

The sine and cosine integrals

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The complete sine integral: first method

In these notes, we consider the integrals of $\sin t/t$ and $\cos t/t$ on intervals like $(0, \infty)$, $(0, x)$ and (x, ∞) . Most of the material appeared in [Jam1]. Companion notes [Jam2], [Jam3] deal with integrals of e^{it}/t^p and, more generally, $f(t)e^{it}$.

We start with the “complete sine integral”:

THEOREM 1. *We have*

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}. \quad (1)$$

Note first that there is no problem of convergence at 0, because $\frac{\sin t}{t} \rightarrow 1$ as $t \rightarrow 0$.

A very quick and neat proof of (1) (already to be seen, for example, in the 1909 note [Har]) lies to hand if we assume the following well-known series identity: for $x \neq k\pi$,

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x + n\pi} \quad (2)$$

One proof of (2) [Wa, p. 17–18] is by taking $x = 0$ in the Fourier series for $\cos ax$ on $[-\pi, \pi]$.

To derive (1), note first that, since $\sin t/t$ is an even function,

$$\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = 2 \int_0^{\infty} \frac{\sin t}{t} dt.$$

Denote this by I . The substitution $t = x + n\pi$ gives

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t} dt = (-1)^n \int_0^\pi \frac{\sin x}{x + n\pi} dx.$$

Assuming that termwise integration of the series is valid, we add these identities for all integers n to obtain at once

$$I = \int_0^\pi \sin x \frac{1}{\sin x} dx = \pi.$$

The termwise integration (for any readers who care) is easily justified by uniform convergence, as follows. By combining the terms for n and $-n$ and multiplying by $\sin x$, we can rewrite the series (2) as

$$\frac{\sin x}{x} + 2x \sin x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2\pi^2} = 1.$$

For $0 < x < \pi$ and $n \geq 2$,

$$\left| \frac{2x \sin x}{x^2 - n^2 \pi^2} \right| \leq \frac{2\pi}{(n^2 - 1)\pi^2}.$$

Since $\sum_{n=2}^{\infty} 1/(n^2 - 1)$ is convergent, it follows, by Weierstrass's "M-test", that the series converges uniformly on the open interval $(0, \pi)$: this is all we need. \square

We note some immediate variants and consequences of (1). First, for any $a > 0$, the substitution $at = u$ gives

$$\int_0^{\infty} \frac{\sin at}{t} dt = \int_0^{\infty} \frac{\sin u}{u} du = \frac{\pi}{2}.$$

Hence also the value of this integral is $-\frac{\pi}{2}$ for $a < 0$.

For $a > 0$, we deduce

$$\int_0^{\infty} \frac{\sin at \cos at}{t} dt = \frac{1}{2} \int_0^{\infty} \frac{\sin 2at}{t} dt = \frac{\pi}{4}. \quad (3)$$

We will use this several times later.

Since $\sin(a + b)t + \sin(a - b)t = 2 \sin at \cos bt$, we can also deduce

$$\int_0^{\infty} \frac{\sin at \cos bt}{t} dt = \begin{cases} \frac{\pi}{2} & \text{if } a > b \geq 0, \\ 0 & \text{if } b > a \geq 0. \end{cases} \quad (4)$$

Integrating by parts, and using (3) and the fact that $\frac{\sin^2 t}{t} \rightarrow 0$ as $t \rightarrow 0$, we obtain

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \left[-\frac{\sin^2 t}{t} \right]_0^{\infty} + \int_0^{\infty} \frac{2 \sin t \cos t}{t} dt = \frac{\pi}{2}. \quad (5)$$

This argument is reversible, so (4) equally implies (1). This is a viable alternative, because one can prove (4) in a similar way to (1), using the series $1/\sin^2 x = \sum_{n=-\infty}^{\infty} [1/(x - n\pi)^2]$; this method is followed in [Wa, p. 186–187].

By the cosine series, $\cos at - \cos bt = \frac{1}{2}(b-a)t^2 + O(t^4)$ for small t , so $\frac{1}{t^2}(\cos at - \cos bt) \rightarrow \frac{1}{2}(b-a)$ and $\frac{1}{t}(\cos at - \cos bt) \rightarrow 0$ as $t \rightarrow 0$. We deduce, for $a, b \geq 0$,

$$\int_0^{\infty} \frac{\cos at - \cos bt}{t^2} dt = \left[-\frac{\cos at - \cos bt}{t} \right]_0^{\infty} + \int_0^{\infty} \frac{-a \sin at + b \sin bt}{t} dt = \frac{\pi}{2}(b-a). \quad (6)$$

Since $\cos(a - b)t - \cos(a + b)t = 2 \sin at \sin bt$, we deduce

$$\int_0^{\infty} \frac{\sin at \sin bt}{t^2} dt = \frac{\pi b}{2} \quad (7)$$

if $a \geq b \geq 0$ (hence also $\frac{\pi a}{2}$ if $b \geq a \geq 0$; of course, (5) is a special case).

