The integral $\int_x^\infty \frac{e^{it}}{t^p}\,dt$; Fresnel-type integrals

Notes by G.J.O. Jameson

Introduction

We shall consider integrals of the form

$$i_p(x) = \int_0^x \frac{e^{it}}{t^p}\,dt,$$

with real and imaginary parts

$$c_p(x) = \int_0^x \frac{\cos t}{t^p}\,dt, \qquad s_p(x) = \int_0^x \frac{\sin t}{t^p}\,dt,$$

together with the complementary integrals

$$I_p(x) = \int_x^\infty \frac{e^{it}}{t^p}\,dt, \qquad C_p(x) = \int_x^\infty \frac{\cos t}{t^p}\,dt, \qquad S_p(x) = \int_x^\infty \frac{\sin t}{t^p}\,dt,$$

where $p > 0$ and $x \geq 0$.

Further, we write $I_p = I_p(0)$, and similarly C_p, S_p, in the cases where these integrals converge (so $I_p = i_p(x) + I_p(x)$ for all $x > 0$): these are the "complete" integrals of this kind.

The case $p = 1$ is special: in particular, S_1 is the "sine integral", with well-known value $\pi/2$. This case, which is not characteristic of other p, is considered in companion notes [Jam1]. Other results apply more generally to integrals of $f(t)e^{it}$: these are presented in [Jam2]. Some of the material in the present notes has appeared in [Jam3].

Another case of particular interest is $p = \frac{1}{2}$, defining the "Fresnel" integrals. A very neat and elegant method is available for the evaluation of $I_{1/2}$, but it does not adapt to other p. We will present two alternative methods for the evaluation of I_p for $0 < p < 1$, respectively using double integration and contour integration. A variant of the first method gives yet another alternative proof of the sine integral $S_1 = \pi/2$.

We will also describe some other properties of the functions $i_p(x)$ and $I_p(x)$, and evaluate some integrals involving $I_p(x)$.

Results for $I_p(x)$ will of course incorporate simultaneous results for $C_p(x)$ and $S_p(x)$: there is a neat economy in using complex numbers in this way. The reader just needs to accept that $\frac{d}{dt}e^{it} = ie^{it}$, and that the usual processes of calculus, such as integration by parts, work in the same way for complex functions of a real variable.

By the fundamental theorem of calculus, $i_p'(x) = -I_p'(x) = e^{ix}/x^p$.

We note two simple substitutions. For $a > 0$, the substitution $at = u$ gives

$$\int_0^x \frac{e^{iat}}{t^p}\, dt = \int_0^{ax} \frac{a^p e^{iu}}{u^p} \frac{1}{a}\, du = a^{p-1} i_p(ax), \tag{1}$$

and a similar statement for $I_p(ax)$, presupposing convergence. Also presupposing convergence, the substitution $t = u + x$ gives

$$I_p(x) = e^{ix} \int_0^\infty \frac{e^{iu}}{(u+x)^p}\, du. \tag{2}$$

Some properties of $c_p(x)$ and $s_p(x)$

The integrals defining $i_p(x)$ and $c_p(x)$ are convergent at 0, hence well-defined, for $0 < p < 1$, while $s_p(x)$ is well-defined for $0 < p < 2$.

From $\cos t \le 1$ and $\sin t \le t$, we have the following inequalities, effective for small x:

$$c_p(x) \le \frac{x^{1-p}}{1-p}, \qquad s_p(x) \le \frac{x^{2-p}}{2-p}.$$

By inserting the series for $\cos t$ and $\sin t$ and integrating termwise, we obtain explicit series expressions for $c_p(x)$ (for $0 < p < 1$) and $s_p(x)$ (for $0 < p < 2$):

$$c_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1-p}}{(2n)!(2n-p+1)} = x^{1-p}\left(\frac{1}{1-p} - \frac{x^2}{2!(3-p)} + \cdots\right),$$

$$s_p(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-p}}{(2n-1)!(2n-p)} = x^{2-p}\left(\frac{1}{2-p} - \frac{x^2}{3!(4-p)} + \cdots\right).$$

These series can be used to calculate values, though in practice the calculation is only pleasant for fairly small x. Some values are

$$s_1(\pi) \approx 1.85194, \quad s_1(2\pi) \approx 1.41816, \quad s_{1/2}(\pi) \approx 1.78967,$$

$$c_{1/2}(\tfrac{\pi}{2}) \approx 1.95490, \quad c_{1/2}(\tfrac{3\pi}{2}) \approx 0.80476.$$

Clearly, $s_p(x)$ is increasing on intervals $[2n\pi, (2n+1)\pi]$ and decreasing on intervals $[(2n-1)\pi, 2n\pi]$, so has maxima at the points $(2n+1)\pi$ and minima at the points $2n\pi$. Similarly, $c_p(x)$ has maxima at $(2n + \frac{1}{2})\pi$ and minima at $(2n - \frac{1}{2})\pi$.

