Binomial power sums and Stirling numbers

Notes by G.J.O. Jameson

These notes are an elementary account of the sums and numbers in the title, along with some related expressions and some applications. Some of the numbers in question can be defined either by an explicit formula or by a combinatorial meaning. We will allow for both of these notions. As a consequence, alternative proofs will be presented for several of the results, reflecting this dichotomy.

Binomial power sums

We start with sums of the following form, for which the term "binomial power sums" seems appropriate, though no standard name, and no standard notation, is established. For $n \geq 0, \;\; k \geq 1$, define

$$S_{n,k} = \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n. \tag{1}$$

For $n = 0$, we count 0^0 as 1, so that $S_{0,k} = \sum_{r=0}^{k} (-1)^r \binom{k}{r} = (1-1)^k = 0$. For $n > 0$, the summation in (1) can equally be taken as $1 \leq r \leq k$.

Note that $S_{n,1} = -1$ and $S_{n,2} = 2^n - 2$ for all $n \geq 1$.

The numbers $S_{n,k}$ satisfy an identity analogous to Pascal's triangle:

THEOREM 1. *For $n \geq 0$ and $k \geq 2$,*

$$S_{n+1,k} = k(S_{n,k} - S_{n,k-1}). \tag{2}$$

Proof. We use the elementary identities $\binom{k}{r} = \binom{k-1}{r} + \binom{k-1}{r-1}$ and $r\binom{k}{r} = k\binom{k-1}{r-1}$. For $n \geq 1$, we have

$$\begin{aligned} S_{n+1,k} &= \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{n+1} \\ &= k \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} r^n \\ &= k \sum_{r=1}^{k} (-1)^r \left[\binom{k}{r} - \binom{k-1}{r} \right] r^n \\ &= k(S_{n,k} - S_{n,k-1}). \end{aligned}$$

For $n = 0$, we can change the summation at the last step to $\sum_{r=0}^{k}$. Of course, the conclusion is actually $S_{1,k} = 0$. □

By induction on n (as a statement for all $k > n$), we deduce the following basic fact about $S_{n,k}$:

THEOREM 2. *We have $S_{n,k} = 0$ for $n < k$.* □

Several alternative proofs will emerge in the ensuing notes.

THEOREM 3. *For all $k \geq 1$ and $n \geq 1$, $S_{n,k}$ is a multiple of $k!$. Further, $S_{k,k} = (-1)^k k!$*

Proof. Again we prove the first statement by induction on n, as a statement for all $k \geq 1$ (including the values $k > n$, for which we know $S_{n,k} = 0$). Note that the case $k = 1$ is trivial for all n. For $n = 1$, we have $S_{1,k} = 0$ for $k \geq 2$. The induction step follows at once from (2).

By (2) and Theorem 2, $S_{k,k} = -kS_{k-1,k-1}$, hence $S_{k,k} = (-1)^k k!$. □

The numbers $S_{n,k}$ for $n \geq k$ have a simple combinatorial meaning, which we now establish. We write $E_n = \{1, 2, \ldots, n\}$.

THEOREM 4. *For $n \geq k$, $(-1)^k S_{n,k}$ is the number of surjections from E_n onto E_k.*

We use the inclusion-exclusion principle in the following form [Cam, p. 76]. We denote the number of members of a set E by $|E|$.

Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set X. For subsets I of E_n, let $A_I = \cap_{i \in I} A_i$, also $A_\emptyset = X$. Then

$$|X \setminus (A_1 \cup \ldots \cup A_n)| = \sum_{I \subseteq E_n} (-1)^{|I|} |A_I|.$$

Proof of Theorem 4. Let X be the set of all mappings from E_n to E_k: then $|X| = k^n$. For each i , let A_i be the set of mappings from E_n to $E_k \setminus \{i\}$. Then $\cup_{i=1}^n A_i$ is exactly the set of mappings from E_n to E_k that are *not* surjections. If I is a subset of E_k with r members, then (with notation as above) A_I is the set of mappings from E_n to $E_k \setminus I$: hence $|A_I| = (k-r)^n$. Taking together the $\binom{k}{r}$ subsets I with r members and applying the inclusion-exclusion principle, we see that the number of surjections is

$$\sum_{r=0}^{k-1} (-1)^r \binom{k}{r} (k-r)^n = \sum_{s=1}^{k} (-1)^{k-s} \binom{k}{s} s^n = (-1)^k S_{n,k}. \qquad \square$$

Of course, Theorem 4 also holds when $n < k$, in the trivial sense that both quantities are 0 (and in fact the proof is valid for this case, thereby amounting to another proof of

Theorem 2). Clearly, it shows that $(-1)^k S_{n,k} > 0$ for $n \geq k$. In fact, it gives a combinatorial proof of (2), as follows.