The incomplete sine integral

The “incomplete” sine integral is the function

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$$

First, some simple facts about it. Since $\frac{\sin t}{t} \leq 1$ for $t > 0$, we have $\text{Si}(x) \leq x$ for all $x > 0$. Of course, (1) says that $\text{Si}(x) \rightarrow \frac{\pi}{2}$ as $x \rightarrow \infty$.

The substitution $at = u$ gives $\int_0^x \frac{\sin at}{t} dt = \text{Si}(ax)$. In particular,

$$\text{Si}(2x) = \int_0^x \frac{\sin 2t}{t} dt = 2 \int_0^x \frac{\sin t \cos t}{t} dt.$$

Hence $\text{Si}(2x) \leq 2 \int_0^x \cos t dt = 2 \sin x$ for $0 \leq x \leq \frac{\pi}{2}$, equivalently $\text{Si}(x) \leq 2 \sin \frac{1}{2}x$ for $0 \leq x \leq \pi$ (this is stronger than $\text{Si}(x) \leq x$).

By the fundamental theorem of calculus, the derivative $\text{Si}'(x)$ is $\sin x/x$. Hence $\text{Si}(x)$ is increasing on intervals $[2n\pi, (2n+1)\pi]$ and decreasing on intervals $[(2n-1)\pi, 2n\pi]$, so it has maxima at the points $(2n+1)\pi$ and minima at the points $2n\pi$.

PROPOSITION 1. $\text{Si}(x) \geq 0$ for all $x > 0$, and its greatest value occurs at $x = \pi$.

Proof. Write

$$A_n = \int_{n\pi}^{(n+2)\pi} \frac{\sin t}{t} dt.$$

By substituting $t + \pi = u$ on $[n\pi, (n+1)\pi]$ and recombining, we see that

$$A_n = \int_{n\pi}^{(n+1)\pi} \left(\frac{1}{t} - \frac{1}{t+\pi} \right) \sin t dt,$$

in which $\frac{1}{t} - \frac{1}{t+\pi} > 0$. If n is even, then $\sin t \geq 0$ on $[n\pi, (n+1)\pi]$, so $A_n \geq 0$ and $\text{Si}[(n+2)\pi] \geq \text{Si}(n\pi)$. Hence $\text{Si}(2n\pi) \geq \dots \geq \text{Si}(2\pi) \geq \text{Si}(0) = 0$ for all n . Since $\text{Si}(x)$ increases on $[2n\pi, (2n+1)\pi]$ and decreases on $[(2n+1)\pi, (2n+2)\pi]$, it follows that $\text{Si}(x) \geq 0$ for all $x \geq 0$. Meanwhile, if n is odd, then $A_n \leq 0$, so that $\text{Si}(\pi) \geq \text{Si}(3\pi) \geq \dots$, hence the greatest value is $\text{Si}(\pi)$. \square

By integrating the series

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!},$$

we obtain the explicit series expression

$$\text{Si}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)} = x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \dots, \quad (8)$$

from which, in principle, $\text{Si}(x)$ can be calculated, though in practice the calculation is only pleasant for fairly small x . One finds, for example, $\text{Si}(\pi) \approx 1.85194$ (recall that this is the greatest value) and $\text{Si}(2\pi) \approx 1.41816$.

The complementary sine and cosine integrals

We cannot simply replace $\sin t$ by $\cos t$ in (1), or in the definition of $\text{Si}(x)$, because the resulting integral would be divergent at 0. To formulate results that make sense for both $\sin t$ and $\cos t$, we consider instead the complementary integrals

$$S(x) = \int_x^\infty \frac{\sin t}{t} dt, \quad C(x) = \int_x^\infty \frac{\cos t}{t} dt.$$

(Here I am departing from the established notation, which is $\text{si}(x)$ and $\text{ci}(x)$ where we have $-S(x)$ and $-C(x)$).

By (1), we have $S(0) = \frac{\pi}{2}$ and $S(x) = \frac{\pi}{2} - \text{Si}(x)$. By the remarks above, $S(x)$ has maxima at $2n\pi$ and minima at $(2n - 1)\pi$, with greatest value $\frac{\pi}{2}$ and least value $S(\pi)$. Also, $S(\pi) \approx -0.28114$ and $S(2\pi) \approx 0.15264$.

Meanwhile, $C(x)$ is defined for $x > 0$, but not at $x = 0$. It has maxima at $(2n - \frac{1}{2})\pi$ and minima at $(2n + \frac{1}{2})\pi$, with overall least value at $\frac{\pi}{2}$

The nature of $S(x)$ and $C(x)$, especially for large x , is revealed by a simple integration by parts. It adds to clarity to describe the results more generally. For a function f , define

$$I_f(x) = \int_x^\infty f(t)e^{it} dt, \quad C_f(x) = \int_x^\infty f(t) \cos t dt, \quad S_f(x) = \int_x^\infty f(t) \sin t dt,$$

assuming that these integrals converge. So $I_f(x) = C_f(x) + iS_f(x)$. Results for $I_f(x)$ will of course deliver simultaneous results for $C_f(x)$ and $S_f(x)$. The reader just needs to accept that $\frac{d}{dt}e^{it} = ie^{it}$ and that the usual processes of calculus, such as integration by parts, work in the same way for complex functions of a real variable.

