

Easy algebra shows that this is equivalent to (2).

A related integral. Method 4 also gives a quick evaluation of

$$J(r) = \int_{-\pi}^{\pi} \log F_r(\theta) d\theta.$$

Again by easy algebra, (1) implies that

$$\frac{2r - 2 \cos \theta}{F_r(\theta)} = -2 \sum_{n=1}^{\infty} r^{n-1} \cos n\theta.$$

Writing s for r and integrating with respect to s from 0 to r , we deduce

$$\log F_r(\theta) = -2 \sum_{n=1}^{\infty} \frac{r^n}{n} \cos n\theta. \quad (3)$$

Alternatively, (3) can be derived from the complex logarithmic series $\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$, valid for $|z| < 1$. Either way, it follows at once that $J(r) = 0$ for all r in $[0, 1)$.

With a and r related as before, it follows that $\int_{-\pi}^{\pi} \log(1 - a \cos \theta) d\theta = -2\pi \log(1 + r^2)$, which can then be expressed in terms of a (we leave this to the sufficiently determined reader).

Again, other methods are possible. Some can be seen in [1]. We are grateful to the referee for this reference.

Reference

1. H. Chen, Four ways to evaluate a Poisson integral, *Math. Magazine* 75 (2002), 290–294.