The probability integral by volume of revolution

Several notes on the evaluation of the probability integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ have appeared in recent volumes of the *Gazette* (see the references below). Here is a very simple method based on volumes of revolution and double integrals (but without any change of variables). Has anyone seen it before ?

Denote the required integral by I. Let A be the region in the (x, z)-plane defined by

$$0 \le z \le e^{-x^2}, \quad x \ge 0.$$

Consider the 3-dimensional region obtained when A is rotated about the z-axis. This region is obviously defined by

$$0 \le z \le e^{-x^2-y^2},$$

so its volume is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = I^2.$$

But this volume was obtained by rotating the curve $x^2 = -\ln z$ (for $0 < z \le 1$) about the z-axis, so it equals

$$\int_0^1 \pi x^2\,dz = \pi \int_0^1 (-\ln z)\,dz = \pi.$$

Hence $I = \sqrt{\pi}$. The limiting process for both integrals is taken care of by the fact that $\delta \ln \delta \to 0$ as $\delta \to 0$ from above.

References

1. Darrell Desbrow, Evaluating the probability integral, *Math. Gazette* note 74.28 (1990).
2. N. Gauthier, Evaluating the probability integral, *Math. Gazette* note 72.22 (1988).
3. Robert M. Young, On the evaluation of certain improper integrals, *Math. Gazette* note 75.4 (1991).

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