We assume that $f(t)$ is *completely monotonic*, that is:

$$(CM) \quad \text{for all } n \geq 0, \quad (-1)^n f^{(n)}(t) \geq 0 \text{ for } t > 0 \text{ and } f^{(n)}(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Note that this implies that $(-1)^n f^{(n)}(t)$ is decreasing and that $(-1)^n f^{(n)}(t)$ also satisfies (CM). Of course, $f(t) = 1/t^p$ is completely monotonic for all $p > 0$.

For such a function, consider $I_{f'}(x)$ (distinguish between this and $I_f'(x)$!). Condition (CM) implies that $f'(t) \leq 0$ and $\int_x^\infty (-f'(t)) dt = f(x)$. Since $|f'(t)e^{it}| \leq -f'(t)$, the integral defining $I_{f'}(x)$ is convergent, and we have

$$|I_{f'}(x)| \leq f(x). \tag{9}$$

Now integrate by parts:

$$I_f(x) = \left[-if(t)e^{it} \right]_x^\infty + i \int_x^\infty f'(t)e^{it} dt = if(x)e^{ix} + iI_{f'}(x).$$

We summarise this information in the following result

PROPOSITION 2. *If f is completely monotonic, then the integrals defining $I_f(x)$ and $I_{f'}(x)$ are convergent for all $x > 0$, and the following statements apply:*

$$I_f(x) = if(x)e^{ix} + iI_{f'}(x), \quad (10)$$

$$C_f(x) = -f(x) \sin x - S_{f'}(x), \quad S_f(x) = f(x) \cos x + C_{f'}(x), \quad (11)$$

Further,

$$|I_f(x)| \leq 2f(x). \quad \square \quad (12)$$

Note that (12) follows at once from (9) and (10). Also, it applies to f' (since $-f'$ is completely monotonic) to give $|I_{f'}(x)| \leq -2f'(x)$. Actually, (12) can be improved to $|I_f(x)| \leq f(x)$: see [Jam3].

We restate these results for our case $f(x) = 1/x$. Taking a slight liberty with the notation, we write

$$I_n(x) = \int_x^\infty \frac{e^{it}}{t^n} dt,$$

and similarly $C_n(x)$, $S_n(x)$ (also, write $I(x)$ for $I_1(x)$).

PROPOSITION 3. *For all $x > 0$,*

$$I(x) = \frac{ie^{ix}}{x} - iI_2(x), \quad (13)$$

$$C(x) = -\frac{\sin x}{x} + S_2(x), \quad S(x) = \frac{\cos x}{x} - C_2(x), \quad (14)$$

Also, $|I(x)| \leq 2/x$ and $|I_2(x)| \leq 2/x^2$. Hence $xI(x) - ie^{ix}$, $xS(x) - \cos x$ and $xC(x) + \sin x$ tend to 0 as $x \rightarrow \infty$. \square

By repeating the process, we can derive increasingly accurate approximations, as follows. (At this point, the reader could skip to “The function $C^*(x)$ ”.)

PROPOSITION 4. *Let f be completely monotonic. Then for all $x > 0$,*

$$I_f(x) = if(x)e^{ix} - f'(x)e^{ix} - I_{f''}(x),$$

$$C_f(x) = -f(x) \sin x - f'(x) \cos x - C_{f''}(x),$$

$$S_f(x) = f(x) \cos x - f'(x) \sin x - S_{f''}(x). \quad (15)$$

Further,

$$\begin{aligned} I_f(x) &= [f(x) - f''(x)] ie^{ix} - [f'(x) - f^{(3)}(x)] e^{ix} + I_{f^{(4)}}(x), \\ C_f(x) &= -[f(x) - f''(x)] \sin x - [f'(x) - f^{(3)}(x)] \cos x + C_{f^{(4)}}(x), \\ S_f(x) &= [f(x) - f''(x)] \cos x - [f'(x) - f^{(3)}(x)] \sin x + S_{f^{(4)}}(x). \end{aligned} \quad (16)$$

Proof. Applying (10) to $-f'(t)$ and substituting back into (10), we obtain (15) for $I_f(x)$. Now apply (15) to $f''(t)$ and substitute, obtaining:

$$I_f(x) = if(x)e^{ix} - f'(x)e^{ix} - if''(x)e^{ix} + f^{(3)}(x)e^{ix} + I_{f^{(4)}}(x).$$

which equates to (16). □

We have alternative bounds for the remainder terms, from (9) and (12). For example, $|I_{f^{(4)}}(x)|$ is bounded both by $-f^{(3)}(x)$ and by $2f^{(4)}(x)$.

Of course, the process can be continued: successive derivatives of $f(x)$ appear in the expressions multiplying e^{ix} and ie^{ix} . The outcome is an asymptotic expansion for $I_f(x)$. However, this does not simply deliver ever-closer approximations, because for a fixed x , the derivatives $f^{(n)}(x)$ will ultimately grow large in magnitude.