Deduction of Theorem 1 from Theorem 4. Write $S'_{n,k} = (-1)^k S'n, k$. Then (2) says $S'_{n+1,k} = k(S'_{n,k} + S'_{n,k-1})$. For each surjection f from E_n onto E_k, we can extend f to E_{n+1} in k different ways by defining $f(n+1)$ to be any element of E_k. For each $j \in E_k$, there are $S'_{n,k-1}$ surjections of E_n onto $E_k \setminus \{j\}$: such a surjection g extends to a surjection $E_{n+1} \to E_k$ by taking $g(n+1) = j$. However, any mapping from E_n to E_k that misses out at least two elements of E_k cannot extend to a surjection from E_{n+1} onto E_k, so all such surjections are one of the two types described. □

A related sum and an inversion theorem for binomial coefficients

We now consider the related sum (again not standard notation)

$$T_{n,k} = \sum_{r=n}^{k} (-1)^r \binom{k}{r} \binom{r}{n}. \tag{3}$$

This sum can also be written as $\sum_{r=0}^{k}$, with the understanding that $\binom{r}{n} = 0$ when $r < n$. Clearly, $T_{0,k} = (1-1)^k = 0$, $T_{1,k} = S_{1,k}$ and $T_{k,k} = (-1)^k$.

THEOREM 5. *We have $T_{n,k} = 0$ for $n < k$.*

Proof. We have seen that this holds when $n = 0$, so assume that $n \geq 1$. Consider the identity $(1-x)^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} x^r$. Differentiate both sides n times and put $x = 1$: we obtain

$$0 = \sum_{r=n}^{k} (-1)^r \binom{k}{r} r(r-1)\ldots(r-n+1) = n! T_{n,k}. \qquad \square \tag{4}$$

This gives another proof of Theorem 2: for fixed k, we can rewrite (4) as

$$0 = \sum_{r=1}^{k} (-1)^r \binom{k}{r} (c_1 r + \cdots + c_{n-1} r^{n-1} + r^n) = \sum_{j=1}^{n-1} c_j S_{j,k} + S_{n,k}$$

for certain coefficients c_j, so induction on n shows that $S_{n,k} = 0$ for $n < k$. Also, for $n = k$, the sum in (4) is obviously $(-1)^k k!$ (without differentiation!), so we can conclude that $S_{k,k}$ has this value.

Theorem 5 can be restated in matrix form, as follows:

THEOREM 6. *Let A be the matrix $a_{k,r} = \binom{k}{r}$ for $0 \leq k \leq N$, $0 \leq r \leq N$. Then the inverse of A is the matrix $b_{k,r} = (-1)^{k+r} \binom{k}{r}$.*

Proof. For $n \le k$, element (k,n) of the product AB is

$$c_{k,n} = \sum_{r=0}^{N} a_{k,r} b_{r,n} = \sum_{r=0}^{k} (-1)^{n+r} \binom{k}{r} \binom{r}{n} = (-1)^n T_{n,k}.$$

Hence $c_{k,n}$ equals 0 for $n < k$ and 1 for $n = k$. Also, clearly, $c_{k,n} = 0$ for $n > k$. □

COROLLARY 6.1. *Let $(x_j)_{j\ge 1})$ be any sequence, and let $y_k = \sum_{r=0}^{k} \binom{k}{r} x_r$ for each $k > 0$. Then $x_k = (-1)^k \sum_{r=0}^{k} (-1)^r \binom{k}{r} y_r$ for each k.*

Proof. Restricted to $k \le N$, $r \le N$ and allowing for zero terms, the two statements equate to $y = Ax$ and $x = By$. (Alternatively, substitute for y_r in the second sum and apply Theorem 5 directly.) □

Stirling numbers (second kind)

The combinatorial definition of the *Stirling numbers of the second kind* is: $S(n,k)$ is the number of partitions of E_n into k non-empty subsets. (Distinguish carefully the notation $S(n,k)$ and $S_{n,k}$.)