TPE1. *The function $s_p(x)$ has least value 0 and greatest value at $x = \pi$. The function $c_p(x)$ has greatest value at $\pi/2$ and least value either at 0 or at $3\pi/2$.*

Proof. Write $f(t) = 1/t^p$: the following only needs the fact that $f(t)$ is decreasing. Let

$$A_n = \int_{n\pi}^{(n+2)\pi} f(t) \sin t \, dt.$$

By substituting $t + \pi = u$ on $[n\pi, (n+1)\pi]$ and recombining, we see that

$$A_n = \int_{n\pi}^{(n+1)\pi} [f(t) - f(t+\pi)] \sin t \, dt,$$

in which $f(t) - f(t+\pi) \geq 0$. If n is even, then $\sin t \geq 0$ on $[n\pi, (n+1)\pi]$, so $A_n \geq 0$, hence $s_p[(n+2)\pi] \geq s_p(n\pi)$. It follows that $s_p(2n\pi) \geq \ldots \geq s_p(2\pi) \geq s_p(0) = 0$ for all n, so, by the preceding remarks, $s_p(x) \geq 0$ for all $x \geq 0$. Meanwhile, if n is odd, then $A_n \leq 0$, so that $s_p(\pi) \geq s_p(3\pi) \geq \ldots$, so the greatest value of $s_p(x)$ occurs at $x = \pi$.

Similarly, if

$$B_n = \int_{(n-\frac{1}{2})\pi}^{(n+\frac{3}{2})\pi} f(t) \cos t \, dt,$$

then $B_n \geq 0$ for even n and $B_n \leq 0$ for odd n. The statements for $c_p(x)$ follow in the same way. $\square$

Simple estimations show that $c_p(\frac{3\pi}{2}) < 0$ for sufficiently small p, while $c_{1/2}(\frac{3\pi}{2}) > 0$.

$I_p(x)$: convergence and some estimations

Convergence of $I_p(x)$, and an approximation effective for large x, is revealed by a simple integration by parts. It adds to clarity to describe these results in the more general context where $1/t^p$ is replaced by $f(t)$. Taking a slight liberty with our notation, we write

$$I_f(x) = \int_x^\infty f(t) e^{it} \, dt, \qquad C_f(x) = \int_x^\infty f(t) \cos t \, dt, \qquad S_f(x) = \int_x^\infty f(t) \sin t \, dt.$$

We assume that $f(t)$ is *completely monotonic*, that is:

(CM) for all $n \geq 0$, $(-1)^n f^{(n)}(t) \geq 0$ for $t > 0$ and $f^{(n)}(t) \to 0$ as $t \to \infty$.

Note that this implies that $(-1)^n f^{(n)}(t)$ is decreasing and that $(-1)^n f^{(n)}(t)$ also satisfies (CM). Of course, $f(t) = 1/t^p$ is completley monotonic for all $p > 0$.

We start with $I_{f'}(x)$ (distinguish between this and $I'_f(x)$!). If f is completely monotonic, then $f'(t) \leq 0$ and $\int_x^\infty (-f'(t)) \, dt = f(x)$. Since $|f'(t)e^{it}| \leq -f'(t)$, the integral defining $I_{f'}(x)$ is convergent, and we have

$$|I_{f'}(x)| \leq f(x). \tag{3}$$

Now integrate by parts:

$$I_f(x) = \Big[-if(t)e^{it} \Big]_x^\infty + i \int_x^\infty f'(t) e^{it} \, dt = if(x)e^{ix} + iI_{f'}(x).$$

We summarise this information in the following result.

TPE 2. *If f is completely monotonic, then the integrals defining $I_f(x)$ and $I_{f'}(x)$ are convergent for all $x > 0$, and the following statements apply:*

$$I_f(x) = if(x)e^{ix} + iI_{f'}(x), \tag{4}$$

$$C_f(x) = -f(x)\sin x - S_{f'}(x), \qquad S_f(x) = f(x)\cos x + C_{f'}(x). \tag{5}$$

Further,

$$|I_f(x)| \le 2f(x). \qquad \square \tag{6}$$

Note that (6) follows at once from (3) and (4). Also, it applies to f' (since $-f'$ is completly monotonic) to give $|I_{f'}(x) \le -2f'(x)$, which can then be inserted in (4).

