An inequality for $\mathrm{Si}(x)$

Graham Jameson, Nick Lord and James McKee

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The "sine integral" is

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt.$$

The aim of this note is to establish the pair of inequalities

$$\tan^{-1} x < \mathrm{Si}(x) < \pi - \tan^{-1} x \qquad \text{for } x > 0. \tag{1}$$

These bounds imply the well-known integral

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

Our proof assumes this result; of course, it would be more satisfying to find a proof that does not rely on it.

An unusual feature of the bounds is that the upper and lower bounds are monotone functions (whose graphs are symmetrical about the line $y = \pi/2$), whereas $\mathrm{Si}(x)$ is an oscillatory function. In fact, by the fundamental theorem of calculus, its derivative is $\sin x/x$, so $\mathrm{Si}(x)$ is increasing on intervals $[2n\pi, (2n+1)\pi]$ and decreasing on intervals $[(2n-1)\pi, 2n\pi]$, hence it has maxima at the points $(2n+1)\pi$ and minima at the points $2n\pi$. (Diagrams appearing in the *Math. Gazette* version of this article are not reproduced here).

Weaker bounds for $\mathrm{Si}(x)$ can be found in the literature. For example, Burckel in [1, p. 123] estimates a contour integral to establish bounds equivalent to

$$\frac{\pi}{2}\left(1 - \frac{1}{x}\right) < \mathrm{Si}(x) < \frac{\pi}{2}\left(1 + \frac{1}{x}\right).$$

This note had an unusual genesis. James McKee is a sixth form student of Nick Lord's. He conjectured that $\tan^{-1} x < \mathrm{Si}(x)$ by comparing Maclaurin expansions of each side (which works for $0 < x \le 1$), then graphing both sides for larger and larger values of x. He then experimented with graphs for the upper bound, initially arriving at $\mathrm{Si}(x) < \sec^{-1}(-x)$ In discussion with Nick Lord, the upper bound was refined:

$$\sec^{-1}(-x) = \cos^{-1}\left(-\frac{1}{x}\right) = \frac{\pi}{2} + \sin^{-1}\frac{1}{x} > \frac{\pi}{2} + \tan^{-1}\frac{1}{x} = \pi - \tan^{-1} x.$$

At this point, highly intrigued but lacking a definitive proof, Nick Lord contacted Graham Jameson, who produced the elegant proof which follows.

Write

$$I(x) = \int_x^\infty \frac{\sin t}{t}\,dt,$$
$$J(x) = \int_x^\infty \frac{1}{t^2+1}\,dt.$$

Given the integrals

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{1}{t^2+1}\,dt = \frac{\pi}{2},$$

we have $I(x) = \frac{\pi}{2} - \mathrm{Si}(x)$ and $J(x) = \frac{\pi}{2} - \tan^{-1} x$, so (1) is equivalent to

$$-J(x) < I(x) < J(x). \tag{2}$$

We shall prove (2). We note in passing that the notation $\mathrm{si}(x)$ is sometimes used for $-I(x)$, and that $J(x)$ also equates to $\tan^{-1}\frac{1}{x}$.

First, a simple estimation of $J(x)$. Since

$$\frac{1}{t^2+1} = \frac{1}{t^2} - \frac{1}{t^2(t^2+1)},$$

we have

$$J(x) = \frac{1}{x} - \int_x^\infty \frac{1}{t^2(t^2+1)}\,dt > \frac{1}{x} - \int_x^\infty \frac{1}{t^4}\,dt = \frac{1}{x} - \frac{1}{3x^3}. \tag{3}$$

Also, $J(x) < \frac{1}{x}$, so (2), once proved, will also imply the inequality $|I(x)| < \frac{1}{x}$.

We now work on the estimation of $I(x)$. It adds to clarity to describe it more generally, for an integral of the form

$$I_f(x) = \int_x^\infty f(t)\sin t\,dt.$$

We assume that $f(t)$ is non-negative and that $f(t)$, together with all its derivatives, is decreasing in magnitude and tends to 0 as $t \to \infty$. This is satisfied by $f(t) = 1/t^p$ for any $p > 0$. It implies that the odd-numbered derivatives are negative and the even-numbered ones positive (so in fact these derivatives satisfy the same hypothesis as $f(t)$ itself).