Clearly $S(n,k) = 0$ when $n < k$. Also, $S(n,1) = S(n,n) = 1$. Further, $S(n,n-1) = \binom{n}{2}$, since this requires just one subset to have 2 elements.

The numbers $S(n,k)$ and $S_{n,k}$ are very closely related, as follows:

THEOREM 7. *We have*

$$S(n,k) = \frac{(-1)^k S_{n,k}}{k!}. \tag{5}$$

Proof. Let $A_1 \cup \ldots \cup A_k$ be a partition of E_n. For each of the $k!$ permutations σ of E_k, a surjection $f : E_n \to E_k$ is defined by putting $f(i) = \sigma(j)$ for all $i \in A_j$. All surjections are obtained in this way, since if f is a surjection, then the k sets $A_j = f^{-1}(j)$ partition E_n. So by Theorem 4, $(-1)^k S_{n,k} = k! S(n,k)$. □

Theorem 3 follows at once, since the numbers $S(n,k)$ are clearly integers.

THEOREM 8. *For $n \ge 1$ and $k \ge 2$,*

$$S(n+1,k) = kS(n,k) + S(n,k-1). \tag{6}$$

Proof 1. For each partition of E_n into k subsets, we obtain k different partitions of E_{n+1} into k subsets by inserting $n+1$ into one of the subsets. For each partition of E_n into $k-1$ subsets, a corresponding partition of E_{n+1} into k subsets is obtained by taking

$\{n+1\}$ as a further subset. These two types account for all partitions of E_{n+1} into k subsets, depending on whether $n+1$ is on its own or combined with other elements. □

Proof 2 (assuming Theorems 1 and 7). Substituting (5) into (2), we obtain

$$(-1)^k k! S(n+1,k) = k(-1)^k k! S(n,k) - k(-1)^{k-1}(k-1)! S(n,k-1),$$

which equates to (6). □

Conversely, Proof 1 can be combined with Theorem 1 to prove Theorem 7 without Theorem 4.

One can use (6) to build a table of values of $S(n,k)$ in the same style as Pascal's triangle. We record the values for $n \leq 8$:

k	1	2	3	4	5	6	7	8
$S(1,k)$	1							
$S(2,k)$	1	1						
$S(3,k)$	1	3	1					
$S(4,k)$	1	7	6	1				
$S(5,k)$	1	15	25	10	1			
$S(6,k)$	1	31	90	65	15	1		
$S(7,k)$	1	63	301	350	140	21	1	
$S(8,k)$	1	127	966	1701	1050	266	28	1

We now describe an algebraic characterisation of the Stirling numbers, which can be used as an alternative definition. It is clear that one can express x^n as follows:

$$x^n = c_{n,1}x + c_{n,2}x(x-1) + \cdots + c_{n,n}x(x-1)\ldots(x-n+1) \tag{7}$$

for certain numbers $c_{n,k}$. In fact, the coefficient of x^n shows that $c_{n,n} = 1$, and the numbers $c_{n,k}$ are determined by the coefficient of x^k in descending order of k. Putting $x = 1$ gives $c_{n,1} = 1$. This is a special case of Newton's form of the interpolating polynomial, and the $c_{n,k}$ equate to "divided differences".

THEOREM 9. *Identity (7) holds with $c_{n,k} = S(n,k)$.*

Proof 1. This is trivial for $n = 1$. Assume it for some $n \geq 1$. Write the term $S(n,k)x(x-1)\ldots(x-k+1)$ as u_k. Then

$$xu_k = [(x-k)+k]u_k = S(n,k)x(x-1)\ldots(x-k) + kS(n,k)x(x-1)\ldots(x-k+1).$$

In the expression for x^{n+1}, combining this with the term xu_{k-1} (for $2 \leq k \leq n$), we see that the coefficient of $x(x-1)\ldots(x-k+1)$ is $S(n,k-1) + kS(n,k)$, which equals $S(n+1,k)$, by Theorem 8. Also, the coefficients of x and $x(x-1)\ldots(x-n)$ are 1. □

Proof 2. Both sides of (7) are polynomials in x of degree n. We show that equality holds for all positive integer values of x: it then follows that it holds for all real values of x. Take a positive integer r. There are r^n mappings from E_n to E_r. For $1 \leq k \leq n$, we enumerate the mappings with k elements in their range. There are $S(n,k)$ partitions $A_1 \cup \ldots \cup A_k$ of E_n into k non-empty subsets. Given such a partition, choose distinct elements $p_1, p_2, \ldots, p_k$ of E_r in order, and define $f(i) = p_j$ for $i \in A_j$: then f has k elements in its range. The number of ways to choose the p_j is $r(r-1)\ldots(r-k+1)$. The required identity follows. □

Yet another proof of Theorem 9, leading directly to the explicit expression (5) for $S(n,k)$, can be given, with some effort, by the standard procedure for constructing interpolating polynomials.