We restate this for our case $f(x) = 1/x^p$, noting that $I_{f'}(x) = -pI_{p+1}(x)$:

TPE3. *The integral defining $I_p(x)$ is convergent for all $p > 0$ and $x > 0$, and we have:*

$$I_p(x) = \frac{ie^{ix}}{x^p} - ipI_{p+1}(x), \tag{7}$$

$$C_p(x) = -\frac{\sin x}{x^p} + pS_{p+1}(x), \qquad S_p(x) = \frac{\cos x}{x^p} - pC_{p+1}(x). \tag{8}$$

Also,

$$|I_p(x)| \le \frac{2}{x^p}. \qquad \square \tag{9}$$

(7) and (8) give estimations that are effective for large x. In particular, $I_p(x) \sim ie^{ix}/x^p$ as $x \to \infty$. For the magnitude of the remainder term $pI_{p+1}(x)$, we have alternative bounds $2p/x^{p+1}$ from (9), and $1/x^p$ from (3). By (7) and (9), we have $|x^p I_p(x) - ie^{ix}| \le 2p/x$.

The process that gave (4) can be repeated. Applying (4) to $I_{f'}(x)$ and substituting back into (4) (which amounts to repeating the integration by parts), we obtain

$$I_f(x) = if(x)e^{ix} - f'(x)e^{ix} - I_{f''}(x), \tag{10}$$

Applying this to $f''(x)$ and substituting again, we obtain

$$I_f(x) = i\left[f(x) - f''(x)\right]e^{ix} - \left[f'(x) - f^{(3)}(x)\right]e^{ix} + I_{f^{(4)}}(x). \tag{11}$$

We leave it to the reader to write out the corresponding formulae for $C_f(x)$ and $S_f(x)$. Of course, the explicit versions for $f(x) = 1/x^p$ are quite complicated. Here we just state (11) for the case $p = 1$:

$$I_1(x) = \left(\frac{1}{x} - \frac{2}{x^3}\right)ie^{ix} + \left(\frac{1}{x^2} - \frac{6}{x^4}\right)e^{ix} + 24I_5(x),$$

in which $24I_5(x)$ is bounded both by $6/x^4$ and $48/x^5$. Implications of this expression are discussed in [Jam1].

Of course, the process can be continued: successive derivatives of $f(x)$ appear in the expressions multiplying e^{ix} and ie^{ix}. The outcome is an asymptotic expansion for $I_f(x)$. However, this does not simply deliver ever-closer approximations, because for a fixed x, the derivatives $f^{(n)}(x)$ will ultimately grow large in magnitude.

Now consider x close to 0. Of course, if $0 < p < 1$, then $I_p(x)$ tends to the finite limit I_p as $x \to 0^+$. For $p > 1$, (3) gives $|I_p(x)| \leq 1/[(p-1)x^{p-1}]$, which is better than (9) for small x.

The case $p = 1$ is special. We consider $C_1(x)$ and $S_1(x)$ separately. We know that $S_1(x) \to S_1(0) = \pi/2$ as $x \to 0^+$. For $C_1(x)$, we have:

TPE4. *There is a constant c such that $C_1(x) + \log x \to c$ as $x \to 0^+$.*

Proof. Define

$$c_1^*(x) = \int_0^x \frac{1 - \cos t}{t}\, dt.$$

Since $0 \leq 1 - \cos t \leq \frac{1}{2}t^2$, we have $|c_1^*(x)| \leq \frac{1}{4}x^2$ for $x > 0$. Now

$$c_1^*(x) - c_1^*(1) = \int_1^x \frac{1 - \cos t}{t}\, dt = \log x - \int_1^x \frac{\cos t}{t}\, dt = \log x - C_1(1) + C_1(x),$$

so $C_1(x) + \log x = c + c_1^*(x)$, where $c = C_1(1) - c_1^*(1)$. The statement follows. □

One can show that in fact $c = -\gamma$: see [Jam1].

The auxiliary functions. Formulae like (4) and (10) simplify pleasantly when expressed in terms of $K_f(x) = e^{-ix}I_f(x)$, because real and imaginary terms are separated. We also write $K_f(x) = V_f(x) + iU_f(x)$, so that

$$U_f(x) = S_f(x)\cos x - C_f(x)\sin x,$$

$$V_f(x) = C_f(x)\cos x + S_f(x)\sin x.$$

U_f and V_f are called the "auxiliary functions". (We write U_f, V_f this way round in a gesture to the customary notation.) Of course, we use the notation $U_p(x)$, $V_p(x)$ for the case where $f(x) = 1/x^p$.

Note that $U_f(n\pi) = (-1)^n S_f(n\pi)$ and $V_f(n\pi) = (-1)^n C_f(n\pi)$, also $U_f(u_n) = (-1)^n C_f(u_n)$ and $V_f(u_n) = (-1)^{n+1} S_f(u_n)$, where $u_n = (n - \frac{1}{2})\pi$.