Integrating by parts, we have

$$I_f(x) = \Big[-f(t)\cos t\Big]_x^\infty + r_1(x) = f(x)\cos x + r_1(x),$$

where

$$r_1(x) = \int_x^\infty f'(t)\cos t\,dt.$$

We can estimate $r_1(x)$ as follows: since $f'(t) \le 0$, we have $|f'(t)\cos t| \le -f'(t)$, hence

$$|r_1(x)| \le \int_x^\infty (-f'(t))\,dt = f(x).$$

This is essentially the proof that the integral $I_f(x)$ converges in the first place. It also gives the estimation

$$f(x)(\cos x - 1) \le I_f(x) \le f(x)(\cos x + 1), \tag{4}$$

so in particular $|I_f(x)| \le 2f(x)$. However, this is well short of the inequality we have set out to prove. To get a better approximation, we repeat the integration by parts, obtaining

$$\begin{aligned} I_f(x) &= f(x)\cos x + \Big[f'(t)\sin t\Big]_x^\infty - I_{f''}(x) \\ &= f(x)\cos x - f'(x)\sin x - I_{f''}(x) \end{aligned} \tag{5}$$

Now applying this to $f''(t)$ and substituting, we have:

$$\begin{aligned} I_f(x) &= f(x)\cos x - f'(x)\sin x - f''(x)\cos x + f^{(3)}(x)\sin x + I_{f^{(4)}}(x) \\ &= [f(x) - f''(x)]\cos x - \left[f'(x) - f^{(3)}(x)\right]\sin x + I_{f^{(4)}}(x). \end{aligned} \tag{6}$$

Furthermore, as in the estimate of $r_1(x)$, we have $|I_{f''}(x)| \le -f'(x)$ and $|I_{f^{(4)}}(x)| \le -f^{(3)}(x)$.

Of course, the process can be continued: successive derivatives of $f(x)$ appear in the expressions multiplying $\cos x$ and $\sin x$. However, this does not simply deliver ever-closer approximations, because for a fixed x, the derivatives $f^{(n)}(x)$ ultimately grow large in magnitude. For our purposes, (6) is exactly what we want.

We mention the fact (although we won't actually use it) that these results simplify pleasantly for $x = n\pi$, where n is an integer. For example, for even n, (4) shows that $I_f(n\pi) \ge 0$. Applying this to $I_{f''}$, we see from (5) that $I_f(n\pi) \le f(n\pi)$. In both cases, the opposite holds for odd n. So we can state (for all n) that $0 \le (-1)^n I_f(n\pi) \le f(n\pi)$.

The reader might like to work out the corresponding expressions for $\int_x^\infty f(t)\cos t\, dt$.

Now let us return to the case we want, given by $f(t) = 1/t$. Statement (6) becomes

$$I(x) = \left(\frac{1}{x} - \frac{2}{x^3}\right)\cos x + \left(\frac{1}{x^2} - \frac{6}{x^4}\right)\sin x + r_4(x), \tag{7}$$

where

$$|r_4(x)| \le \frac{6}{x^4}.$$

Write this as $F(x) + r_4(x)$. Clearly, $F(x)$ is a highly accurate approximation to $I(x)$ for large x. We now estimate its absolute value. By the Cauchy-Schwarz inequality, $|a\cos x + b\sin x| \le (a^2 + b^2)^{1/2}$. Assume that $x \ge 2$ and write $y = 1/x$, so that $0 < y \le \frac{1}{2}$. Then

$$\begin{aligned} F(x)^2 &\le (y - 2y^3)^2 + (y^2 - 6y^4)^2 \\ &= y^2 - 3y^4 - 8y^6 + 36y^8 \\ &= (y - \tfrac{3}{2}y^3)^2 - 10\tfrac{1}{4}y^6 + 36y^8 \\ &< (y - \tfrac{3}{2}y^3)^2. \end{aligned}$$

So $|F(x)| < y - \frac{3}{2}y^3$, and hence

$$|I(x)| < \frac{1}{x} - \frac{3}{2x^3} + \frac{6}{x^4}. \tag{8}$$

With (3), this gives

$$J(x) - |I(x)| > \frac{7}{6x^3} - \frac{6}{x^4} = \frac{1}{6x^4}(7x - 36),$$

which is positive for $x \geq \frac{36}{7}$. This proves (2) for such x.

It remains to show that (2) also holds for $x \leq \frac{36}{7}$ (in fact, we consider $x \leq 2\pi$). Unfortunately, with all the details included, this requires nearly as much work as the main case! However, readers have the option of leaving out some of the details, as indicated. Write

$$f(t) = \frac{\sin t}{t}, \qquad g(t) = \frac{1}{t^2 + 1}, \qquad h(t) = \frac{t}{t^2 + 1}.$$

We need two Lemmas comparing $f(t)$ with $g(t)$ and $-g(t)$. These Lemmas are graphically convincing (though the diagrams are not reproduced here), and some readers may prefer to skip the formal proofs and go straight to the proofs of $I(x) < J(x)$ and $I(x) + J(x) > 0$.

Lemma 1: There is a point z_0 in $(0, \pi)$ such that $f(t) > g(t)$ on $(0, z_0)$ and $f(t) < g(t)$ on $(z_0, 2\pi)$.