The series $(e^x - 1)^k$ and applications

The "exponential generating function" of a sequence (a_n) is $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$. For $(S_{n,k})$ (for fixed k), we find:

THEOREM 10. *For $k \geq 1$, we have*

$$(e^x - 1)^k = (-1)^k \sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} x^n, \tag{8}$$

Proof. By the binomial theorem,

$$(e^x - 1)^k = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} e^{rx}.$$

To avoid terms like 0^0, we use the fact that $\sum_{r=0}^{k} (-1)^{k-r}\binom{k}{r} = (1-1)^k = 0$ to rewrite this as

$$\begin{aligned}
(e^x - 1)^k &= \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} (e^{rx} - 1) \\
&= (-1)^k \sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{n=1}^{\infty} \frac{r^n}{n!} x^n \\
&= (-1)^k \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n \\
&= (-1)^k \sum_{n=1}^{\infty} \frac{S_{n,k}}{n!} x^n.
\end{aligned}$$

At the same time,

$$(e^x - 1)^k = \left(\sum_{r=1}^{\infty} \frac{x^r}{r!} \right)^k,$$

which is of the form $\sum_{n=0}^{\infty} c_{n,k} x^n$, with $c_{n,k} = 0$ for $n < k$. Equating coefficients, we see that $S_{n,k} = 0$ for $n < k$ (yet another proof of Theorem 2!), so the sum in (8) can be restricted to $n \geq k$. Also, we can conclude (again) that $S_{k,k} = (-1)^k k!$. □

By substituting $e^x - 1$ into other power series and equating coefficients, we can derive further identities involving $S_{n,k}$.

THEOREM 11. *We have*

$$\sum_{k=1}^{n} S_{n,k} = (-1)^n. \tag{9}$$

Proof. Substitute $y = e^x - 1$ in the geometric series $(1+y)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k y^k$. We obtain (for sufficiently small x)

$$\begin{aligned} e^{-x} &= 1 + \sum_{k=1}^{\infty} (-1)^k (e^x - 1)^k \\ &= 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} S_{n,k}. \end{aligned}$$

But also $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$. Equating coefficients, we see that $\sum_{k=1}^{n} S_{n,k} = (-1)^n$. □

The reversal of summation can be justified as follows. Recall that $(-1)^k S_{n,k} \geq 0$. Now using the series for $(1-y)^{-1}$ instead of $(1+y)^{-1}$, we see that the double series with $S_{n,k}$ replaced by $|S_{n,k}|$ converges when $|e^x - 1| < 1$.

THEOREM 12. *For* $n \geq 2$,

$$\sum_{k=1}^{n} \frac{S_{n,k}}{k} = 0. \tag{10}$$

Proof. Note that if $y = e^x - 1$, then $\log(1+y) = x$. Substitute this in the series $\log(1+y) = \sum_{k=1}^{\infty} (-1)^{k-1} y^k / k$. We obtain

$$\begin{aligned} x &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (e^x - 1)^k \\ &= -\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} x^n \\ &= -\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} \frac{S_{n,k}}{k}. \end{aligned}$$

So for $n \geq 2$, the coefficient of x^n is 0, hence (10). □

Recall that the *Bernoulli numbers* B_n are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n. \tag{11}$$

THEOREM 13. *For all* $n \geq 1$,

$$B_n = \sum_{k=1}^{n} \frac{S_{n,k}}{k+1}. \tag{12}$$

Proof. Again with $y = e^x - 1$, we use the series $\frac{1}{y} \log(1+y) = \sum_{k=0}^{\infty} (-1)^k y^k/(k+1)$. Substituting, we obtain

$$\begin{aligned} \frac{x}{e^x - 1} &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} (e^x - 1)^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} \frac{S_{n,k}}{k+1}. \end{aligned}$$

Equating the coefficients to those in (11), we obtain (12). □

Other proofs are known (see [G], [Car]), but this one is arguably as good as any.

Divisibility by primes

Using Fermat's "little" theorem, we can derive some results on the divisibility of $S_{n,k}$ and $S(n,k)$ by primes.