In terms of these functions, (4) and (5) become

$$K_f(x) = if(x) + iK_{f'}(x),$$

$$U_f(x) = f(x) + V_{f'}(x), \qquad V_f(x) = -U_{f'}(x),$$

and (10) becomes

$$K_f(x) = if(x) - f'(x) - K_{f''}(x),$$

$$U_f(x) = f(x) - U_{f''}(x), \qquad V_f(x) = -f'(x) - V_{f''}(x).$$

One can show (for completely monotonic f) that $0 \le U_f(x) \le f(x)$ and $0 \le V_f(x) \le -f'(x)$, and hence that $|I_f(x)| \le f(x)$, thereby improving (6): see [Jam2] or [Jam4]. Below, we will give a proof of these statements for our case $f(x) = 1/x^p$.

The case $p = \frac{1}{2}$: the Fresnel integrals

The substitution $t = u^2$ gives

$$I_{1/2} = 2\int_0^\infty e^{iu^2}\,du = \int_{-\infty}^\infty e^{iu^2}\,du, \tag{12}$$

also, for example, $I_{1/2}(x) = 2\int_{x^{1/2}}^\infty e^{iu^2}\,du$. The Fresnel integrals are often presented in this form, and it is the form we will use for the evaluation.

We assume the following elementary form of the theorem on pointwise convergence of Fourier series. Suppose that f is differentiable on $[0,1]$ and $f(0) = f(1)$. Let $c_n = \int_0^1 f(x)e^{-2n\pi ix}\,dx$. Then for all x in $[0,1]$, we have $f(x) = \sum_{-\infty}^\infty c_n e^{2n\pi ix}$, where the notation $\sum_{n=-\infty}^\infty a_n$ means $\lim_{N\to\infty}\sum_{n=-N}^N a_n$. (For readers more familiar with real Fourier series, observe that the substitution $a_n = c_n + c_{-n},\ \ b_n = i(c_n - c_{-n})$ translates the series into the form $\frac{1}{2}a_0 + \sum_{n=1}^\infty (a_n\cos 2n\pi x + b_n \sin 2n\pi x)$.)

TPE5 THEOREM. *We have*

$$I_{1/2} = \left(\frac{\pi}{2}\right)^{1/2}(1+i), \tag{13}$$

equivalently

$$\int_{-\infty}^\infty e^{2\pi ix^2}\,dx = \tfrac{1}{2}(1+i), \tag{14}$$

so that

$$\int_{-\infty}^\infty \cos 2\pi x^2\,dx = \int_{-\infty}^\infty \sin 2\pi x^2\,dx = \tfrac{1}{2}. \tag{15}$$

Proof. The equivalence of (13) and (14) is clear from (12) and the futher substitution $u = (2\pi)^{1/2}x$. Let $f(x) = e^{2\pi ix^2}$ and $I = \int_{-\infty}^\infty f(x)\,dx$. Consider the Fourier series

$\sum_{n=-\infty}^{\infty} c_n e^{2n\pi ix}$ for $f(x)$ on $[0,1]$. Note that $f(0) = f(1) = 1$. Convergence at the points 0 and $\frac{1}{2}$ gives

$$\sum_{n=-\infty}^{\infty} c_n = f(0) = 1,$$

$$\sum_{n=-\infty}^{\infty} (-1)^n c_n = f(\tfrac{1}{2}) = e^{\pi i/2} = i,$$

hence

$$\sum_{n=-\infty}^{\infty} c_{2n} = \tfrac{1}{2}(1+i).$$

This sum can equally be written as $\sum_{n=-\infty}^{\infty} c_{-2n}$. Now

$$c_{-2n} = \int_0^1 e^{2\pi i(x^2+2nx)}\,dx,$$

and $x^2 + 2nx = (x+n)^2 - n^2$, so

$$c_{-2n} = \int_0^1 e^{2\pi i(x+n)^2}\,dx = \int_n^{n+1} e^{2\pi iy^2}\,dy.$$

Hence

$$\sum_{n=-\infty}^{\infty} c_{-2n} = I. \qquad \square$$

Alternatively, this proof can be presented in terms of real Fourier series, resulting in separate evaluations of the cosine and sine integrals.

Evaluation for general p: double integral method

We will establish an integral expression for $I_p(x)$, which has other applications as well as the evaluation of I_p. We assume a degree of familiarity with the gamma function. The starting point is the observation that for any $p > 0$, the substitution $tu = v$ gives

$$\int_0^\infty u^{p-1}e^{-tu}\,du = \frac{1}{t^p}\int_0^\infty v^{p-1}e^{-v}\,dv = \frac{\Gamma(p)}{t^p}. \tag{16}$$

Though it might seem a strange way to proceed, we substitute the implied integral expression for $1/t^p$ in $I_p(x)$. In the case $p = 1$, it simplifies to $\int_0^\infty e^{-tu}\,du = 1/t$.

