The probability integral by volume of revolution

Several notes on the evaluation of the probability integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) have appeared in recent volumes of the *Gazette* (see the references below). Here is a very simple method based on volumes of revolution and double integrals (but without any change of variables). Has anyone seen it before?

Denote the required integral by \( I \). Let \( A \) be the region in the \((x, z)\)-plane defined by

\[
0 \leq z \leq e^{-x^2}, \quad x \geq 0.
\]

Consider the 3-dimensional region obtained when \( A \) is rotated about the \( z \)-axis. This region is obviously defined by

\[
0 \leq z \leq e^{-x^2-y^2},
\]

so its volume is

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx \, dy = I^2.
\]

But this volume was obtained by rotating the curve \( x^2 = -\ln z \) (for \( 0 < z \leq 1 \)) about the \( z \)-axis, so it equals

\[
\pi \int_{0}^{1} x^2 \, dz = \pi \int_{0}^{1} (-\ln z) \, dz = \pi.
\]

Hence \( I = \sqrt{\pi} \). The limiting process for both integrals is taken care of by the fact that \( \delta \ln \delta \to 0 \) as \( \delta \to 0 \) from above.

References


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