We restate (16) explicitly for the case $f(t) = 1/t$:

PROPOSITION 5. For $x > 0$,

$$\begin{aligned} I(x) &= \left(\frac{1}{x} - \frac{2}{x^3}\right) ie^{ix} + \left(\frac{1}{x^2} - \frac{6}{x^4}\right) e^{ix} + 24I_5(x), \\ C(x) &= -\left(\frac{1}{x} - \frac{2}{x^3}\right) \sin x + \left(\frac{1}{x^2} - \frac{6}{x^4}\right) \cos x + 24C_5(x), \\ S(x) &= \left(\frac{1}{x} - \frac{2}{x^3}\right) \cos x + \left(\frac{1}{x^2} - \frac{6}{x^4}\right) \sin x + 24S_5(x). \end{aligned} \quad (17)$$

Further, $24|I_5(x)|$ is bounded by both $6/x^4$ and $48/x^5$. □

We deduce a bound for $|I(x)|$:

PROPOSITION 6. For $x \geq 2$,

$$|I(x)| < \frac{1}{x} - \frac{3}{2x^3} + \frac{6}{x^4}.$$

Proof. Denote the expression for $I(x)$ in (17) by $F(x) + 24I_5(x)$. Assume that $x \geq 2$ and write $y = 1/x$, so that $y \leq \frac{1}{2}$. Then

$$\begin{aligned} |F(x)|^2 &= (y - 2y^3)^2 + (y^2 - 6y^4)^2 \\ &= y^2 - 3y^4 - 8y^6 + 36y^8 \\ &= (y - \frac{3}{2}y^3)^2 - 10\frac{1}{4}y^6 + 36y^8 \\ &< (y - \frac{3}{2}y^3)^2, \end{aligned}$$

so $|F(x)| < y - \frac{3}{2}y^3$. The stated inequality follows. \square

This bound is smaller than $1/x$ when $x > 4$. It is used in [JLM] as a stage in the proof of the stronger inequality $|S(x)| \leq \frac{\pi}{2} - \tan^{-1} x$ (note that this is exact at 0).

The function $C^(x)$*

Can we find a formula that enables us to calculate $C(x)$, and that opens the way to some kind of analogue of (1)? The key is to introduce the function

$$C^*(x) = \int_0^x \frac{1 - \cos t}{t} dt$$

(This function is sometimes denoted by $\text{Cin}(x)$). It is elementary that $0 \leq 1 - \cos t \leq \frac{1}{2}t^2$, hence $0 \leq \frac{1 - \cos t}{t} \leq \frac{1}{2}t$, for $t > 0$. So there is no problem of convergence of the integral at 0, and we have $0 \leq C^*(x) \leq \frac{1}{4}x^2$ for all $x > 0$. By integrating the series

$$\frac{1 - \cos t}{t} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{(2n)!},$$

we obtain the power series expression

$$C^*(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(2n)!(2n)} = \frac{x^2}{2!2} - \frac{x^4}{4!4} + \dots \quad (18)$$

For example, $C^*(\frac{\pi}{2}) \approx 0.55680$ and $C^*(\pi) \approx 1.64828$.

We now relate $C^*(x)$ and $C(x)$. We have

$$C^*(x) - C^*(1) = \int_1^x \frac{1 - \cos t}{t} dt = \log x - \int_1^x \frac{\cos t}{t} dt = \log x - C(1) + C(x),$$

so

$$C(x) = C^*(x) - \log x + c, \quad (19)$$

where c is constant, in fact $c = C(1) - C^*(1)$.

Even without knowing c , we can draw some conclusions from (6). One, which we will use later, is $C(x) \sim -\log x$, hence $x C(x) \rightarrow 0$, as $x \rightarrow 0^+$. Another is the following integral, which can be compared with (6). It is a special case of the ‘‘Frullani integral’’: see [Fer, p. 133–135], [Jam4] or [Tr], where it is used in the evaluation of the integral of $\sin^n x/x^m$ (I am grateful to Nick Lord for the references [Fer] and [Tr]).

PROPOSITION 7. For $a, b > 0$,

$$\int_0^\infty \frac{\cos at - \cos bt}{t} dt = \log b - \log a. \quad (20)$$

Proof. The substitution $at = u$ gives

$$\int_0^x \frac{1 - \cos at}{t} dt = \int_0^{ax} \frac{1 - \cos u}{u} du = C^*(ax).$$

Hence

$$\begin{aligned} \int_0^x \frac{\cos at - \cos bt}{t} dt &= C^*(bx) - C^*(ax) \\ &= C(bx) - C(ax) + \log bx - \log ax \\ &= C(bx) - C(ax) + \log b - \log a \\ &\rightarrow \log b - \log a \quad \text{as } x \rightarrow \infty. \quad \square \end{aligned}$$

However, for a fully satisfactory version of (19), and for the calculation of $C(x)$, of course we need to know the value of c . The answer turns out to be that $c = -\gamma$, where γ is Euler’s constant. Let us state this fact as a theorem:

THEOREM 2. We have

$$C(x) = C^*(x) - \log x - \gamma. \quad (21)$$

Surprisingly, this result is not mentioned in the comprehensive article [Lag] on Euler’s constant. It can be seen stated without proof in compilations of formulae, such as Wikipedia or [DLMF, chapter 6] However, it is not easy to find accessible references with a proof. At the same time, the method will also give a second proof of Theorem 1. Later, we describe an alternative route to both theorems using contour integration.