TPE6 THEOREM. *For $x > 0$ and $p > 0$, we have*

$$I_p(x) = \frac{e^{ix}}{\Gamma(p)}\int_0^\infty \frac{u+i}{u^2+1}\,u^{p-1}e^{-ux}\,du \tag{17}$$

and

$$|I_p(x)| \le \frac{1}{x^p}. \tag{18}$$

For $0 < p < 1$, this holds with $x = 0$, so that

$$I_p = \frac{1}{\Gamma(p)} \int_0^\infty \frac{(u+i)u^{p-1}}{u^2+1}\, du. \tag{19}$$

Also, for $0 < p < 2$,

$$S_p = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^{p-1}}{u^2+1}\, du. \tag{20}$$

Proof. By (16), we have

$$\Gamma(p)[I_p(x) - I_p(y)] = \int_x^y e^{it} \int_0^\infty u^{p-1} e^{-ut}\, du\, dt.$$

Assuming reversal of the integral is valid, we have

$$\begin{aligned}
\Gamma(p)[I_p(x) - I_p(y)] &= \int_0^\infty u^{p-1} \int_x^y e^{-(u-i)t}\, dt\, du \\
&= \int_0^\infty \frac{u^{p-1}}{u-i} \left(e^{-(u-i)x} - e^{-(u-i)y}\right) du \\
&= H(x) - H(y),
\end{aligned}$$

where

$$H(x) = e^{ix} \int_0^\infty \frac{u^{p-1}}{u-i} e^{-ux}\, du.$$

Since $|u - i| \geq 1$, we have

$$|H(x)| \leq \int_0^\infty u^{p-1} e^{-ux}\, du = \frac{\Gamma(p)}{x^p}$$

for $x > 0$. Hence $H(y) \to 0$ as $y \to \infty$. Taking the limit as $y \to \infty$, we conclude that $\Gamma(p) I_p(x) = H(x)$, hence (17), (18) and (19).

To justify reversal of the double integral, replacing the integrand by its absolute value, we obtain

$$\int_x^y \int_0^\infty u^{p-1} e^{-ut}\, du\, dt = \Gamma(p) \int_x^y \frac{1}{t^p}\, dt,$$

which is finite if $x > 0$ and $p > 0$, also if $x = 0$ and $0 < p < 1$.

For (20), we reason similarly with $x = 0$ and e^{it} replaced by $\sin t$. Since $|\sin t| \leq t$, reversal is valid provided that

$$\int_0^y t \int_0^\infty u^{p-1} e^{-ut}\, du\, dt = \Gamma(p) \int_0^y \frac{1}{t^{p-1}}\, dt$$

is finite, which is true for $0 < p < 2$. □

TPE7 COROLLARY. *We have $S_1 = \pi/2$.*

Proof. By (20),

$$S_1 = \int_0^\infty \frac{1}{1+u^2}\,du = \frac{\pi}{2}. \qquad \square$$

This proof can be compared with the three methods for S_1 described in [Jam1].

As another corollary, we have the following expressions and inequalities for the auxiliary functions. For the case $p = 1$, the expressions can be seen stated without proof in compilations such as Wikipedia and [DLMF, chapter 6].

TPE8 COROLLARY. *The auxiliary functions $U_p(x)$, $V_p(x)$ satisfy*

$$U_p(x) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^{p-1}}{u^2+1} e^{-ux}\,du,$$

$$V_p(x) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^{p}}{u^2+1} e^{-ux}\,du.$$

Both are decreasing functions of x. Also, $0 \le U_p(x) \le 1/x^p$ and $0 \le V_p(x) \le p/x^{p+1}$.

Proof. For the inequalities, just note that $u^2+1 \ge 1$ and apply (16). $\square$

Note. A variant of the proof of TPE6 is as follows. Assuming its validity, differentiation under the integral sign gives

$$H'(x) = -\int_0^\infty u^{p-1} e^{-(u-i)x}\,du = -\Gamma(p)\frac{e^{ix}}{x^p} = \Gamma(p) I_p'(x),$$

so $\Gamma(p)I_p(x) - H(x)$ is constant. Both $I_p(x)$ and $H(x)$ tend to 0 as $x \to \infty$, so the constant is 0. This process is justified for $x > 0$ by uniform convergence of the integral for $H'(x)$. However, to derive the statements for $x = 0$, one must then show directly that the expression for $I_p(x)$ tends to the stated limit as $x \to 0^+$: this requires some work.