Proof: Clearly, $f(\pi) = 0$, while $g(\pi) > 0$. Also, for $\pi \leq t \leq 2\pi$, we have $f(t) \leq 0 < g(t)$. By the sine series, $f(t) > 1 - \frac{1}{6}t^2$ for $0 < t < \sqrt{5}$, while direct multiplication shows that $g(t) \leq 1 - \frac{1}{6}t^2$ for $t \leq \sqrt{5}$. So there is a point z_0 between $\sqrt{5}$ and π with $f(z_0) = g(z_0)$. To show the uniqueness of z_0, we compare the gradients of $\sin t$ and $h(t)$. Now $\cos\sqrt{5} < \cos\frac{2\pi}{3} = -\frac{1}{2}$, so for $\sqrt{5} \leq t \leq \pi$, the magnitude of the gradient of $\sin t$ is at least $\frac{1}{2}$. Meanwhile, $h'(t) = (1 - t^2)/(1 + t^2)^2$, so $|h'(t)| < 1/t^2 < \frac{1}{5}$. Hence z_0 is unique.

Proof of $I(x) < J(x)$ for $0 \leq x \leq 2\pi$: We have shown that $I(2\pi) < J(2\pi)$, and of course $I(0) = J(0) = \frac{\pi}{2}$. By Lemma 1, $J(x) - I(x)$ is increasing on $[0, z_0]$ and decreasing on $[z_0, 2\pi]$, since its derivative is $f(x) - g(x)$. Hence $J(x) - I(x) \geq 0$ on $[0, 2\pi]$.

The proof that $I(x) + J(x) > 0$ is along similar lines, but it requires slightly more work.

Lemma 2: There are points x_1, z_1 in $(\pi, 2\pi)$ at which $f(t) + g(t) = 0$, with $f(t) + g(t) > 0$ on $[0, x_1]$ and $[z_1, 2\pi]$, while $f(t) + g(t) < 0$ on $[x_1, z_1]$.

Proof: Write $H(t) = t[f(t) + g(t)] = \sin t + h(t)$. Clearly, $H(\pi) = h(\pi) > 0$, $H(2\pi) > 0$ and $H(\frac{3}{2}\pi) = h(\frac{3}{2}\pi) - 1 < 0$. So there exist $x_1 < \frac{3}{2}\pi$ and $z_1 > \frac{3}{2}\pi$ with $H(x_1) = H(z_1) = 0$. Also, $H(t)$ is strictly decreasing on $[\pi, \frac{3}{2}\pi]$, so x_1 is unique. For $\frac{3}{2}\pi \leq t \leq \frac{7}{4}\pi$, $\sin t \leq -1/\sqrt{2}$ while $h(t) < 1/\pi$, so $H(t) < 0$. Between $\frac{7}{4}\pi$ and 2π, the gradient of $\sin t$ is at least $1/\sqrt{2}$, while $|h'(t)| < 1/t^2 < 1/\pi^2$, so $H'(t) > 0$. Hence z_1 is unique.

Proof of $I(x) + J(x) > 0$ for $0 \le x \le 2\pi$. By Lemma 2, $I(x) + J(x)$ is decreasing on $[0, x_1]$ and $[z_1, 2\pi]$, and increasing on $[x_1, z_1]$. Since $I(2\pi) + J(2\pi) > 0$, it is enough to show that $I(x_1) + J(x_1) > 0$. We show that $x_1 < \pi + \frac{3}{10}$. Write $\pi + \frac{3}{10} = u$. From the inequality $\sin t > t - \frac{1}{6}t^3$, we have $-\sin u = \sin \frac{3}{10} > 0.295$, while $h(u) < \frac{1}{u} < 0.291$. So $\sin u + h(u) < 0$, hence $f(u) + g(u) < 0$, which shows that $x_1 < u$, as stated. (Calculation actually gives $x_1 \approx 3.4147$.)

Since I is increasing and J is decreasing on $[\pi, 2\pi]$, it is enough to show that $I(\pi) + J(\pi + \frac{3}{10}) > 0$. From (3), we find $J(\pi + \frac{3}{10}) > 0.28238$. To evaluate $I(\pi)$, we use the identity

$$\mathrm{Si}(x) = x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \cdots,$$

obtained by integrating the series for $\sin t/t$. This gives $\mathrm{Si}(\pi) \approx 1.85194$, hence $I(\pi) \approx -0.28114$ to five d.p., so the required inequality holds.

A stronger inequality for $x \ge 1$. We are indebted to the referee for the following observation: $|I(x)| < K(x)$ for $x \ge 1$, where $K(x) = 1/(x^2+1)^{1/2}$. This is stronger than (2), because $K(x) < J(x)$, as one can see by writing $K(x)$ as $-\int_x^\infty K'(t)\,dt$. At the same time, it does not hold for x close to 0, since $K(0) = 1$.

We sketch the proof. By multiplying out, one sees that $(1+u)^{-1/2} \ge 1 - \frac{1}{2}u$ for $0 \le u \le 1$, from which we deduce

$$K(x) \ge \frac{1}{x} - \frac{1}{2x^3}$$

for $x \ge 1$. By (8), it follows that $|I(x)| < K(x)$ for $x \ge 6$. Calculation shows that $I(1) < K(1)$. A suitable modification of the steps above now shows that the inequality holds for $1 \le x \le 6$.

Reference

1. R. B. Burckel, *An Introduction to Classical Complex Analysis, vol. 1*, Academic Press (1979).