Two limit expressions for c follow from (19). Since $C(x) \rightarrow 0$ as $x \rightarrow \infty$, we have

$$C^*(x) - \log x \rightarrow -c \quad \text{as } x \rightarrow \infty. \quad (22)$$

This will be used in our proof of (21). Also, since $C^*(x) \rightarrow 0$ as $x \rightarrow 0^+$, we have

$$C(x) + \log x \rightarrow c \quad \text{as } x \rightarrow 0^+. \quad (23)$$

This (once we know that $c = -\gamma$) describes the nature of $C(x)$ near 0, so can be regarded as the true analogue of (1).

Also, (21), together with (18), enables us to calculate $C(x)$. We find, for example, $C(\frac{\pi}{2}) \approx -0.47200$ (recall that this is the least value) and $C(\pi) \approx -0.07367$.

Second proof of Theorem 1 and a proof of Theorem 2

We will use the following elementary version of the Riemann-Lebesgue Lemma, which is easily proved by integration by parts: *if f is continuous on $[a, b]$ and has a continuous derivative on (a, b) , then*

$$\int_a^b f(t) \sin nt \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and similarly with $\sin nt$ replaced by $\cos nt$. We also use:

LEMMA 1. *Let*

$$h(t) = \frac{1}{t} - \frac{1}{\sin t}.$$

Then $h(t) \rightarrow 0$ as $t \rightarrow 0$.

Proof. By the series for $\sin t$, and the continuity of power series functions, we have

$$\begin{aligned} h(t) = \frac{t - \sin t}{t \sin t} &= \frac{t^3/3! - t^5/5! + \dots}{t^2 - t^4/3! + \dots} \\ &= \frac{t/3! - t^3/5! + \dots}{1 - t^2/3! + \dots} \\ &\rightarrow 0 \quad \text{as } t \rightarrow 0. \quad \square \end{aligned}$$

Second proof of Theorem 1. It is sufficient to show that $I_n \rightarrow \frac{\pi}{2}$ as $n \rightarrow \infty$, where

$$I_n = \int_0^{(n+\frac{1}{2})\pi} \frac{\sin t}{t} \, dt.$$

Substituting $t = (2n + 1)u$ (and then writing t for u), we have

$$I_n = \int_0^{\pi/2} \frac{\sin(2n + 1)t}{t} \, dt.$$

Let $D_n(t) = 1 + 2 \sum_{r=1}^n \cos 2rt$ (applied to $\frac{1}{2}t$, this is the *Dirichlet kernel*). Note that $\int_0^{\pi/2} \cos 2rt \, dt = 0$ for non-zero integers r , so $\int_0^{\pi/2} D_n(t) \, dt = \frac{\pi}{2}$.

Since $\sin(a + b) - \sin(a - b) = 2 \cos a \sin b$, we have

$$\sin(2r + 1)t - \sin(2r - 1)t = 2 \cos 2rt \sin t.$$

Adding for $1 \leq r \leq n$, we obtain

$$\sin(2n+1)t - \sin t = 2 \sin t \sum_{r=1}^n \cos 2rt,$$

hence

$$D_n(t) = \frac{\sin(2n+1)t}{\sin t}.$$

So we have

$$I_n - \frac{\pi}{2} = I_n - \int_0^{\pi/2} D_n(t) dt = \int_0^{\pi/2} h(t) \sin(2n+1)t dt.$$

By Lemma 1, $h(t)$ becomes continuous on $[0, \frac{\pi}{2}]$ if assigned the value 0 at 0. So the Riemann-Lebesgue Lemma applies to show that $I_n - \frac{\pi}{2} \rightarrow 0$ as $n \rightarrow \infty$. \square

Though this proof of (1) is not quite as neat as our first one, it is more self-contained because it does not depend on the series (2). It appears in numerous books, e.g. [Ti, p. 42–43]. For readers familiar with it, we mention that the *Fejér kernel* can be used in a similar way to prove (5) instead of (1).

Proof of Theorem 2. Recall from (22) that $-c = \lim_{x \rightarrow \infty} [C^*(x) - \log x]$. Let $J_n = C^*[(n + \frac{1}{2})\pi]$. We will show that $J_n - \log n\pi \rightarrow \gamma$ as $n \rightarrow \infty$, with n restricted to even values. Since $\log(n + \frac{1}{2})\pi - \log n\pi \rightarrow 0$ as $n \rightarrow \infty$, this will imply that $c = -\gamma$.