We now proceed to the evaluation of I_p. We will use the well-known integral

$$\int_0^\infty \frac{1}{y^a(1+y)}\,dy = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1), \tag{21}$$

(e.g. [Wa, p. 187]). Also, we will use Euler's reflection formula for the gamma function:

$$\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin \pi q} \tag{22}$$

The value of I_p is actually stated more neatly in terms of $q = 1 - p$, as follows:

TPE9 THEOREM. *Let $0 < p < 1$ and $q = 1 - p$. Then*

$$I_p = \Gamma(q) e^{\frac{1}{2}\pi q i}, \qquad C_p = \Gamma(q) \cos \tfrac{1}{2}\pi q, \qquad S_p = \Gamma(q) \sin \tfrac{1}{2}\pi q. \tag{23}$$

The expression for S_p is also valid for $1 < p < 2$.

Proof. For $0 < p < 1$, (19) gives

$$C_p = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^p}{u^2+1}\,du = \frac{1}{\Gamma(1-q)} \int_0^\infty \frac{u^{1-q}}{u^2+1}\,du,$$

while for $0 < p < 2$, (20) gives

$$S_p = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^{p-1}}{u^2+1}\,du = \frac{1}{\Gamma(1-q)} \int_0^\infty \frac{u^{-q}}{u^2+1}\,du.$$

Substituting $u^2 = y$ in (21), we obtain, for $0 < p < 1$,

$$\int_0^\infty \frac{u^{1-q}}{1+u^2}\,du = \int_0^\infty \frac{1}{2y^{\frac{1}{2}q}(1+y)}\,dy = \frac{\pi}{2\sin\frac{1}{2}\pi q}.$$

If $0 < p < 2$, so that $0 < \frac{1}{2}(q+1) < 1$, we can replace q by $q+1$ in this, obtaining

$$\int_0^\infty \frac{u^{-q}}{1+u^2}\,du = \frac{\pi}{2\cos\frac{1}{2}\pi q}.$$

Now substituting for $\Gamma(1-q)$ from (22) and using $\sin\pi q = 2\sin\frac{1}{2}\pi q\cos\frac{1}{2}\pi q$, we obtain

$$C_p = \frac{\Gamma(q)\sin\pi q}{\pi}\,\frac{\pi}{2\sin\frac{1}{2}\pi q} = \Gamma(q)\cos\tfrac{1}{2}\pi q$$

and similarly $S_p = \Gamma(q)\sin\frac{1}{2}\pi q$. □

Of course, (13) is the special case $p = \frac{1}{2}$. For this case, the required integrals $\int_0^\infty u^{\pm 1/2}/(1+u^2)\,du$ can be evaluated by more elementary means.

We have stated C_p and S_p in terms of q, but it is worth recording the expressions in terms of p that emerged on the way:

$$C_p = \frac{\pi}{2\Gamma(p)\cos\frac{1}{2}\pi p}, \qquad S_p = \frac{\pi}{2\Gamma(p)\sin\frac{1}{2}\pi p}.$$

Meanwhile, stated entirely in terms of q, (23) says

$$I_{1-q} = \int_0^\infty t^{q-1}e^{it}\,dt = \Gamma(q)e^{\frac{1}{2}\pi q i},$$

which is pleasantly analogous to the defining formula for $\Gamma(q)$.

Evaluation for general p: contour integral method

We give a second proof of (23) by contour integration. We will actually prove the statement in the conjugate form

$$\int_0^\infty t^{q-1}e^{-it}\,dt = \Gamma(q)e^{-\frac{1}{2}\pi q i}. \tag{24}$$

Before giving the proof, let us dispose of a tempting, but fallacious, short cut. In (24), if we write $it = u$ and formally apply the procedure for integration by substitution, we appear to obtain

$$\int_0^\infty \frac{1}{i^q} u^{q-1} e^{-u}\, du = \frac{1}{i^q}\Gamma(q) = \Gamma(q) e^{-\frac{1}{2}\pi q i}.$$

Apparently, something rather like this was done by Euler himself. However, it has to be said that there is absolutely no general result justifying this procedure!

We turn to the proof. Let $0 < r < R$. Let C_R denote the circular arc of radius R in the positive quadrant, represented by $z = Re^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$, similarly C_r. Denote by Δ the closed contour consisting of the real interval $[r, R]$, C_R described anticlockwise, the imaginary axis from iR to ir and C_r described clockwise. Let $f(z) = z^{q-1}e^{-z}$, where the principal value of $\log z$ is used to define z^{p-1}. In particular, $\log it$ (for $t > 0$) is expressed as $\log t + i\frac{\pi}{2}$, so $(it)^{q-1}$ is expressed as $t^{q-1}\exp(\frac{1}{2}(q-1)\pi i)$. By Cauchy's integral theorem, $\int_\Delta f(z)\, dz = 0$. The contribution of the real axis converges to $\Gamma(q)$ when $r \to 0$ and $R \to \infty$. The contribution of the imaginary axis, taken towards the origin, is

$$-\int_r^R (it)^{q-1} e^{-it} i\, dt = -e^{\frac{1}{2}q\pi i} \int_r^R t^{q-1} e^{-it}\, dt.$$

The conclusion will follow if we can show that the contributions of the circular arcs tend to 0 when $r \to 0$ and $R \to \infty$. (In fact, after taking the limit and multiplying by i^q, the contributions of the two axes equate to the two quantities we wanted to identify in the short cut, and the identification only works because the contributions of the circular arcs tend to zero.)

