

## The probability integral by volume of revolution

Several notes on the evaluation of the probability integral  $\int_{-\infty}^{\infty} e^{-x^2} dx$  have appeared in recent volumes of the *Gazette* (see the references below). Here is a very simple method based on volumes of revolution and double integrals (but without any change of variables). Has anyone seen it before ?

Denote the required integral by  $I$ . Let  $A$  be the region in the  $(x, z)$ -plane defined by

$$0 \leq z \leq e^{-x^2}, \quad x \geq 0.$$

Consider the 3-dimensional region obtained when  $A$  is rotated about the  $z$ -axis. This region is obviously defined by

$$0 \leq z \leq e^{-x^2-y^2},$$

so its volume is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = I^2.$$

But this volume was obtained by rotating the curve  $x^2 = -\ln z$  (for  $0 < z \leq 1$ ) about the  $z$ -axis, so it equals

$$\int_0^1 \pi x^2 dz = \pi \int_0^1 (-\ln z) dz = \pi.$$

Hence  $I = \sqrt{\pi}$ . The limiting process for both integrals is taken care of by the fact that  $\delta \ln \delta \rightarrow 0$  as  $\delta \rightarrow 0$  from above.

### References

1. Darrell Desbrow, Evaluating the probability integral, *Math. Gazette* note 74.28 (1990).
2. N. Gauthier, Evaluating the probability integral, *Math. Gazette* note 72.22 (1988).
3. Robert M. Young, On the evaluation of certain improper integrals, *Math. Gazette* note 75.4 (1991).

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