Substituting $t = (2n+1)u$ (and then writing t for u), we have

$$J_n = \int_0^{\pi/2} \frac{1 - \cos(2n+1)t}{t} dt.$$

Let $\tilde{D}_n(t) = 2 \sum_{r=1}^n \sin 2rt$ (applied to $\frac{1}{2}t$, this is the *conjugate Dirichlet kernel*). Since $\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$, we have

$$\cos(2r-1)t - \cos(2r+1)t = 2 \sin 2rt \sin t,$$

hence by addition

$$\tilde{D}_n(t) = \frac{\cos t - \cos(2n+1)t}{\sin t},$$

so that

$$\frac{1 - \cos(2n+1)t}{t} = \tilde{D}_n(t) + \frac{1}{t} - \frac{\cos t}{\sin t} - h(t) \cos(2n+1)t. \quad (24)$$

The integral of $\tilde{D}_n(t)$, unlike the integral of $D_n(t)$, needs a bit of work. Observe that

$$\int_0^{\pi/2} \sin 2rt dt = \frac{1}{2r} (1 - \cos r\pi) = \begin{cases} 0 & \text{for } r \text{ even,} \\ \frac{1}{r} & \text{for } r \text{ odd.} \end{cases}$$

For even n , the odd numbers less than n can be listed as $2r - 1$ for $1 \leq r \leq \frac{n}{2}$, so

$$\int_0^{\pi/2} \tilde{D}_n(t) dt = \sum_{r=1}^{n/2} \frac{2}{2r-1}.$$

LEMMA 2. *We have*

$$\sum_{r=1}^k \frac{2}{2r-1} = \log k + 2 \log 2 + \gamma + \rho_k,$$

where $\rho_k \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Write $H_k = \sum_{r=1}^k \frac{1}{r}$. Then

$$\sum_{r=1}^k \frac{2}{2r-1} = 2H_{2k} - \sum_{r=1}^k \frac{2}{2r} = 2H_{2k} - H_k.$$

Now $H_k = \log k + \gamma + q_k$, where $q_k \rightarrow 0$ as $k \rightarrow \infty$. So

$$\begin{aligned} 2H_{2k} - H_k &= 2 \log 2k + 2\gamma + 2q_{2k} - \log k - \gamma - q_k \\ &= \log k + 2 \log 2 + \gamma + \rho_k, \end{aligned}$$

where $\rho_k = 2q_{2k} - q_k \rightarrow 0$ as $k \rightarrow \infty$. □

Applying this with $k = \frac{n}{2}$, we have

$$\int_0^{\pi/2} \tilde{D}_n(t) dt = \log n + \log 2 + \gamma + \rho_{n/2}. \quad (25)$$

Completion of the proof of Theorem 2. As before, by the Riemann-Lebesgue Lemma, $\int_0^{\pi/2} h(t) \cos(2n+1)t dt \rightarrow 0$ as $n \rightarrow \infty$: denote this by σ_n . Now

$$\begin{aligned} \int_0^{\pi/2} \left(\frac{1}{t} - \frac{\cos t}{\sin t} \right) dt &= \lim_{\delta \rightarrow 0^+} \left[\log t - \log \sin t \right]_{\delta}^{\pi/2} \\ &= \log \frac{\pi}{2} + \lim_{\delta \rightarrow 0^+} \log \frac{\sin \delta}{\delta} \\ &= \log \frac{\pi}{2}, \end{aligned}$$

since $\lim_{\delta \rightarrow 0^+} \frac{\sin \delta}{\delta} = 1$. Inserting this and (25) into (24), we obtain

$$J_n = \log n + \log 2 + \gamma + \rho_{n/2} + \log \frac{\pi}{2} - \sigma_n = \log n\pi + \gamma + r_n,$$

where $r_n \rightarrow 0$ as $n \rightarrow \infty$. □

A minor variation is to work with $2 \sum_{r=1}^n \sin(2r-1)t$. This avoids the adjustment from k to $\frac{n}{2}$, but requires the evaluation $\int_0^{\pi/2} (\frac{1}{t} - \frac{1}{\sin t}) dt = \log \frac{\pi}{4}$, which is slightly harder.

Contour integral method and the exponential integral

We now present a contour integral method that provides a third proof of Theorem 1, and at the same time establishes the equivalence of Theorem 2 with the *exponential integral*, which we now describe.

Define

$$E(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \quad E^*(x) = \int_0^x \frac{1 - e^{-t}}{t} dt.$$