The arc C_r is given by $z = re^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$. For such z, we have $|e^{-z}| = e^{-r\cos\theta} \leq 1$, hence $|f(z)| \leq r^{q-1}$ and

$$\left| \int_{C_r} f(z)\, dz \right| \leq r^{q-1}\frac{\pi r}{2} = \frac{\pi r^q}{2},$$

which tends to 0 as $r \to 0^+$, since $q > 0$.

The integral on C_R is

$$I_R =: \int_0^{\pi/2} R^{q-1} e^{i(q-1)\theta} e^{-Re^{i\theta}} iRe^{i\theta}\, d\theta.$$

Since $|e^{-Re^{i\theta}}| = e^{-R\cos\theta}$,

$$|I_R| \leq R^q \int_0^{\pi/2} e^{-R\cos\theta}\, d\theta.$$

TPE10 LEMMA. *We have*

$$0 \leq \int_0^{\pi/2} e^{-R\cos\theta}\, d\theta \leq \frac{\pi}{2R}.$$

Proof. Denote this integral by J_R. The substitution $\theta = \frac{\pi}{2} - \phi$ shows that $J_R = \int_0^{\pi/2} e^{-R\sin\theta}\,d\theta$. By the concavity of the sine function, $\sin\theta \geq \frac{2\theta}{\pi}$ on $[0, \frac{\pi}{2}]$. Hence

$$J_R \leq \int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \left[-\frac{\pi}{2R}e^{-2R\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{2R}(1-e^{-R}). \quad \square$$

So we have

$$|I_R| \leq R^q J_r \leq \frac{\pi}{2R^{1-q}},$$

which tends to 0 as $R \to \infty$, since $q < 1$. This concludes the proof of (24).

Some integrals involving $I_p(x)$

For the following integrals, we will repeatedly use the estimations in TPE3. Recall also that $I_p'(x) = -e^{ix}/x^p$.

TPE11. *For all $p > 0$,*

$$\int_0^\infty x^{p-1} I_p(x)\,dx = \frac{i}{p}, \qquad \int_0^\infty x^{p-1} C_p(x)\,dx = 0, \qquad \int_0^\infty x^{p-1} S_p(x)\,dx = \frac{1}{p}. \tag{25}$$

Proof. We have

$$\frac{d}{dx}[x^p I_p(x)] = p x^{p-1} I_p(x) - e^{ix}.$$

hence an antiderivative of $px^{p-1}I_p(x)$ is $x^p I_p(x) - ie^{ix}$. By (7), $|x^p I_p(x) - ie^{ix}| \leq 2p/x$. Also, by (3) (and TPE4 for the case $p = 1$), $x^p I_p(x) \to 0$ as $x \to 0$ for all $p > 0$. Statement (25) follows. $\square$

In particular, $\int_0^\infty I_1(x)\,dx = i$, $\int_0^\infty C_1(x)\,dx = 0$ and $\int_0^\infty S_1(x)\,dx = 1$.

The values of further integrals will be expressed in terms of I_p, C_p and S_p themselves. First, we have the following simple and attractive connection:

TPE12. *For $0 < p < 1$,*

$$\int_0^\infty I_{p+1}(x)\,dx = I_p. \tag{26}$$

Proof. Integrating by parts, we have

$$\int_0^1 1.I_{p+1}(x)\,dx = \Big[xI_{p+1}(x)\Big]_0^\infty + \int_0^\infty x\frac{e^{ix}}{x^{p+1}}\,dx.$$

The integral is I_p. By (3), $|xI_{p+1}(x)| \leq \frac{1}{p}x^{1-p} \to 0$ as $x \to 0^+$, and by (9), $|xI_{p+1}(x)| \leq 2/x^p \to 0$ as $x \to \infty$. The statement follows. $\square$

TPE13. *For* $0 < p < 1$

$$\int_0^\infty I_p(x)\,dx = i(1-p)I_p, \tag{27}$$

$$\int_0^\infty C_p(x)\,dx = -(1-p)S_p, \qquad \int_0^\infty S_p(x)\,dx = (1-p)C_p. \tag{28}$$

Proof. Recall from (7) that

$$I_p(x) = \frac{ie^{ix}}{x^p} - ipI_{p+1}(x).$$

Integrating and applying (26), we obtain (27). □

We now consider products like $e^{ix}I_p(x)$ and $S_p(x)\sin x$. Recall that the substitution $ax = y$ gives $\int_0^\infty x^{-p}e^{iax}\,dx = a^{p-1}I_p$ (and similarly for C_p and S_p).