THEOREM 14. *If* p *is prime and* $2 \leq k \leq p-1$, *then* $S_{p,k}$ *and* $S(p,k)$ *are multiples of* p.

Proof. By Fermat's theorem, if $1 \leq r \leq k$, then $r^p \equiv r \bmod p$. Hence

$$S_{p,k} \equiv \sum_{r=1}^{k} (-1)^r \binom{k}{r} r = S_{1,k} \quad \bmod p.$$

But $S_{1,k} = 0$, so $S_{p,k}$ is a multiple of p. Since $S_{p,k} = (-1)^k k! S(p,k)$ and p does not divide $k!$, it divides $S(p,k)$. □

THEOREM 15. *Let* p *be prime and* $n \geq 2$. *Then:*

(i) *if* $p-1$ *does not divide* n, *then* $S_{n,p-1}$ *and* $S(n,p-1)$ *are multiples of* p;

(ii) *if* $p-1$ *divides* n, *then* $S_{n,p-1} \equiv -1$ *and* $S(n,p-1) \equiv 1$ *mod* p.

Proof. If $p-1$ does not divide n, then $n = k(p-1) + n'$, where k is an integer and $1 \le n' \le p-2$. By Fermat's theorem, $r^n \equiv r^{n'}$ mod p for each r. So $S_{n,p-1} \equiv S_{n',p-1}$ mod p. But by Theorem 2, $S_{n',p-1} = 0$.

If $p-1$ divides n, then, by Fermat's theorem, $r^n \equiv 1$ mod p for $1 \le r \le p-1$, so

$$S_{n,p-1} \equiv \sum_{r=1}^{p-1} (-1)^r \binom{p-1}{r} \quad \text{mod } p.$$

But $\sum_{r=1}^{p-1} (-1)^r \binom{p-1}{r} = (1-1)^{p-1} - 1 = -1$. By Wilson's theorem, $(p-1)! \equiv -1$ mod p, hence $S(n, p-1) \equiv 1$ mod p. □

A neat application is the following result, which is more commonly proved by number-theoretic methods, e. g. [HWr], Theorem 119.

THEOREM 16. *If p is prime and $p-1$ does not divide n, then $\sum_{r=1}^{p-1} r^n$ is a multiple of p.*

Proof. For $1 \le r \le p-1$, $\binom{p}{r}$ is a multiple of p. Since $\binom{p}{r} = \binom{p-1}{r-1} + \binom{p-1}{r}$, it follows easily that $\binom{p-1}{r} \equiv (-1)^r$ mod p. The statement follows, by Theorem 15(i). □

(If $p-1$ divides n, then by Fermat's theorem again, $\sum_{r=1}^{p-1} r^n \equiv -1$ mod p.)

Using the expression for the Bernoulli numbers in Theorem 13, we can deduce the following theorem on these numbers, which was proved by von Staudt and Clausen as long ago as 1840. This proof was outlined in [Car], where it is attributed to Lucas. See also [Jam].

THEOREM 17. *Let $n \ge 2$ be even and let P_n be the set of primes p such that $p-1$ divides n. Then*

$$B_n = A_n - \sum_{p \in P_n} \frac{1}{p},$$

where A_n is an integer. If $B_n = a_n/b_n$ in lowest terms, then $b_n = \prod_{p \in P_n} p$.

Proof. Consider the expression (12) for B_n. Recall from Theorem 3 that $S_{n,k}$ is a multiple of $k!$. If $k \ge 5$ and $k+1$ is composite, then $k+1$ divides $k!$: if $k+1 = ab$, where $a \ne b$, then a and b appear in the product defining $k!$, while if $k+1 = a^2$, with $a \ge 3$, then a and $2a$ appear. So for all such k, $S_{n,k}/(k+1)$ is an integer.

The case $k = 3$ has to be treated separately. Now $S_{n,3} = -3 + 3.2^n - 3^n$. Since n is even, $3^n \equiv 1$ mod 4, so $3^n + 3$, hence also $S_{n,3}$, is a multiple of 4.