$E(x)$, as well as its various mutations, is known as the “exponential integral”. Exactly as for $C(x)$, we have

$$E(x) = E^*(x) - \log x + c', \tag{26}$$

where $c' = E(1) - E^*(1)$. Integration by parts equates c' to $\int_0^\infty e^{-t} \log t dt$. It is a well-known fact that $c' = -\gamma$, so that

$$E(x) = E^*(x) - \log x - \gamma, \tag{27}$$

For a proof, see [BM, p. 176–177], [Lo] or [Jam1]. By (19) and (26),

$$C(x) - E(x) = C^*(x) - E^*(x) + c - c'. \tag{28}$$

We will show that $C^*(x) - E^*(x) \rightarrow 0$ as $x \rightarrow \infty$. By (28), it then follows that $c = c'$, and hence that (21) and (27) are equivalent (without evaluating either). Another way to restate $c = c'$ is $C(x) - E(x) \rightarrow 0$ as $x \rightarrow 0^+$, hence

$$\int_0^\infty \frac{\cos t - e^{-t}}{t} dt = 0. \tag{29}$$

Let C_R be the circular arc of radius R in the positive quadrant, represented by $z = Re^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$. Denote by Γ the closed contour consisting of C_R together with the real interval $[0, R]$, and the imaginary axis from iR to 0 . Let

$$f(z) = \frac{1 - e^{iz}}{z}.$$

Then f has no pole at 0 , since $f(z) = -i + \frac{1}{2}z + \dots$. By Cauchy’s integral theorem, $\int_\Gamma f(z) dz = 0$. The contribution of the real axis is

$$\int_0^R \frac{1 - e^{it}}{t} dt = C^*(R) - i\text{Si}(R).$$

The contribution of the imaginary axis, taken towards the origin, is

$$-\int_0^R \frac{1 - e^{-t}}{t} dt = -E^*(R).$$

Now consider C_R . Here the contribution of the term $\frac{1}{z}$ is, of course, $\frac{\pi}{2}i$. The contribution of the other term is

$$I_R =: -\int_{C_R} \frac{e^{iz}}{z} dz = -\int_0^{\pi/2} i e^{iRe^{i\theta}} d\theta.$$

Its magnitude is estimated by the following Lemma:

LEMMA 3. *We have*

$$0 \leq \int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq \frac{\pi}{2R}.$$

Proof. The function $\sin\theta$ is concave on $[0, \frac{\pi}{2}]$, since its derivative $\cos\theta$ is decreasing. This means that its graph lies above the straight line connecting its values at 0 and $\frac{\pi}{2}$, so $\sin\theta \geq \frac{2\theta}{\pi}$ on $[0, \frac{\pi}{2}]$. Hence

$$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \left[-\frac{\pi}{2R} e^{-2R\theta/\pi} \right]_0^{\pi/2} = \frac{\pi}{2R} (1 - e^{-R}). \quad \square$$

Since $|e^{iRe^{i\theta}}| = e^{-R\sin\theta}$, it follows that $|I_R| \leq \pi/(2R)$.

Considering first the imaginary part, and writing x for R for consistency with our earlier results, we obtain $|\text{Si}(x) - \frac{\pi}{2}| \leq \frac{\pi}{2x}$. This is a third proof of Theorem 1, enhanced by the stated estimate for $|\text{Si}(x) - \frac{\pi}{2}| = |S(x)|$. The method can be seen, for example, in [Bur, p. 123]). However, as already mentioned, a stronger inequality for $S(x)$ was given in [JLM].

Meanwhile, consideration of the real part shows that

$$|C^*(x) - E^*(x)| \leq \frac{\pi}{2x},$$

so that $C^*(x) - E^*(x) \rightarrow 0$ as $x \rightarrow \infty$, as required.

An alternative proof of both (1) and the identity $c = c'$ is by double integrals: see [Jam5].

Some integrals involving $S(x)$ and $C(x)$

We apply our results to some integrals involving $S(x)$ and $C(x)$ (most of which can be seen stated without proof in [DLMF, chapter 6]. These applications will actually use Theorem 1, Proposition 3 and the limit $\lim_{x \rightarrow 0^+} xC(x) = 0$ from (19), but not Theorem 2.

By the fundamental theorem of calculus, we have $S'(x) = -\sin x/x$ and $C'(x) = -\cos x/x$. Hence $\frac{d}{dx}[xS(x)] = S(x) - \sin x$ and $\frac{d}{dx}[xC(x)] = C(x) - \cos x$, so antiderivatives of $S(x)$ and $C(x)$ are as follows:

$$\int S(x) dx = xS(x) - \cos x, \quad \int C(x) dx = xC(x) + \sin x.$$

By Proposition 3, $xS(x) - \cos x \rightarrow 0$ as $x \rightarrow \infty$, so we deduce at once

$$\int_0^\infty S(x) dx = \left[xS(x) - \cos x \right]_0^\infty = 1. \quad (30)$$

Since $xC(x) \rightarrow 0$ as $x \rightarrow 0^+$, we have similarly

$$\int_0^\infty C(x) dx = \left[xC(x) + \sin x \right]_0^\infty = 0. \quad (31)$$

Next, we consider the integrals of $S(x) \sin x$ and $C(x) \cos x$. Integrating by parts and using (3), together with $\lim_{x \rightarrow \infty} S(x) = 0$, we find

$$\int_0^\infty S(x) \sin x dx = \left[-S(x) \cos x \right]_0^\infty - \int_0^\infty \frac{\sin x}{x} \cos x dx = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \quad (32)$$

Now $C(x) \sin x \rightarrow 0$ as $x \rightarrow 0^+$, since $\sin x/x \rightarrow 1$, so we have similarly

$$\int_0^\infty C(x) \cos x dx = \left[C(x) \sin x \right]_0^\infty + \int_0^\infty \frac{\cos x}{x} \sin x dx = 0 + \frac{\pi}{4} = \frac{\pi}{4}. \quad (33)$$

Of course, the integrals in (32) and (33) are really double integrals. Formal reversal of the double integrals duly delivers the stated values. However, the conditions for reversal of improper integrals are not satisfied, and one should really consider the integral on $[0, R]$ of $\int_x^R \frac{\sin t}{t} dt = S(x) - S(R)$. This simply leads, rather less directly, to the same limiting process that we have just considered.