TPE14. *For* $0 < p \le 1$,

$$\int_0^\infty C_p(x)\cos x\,dx = 2^{p-2}S_p, \tag{29}$$

$$\int_0^\infty S_p(x)\sin x\,dx = (1-2^{p-2})S_p. \tag{30}$$

For $0 < p < 1$,

$$\int_0^\infty e^{ix}I_p(x)\,dx = i(1-2^{p-1})I_p, \tag{31}$$

$$\int_0^\infty (C_p(x)\sin x + S_p(x)\cos x)\,dx = (1-2^{p-1})C_p. \tag{32}$$

Proof. We prove (30); (29) and (31) are similar. Integrate by parts:

$$\begin{aligned}\int_0^\infty S_p(x)\sin x\,dx &= \Big[-S_p(x)\cos x\Big]_0^\infty - \int_0^\infty \frac{\cos x\sin x}{x^p}\,dx \\ &= S_p - \tfrac{1}{2}\int_0^\infty \frac{\sin 2x}{x^p}\,dx \\ &= S_p - 2^{p-2}S_p.\end{aligned}$$

For (32), take the imaginary part in (31). □

However, $\int_0^\infty S_p(x)\cos x\,dx$ is divergent, since $\int_0^\infty \sin^2 x/x^p\,dx$ is divergent. Similarly for $\int_0^\infty e^{-ix}I_p(x)\,dx$. Informally, these cases of divergence, and the convergence in (32), are explained by (8): for example, $S_p(x)\cos x \sim \cos^2 x/x^p$ for large x, while $C_p(x)\sin x + S_p(x)\cos x \sim \cos 2x/x^p$.

In particular,

$$\int_0^\infty C_1(x)\cos x\,dx = \int_0^\infty S_1(x)\sin x\,dx = \tfrac{1}{2}S_1 = \frac{\pi}{4}.$$

Using TPE4, one can show that $\int_0^\infty e^{ix} I_1(x)\, dx = i \log 2$ (see [Jam1]).

It is now easy to derive the integrals of expressions like $x^{p-1} I_p(x)^2$:

TPE15. *For* $0 < p \le 1$,

$$\int_0^\infty x^{p-1} C_p(x)^2\, dx = \frac{2^{p-1}}{p} S_p, \tag{33}$$

$$\int_0^\infty x^{p-1} S_p(x)^2\, dx = \frac{2}{p}(1 - 2^{p-2}) S_p. \tag{34}$$

For $0 < p < 1$,

$$\int_0^\infty x^{p-1} I_p(x)^2\, dx = \frac{2i}{p}(1 - 2^{p-1}) I_p, \tag{35}$$

$$\int_0^\infty x^{p-1} C_p(x) S_p(x)\, dx = \frac{1}{p}(1 - 2^{p-1}) C_p. \tag{36}$$

Proof. Again, we write out the details for (34). Note that by (9), $x^p S_p(x)^2 \to 0$ as $x \to \infty$. Integrating by parts and using (30), we have

$$\begin{aligned}
\int_0^\infty x^{p-1} S_p(x)^2\, dx &= \left[\frac{x^p}{p} S_p(x)^2\right]_0^\infty + 2\int_0^\infty \frac{x^p}{p} S_p(x) \frac{\sin x}{x^p}\, dx \\
&= 0 + \frac{2}{p}\int_0^\infty S_p(x)\, \sin x\, dx \\
&= \frac{2}{p}(1 - 2^{p-2}) S_p.
\end{aligned}$$

For (36), take the imaginary part in (35). □

In particular,

$$\int_0^\infty C_1(x)^2\, dx = \int_0^\infty S_1(x)^2\, dx = S_1 = \frac{\pi}{2}.$$

References

[DLMF] *NIST Digital Library of Mathematical Functions*, at dlmf.nist.gov

[Jam1] G. J. O. Jameson, The sine and cosine integrals, at www.maths.lancs.ac.uk/~jameson/

[Jam2] G. J. O. Jameson, Integrals of the form $\int_x^\infty f(t) e^{it}\, dt$, at www.maths.lancs.ac.uk/~jameson/

[Jam3] G. J. O. Jameson, Evaluating Fresnel-type integrals, *Math. Gazette* **99** (2015), 491–498.

[Jam4] G. J. O. Jameson, An inequality for integrals of the form $\int_x^\infty f(t) e^{it}\, dt$, *J. Math. Ineq.* (2015), to appear.

[Wa] P. L. Walker, *The Theory of Fourier Series and Integrals*, John Wiley (1986).

updated 25 November 2015