Now suppose that $k = p-1$, where p is prime. By Theorem 15, if $p-1$ does not divide

n, then $S_{n,k}$ is a multiple of $k+1$. If $p-1$ divides n, then $S_{n,p-1} = mp - 1$ for some integer m, so $\frac{1}{p}S_{n,p-1} = m - \frac{1}{p}$. This proves the first statement, and the second one follows easily. □

A special case. If $n = 2q$, where q is prime and $2q+1$ is composite (which occurs, for example, if q is congruent to 1 mod 3 or to 2 mod 5), then 2 and 3 are the only elements of P_n, so $B_n = A'_n + \frac{1}{6}$ for an integer A'_n.

Stirling numbers (first kind)

We have left these numbers to the end because they are not really connected with the main flow of our results. We treat them rather more briefly.

The Stirling numbers of the first kind divide further into two subspecies, "signed" and "unsigned". The algebraic definition for both is inverse to the definition seen in Theorem 9 for numbers of the first kind. For $1 \le k \le n$, the "signed" numbers $s(n,k)$ and the "unsigned" ones $s'(n,k)$ are defined by

$$x(x-1)\ldots(x-n+1) = \sum_{k=1}^{n} s(n,k)x^k, \tag{13}$$

$$x(x+1)\ldots(x+n-1) = \sum_{k=1}^{n} s'(n,k)x^k. \tag{14}$$

For $k > n$, we define $s(n,k) = s'(n,k) = 0$. Obviously, $s'(n,k) > 0$ whenever $k \le n$. Substituting $-x$ for x in (13), we see that $s'(n,k) = (-1)^{n+k}s(n,k)$, so in fact $s'(n,k) = |s(n,k)|$. The notation $s(n,k)$ is (more or less) standard, but $s'(n,k)$ is not.

Clearly, $s(n,n) = 1$ and $s'(n,1) = (n-1)!$. With a little more thought, one sees that $s'(n,n-1) = \frac{1}{2}n(n-1)$ and $s'(n,2) = (n-1)!H_{n-1}$, where H_n is the harmonic sum $\sum_{r=1}^{n} \frac{1}{r}$.

Taking $x = 1$ in (13) and (14) in turn, we see that $\sum_{k=1}^{n} s(n,k) = 0$ for $n \ge 2$, and $\sum_{k=1}^{n} s'(n,k) = n!$ for all n.

The combinatorial interpretation is: $s'(n,k)$ is the number of permutations of E_n that factorise into k cycles. See [Cam, p. 80].

THEOREM 18. *For $k \ge 2$, we have*

$$s(n+1,k) = -ns(n,k) + s(n,k-1), \tag{15}$$

$$s'(n+1,k) = ns'(n,k) + s'(n,k-1), \tag{16}$$

Proof. The expression for $x(x-1)\ldots(x-n+1)$ contains the terms $s(n,k-1)x^{k-1}$

and $s(n,k)x^k$. Multiplying by $x-n$, we see that the coefficient of x^k in $x(x-1)\dots(x-n)$ is $s(n,k-1) - ns(n,k)$. □

We record the values for $n \leq 6$:

k	1	2	3	4	5	6
$s'(1,k)$	1					
$s'(2,k)$	1	1				
$s'(3,k)$	2	3	1			
$s'(4,k)$	6	11	6	1		
$s'(5,k)$	24	50	35	10	1	
$s'(6,k)$	120	274	225	85	15	1

There is no known explicit formula for $s(n,k)$ or $s'(n,k)$ corresponding to (5).

From the algebraic definitions, it is clear that the two kinds of Stirling numbers are inverse to each other in some sense. The exact statement is:

THEOREM 19. *We have*

$$\sum_{r=1}^{n} S(n,r)s(r,k) = \begin{cases} 0 & \text{for } k < n, \\ 1 & \text{for } k = n. \end{cases} \tag{17}$$

Let A, B be the matrices with entries $S(n,r)$ and $s(n,r)$ for $1 \leq n \leq N$, $1 \leq r \leq N$. Then $B = A^{-1}$.

Proof. By (7) and (13),

$$\begin{aligned} x^n &= \sum_{r=1}^{n} S(n,r)x(x-1)\dots(x-r+1) \\ &= \sum_{r=1}^{n} S(n,r) \sum_{k=1}^{r} s(r,k)x^k \\ &= \sum_{k=1}^{n} x^k \sum_{r=k}^{n} S(n,r)s(r,k). \end{aligned}$$

Equating coefficients, and noting that $s(r,k) = 0$ for $r < k$, we obtain (17).

For $k \leq n$, the sum in (17) is element (n,k) of the matrix AB, since $S(n,r) = 0$ for $r > n$. For $k > n$, the sum is clearly 0. □

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