Since $\frac{d}{dt}S(t)^2 = 2S(t)S'(t) = -2S(t)\frac{\sin t}{t}$, we can express $S(x)^2$ as an integral:

$$S(x)^2 = 2 \int_x^\infty \frac{S(t) \sin t}{t} dt,$$

so in particular we have

$$\int_0^\infty \frac{S(x) \sin x}{x} dx = \frac{1}{2}S(0)^2 = \frac{\pi^2}{8}. \quad (34)$$

Without using this expression, we deduce further

$$\int_0^\infty S(x)^2 dx = \int_0^\infty C(x)^2 dx = \frac{\pi}{2}. \quad (35)$$

To show this, integrate by parts with 1 as one factor:

$$\begin{aligned}\int_0^\infty 1 \cdot S(x)^2 dx &= \left[xS(x)^2 \right]_0^\infty + 2 \int_0^\infty xS(x) \frac{\sin x}{x} dx \\ &= 2 \int_0^\infty S(x) \sin x dx = \frac{\pi}{2},\end{aligned}$$

in which we used (32) and $\lim_{x \rightarrow \infty} [xS(x)^2] = 0$ (recall $|S(x)| \leq 2/x$). The integral of $C(x)^2$ is similar, with the additional remark that $\lim_{x \rightarrow 0^+} [xC(x)^2] = 0$.

We finish with another pair of integrals that require a little more work. The reasoning in (32) shows that $\int_0^\infty S(x) \cos x dx$ is divergent, since $\int_0^\infty \frac{\sin^2 x}{x} dx$ is divergent. So it may seem rather remarkable that the integral of the combination $S(x) \cos x + C(x) \sin x$ is convergent:

PROPOSITION 8. *We have*

$$\int_0^\infty \left(S(x) \cos x + C(x) \sin x \right) dx = \log 2, \quad (36)$$

$$\int_0^\infty S(x)C(x) dx = \log 2. \quad (37)$$

Proof. The equivalence of (36) and (37) is shown by

$$\begin{aligned}\int_0^\infty 1 \cdot S(x)C(x) dx &= \left[xS(x)C(x) \right]_0^\infty + \int_0^\infty x \left(S(x) \frac{\cos x}{x} + C(x) \frac{\sin x}{x} \right) dx \\ &= 0 + \int_0^\infty \left(S(x) \cos x + C(x) \sin x \right) dx.\end{aligned}$$

We now integrate on $[\delta, X]$ and consider (with some care) the limit as $\delta \rightarrow 0$ and $X \rightarrow \infty$. Integrating both products by parts, we have

$$\begin{aligned}\int_\delta^X \left(S(x) \cos x + C(x) \sin x \right) dx &= \left[S(x) \sin x - C(x) \cos x \right]_\delta^X + \int_\delta^X \left(\frac{\sin^2 x}{x} - \frac{\cos^2 x}{x} \right) dx \\ &= \left[S(x) \sin x - C(x) \cos x \right]_\delta^X - \int_\delta^X \frac{\cos 2x}{x} dx.\end{aligned}$$

Now

$$\int_\delta^\infty \frac{\cos 2x}{x} dx = \int_{2\delta}^\infty \frac{\cos y}{y} dy = C(2\delta),$$

so, taking the limit as $X \rightarrow \infty$, we have

$$\int_\delta^\infty \left(S(x) \cos x + C(x) \sin x \right) dx = -S(\delta) \sin \delta + C(\delta) \cos \delta - C(2\delta).$$

Clearly, $\lim_{\delta \rightarrow 0^+} S(\delta) \sin \delta = 0$. To evaluate the limit of the remaining terms, we apply (6) (we do not need to know that $c = -\gamma$): we obtain

$$\begin{aligned}C(\delta) \cos \delta - C(2\delta) &= \log 2\delta - c - C^*(2\delta) + (c - \log \delta + C^*(\delta)) \cos \delta \\ &= \log 2 + (1 - \cos \delta) \log \delta - c(1 - \cos \delta) + C^*(\delta) \cos \delta - C^*(2\delta).\end{aligned}$$

Since $0 \leq 1 - \cos \delta \leq \delta^2$, we have $(1 - \cos \delta) \log \delta \rightarrow 0$ as $\delta \rightarrow 0$. Also, $C^*(\delta) \rightarrow 0$, so $C(\delta) \cos \delta - C(2\delta) \rightarrow \log 2$ as $\delta \rightarrow 0^+$. \square

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