Some remarkable integrals derived from a simple algebraic identity

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The identity in question really is simple: it says, for $u \neq -1$,

$$\frac{1}{1+u} + \frac{1}{1+\frac{1}{u}} = \frac{1}{1+u} + \frac{u}{1+u} = 1. \tag{1}$$

We describe two types of definite integral that look quite formidable, but dissolve into a much simpler form by an application of (1) in a way that seems almost magical.

Both types, or at least special cases, have certainly been mathematical folklore for a long time. For example, case (10) below appears in [1, p. 262], published in 1922 (we are grateful to Donald Kershaw for showing us this example). However, they do not seem to figure in most books on calculus, except possibly tucked away as an exercise. The comprehensive survey [2] mentions the second type on p. 253, but only as a lemma on the way to an identity the authors call the "master formula". We come back to this formula later, but only after describing a number of other more immediate applications.

Recall that an *even* function is one satisfying $f(-x) = f(x)$ for all x. Our first type is the following:

THEOREM 1. *If f is an even function, then for any real a and any $r > 0$,*

$$\int_{-r}^{r} \frac{f(x)}{e^{ax}+1}\,dx = \int_0^r f(x)\,dx. \tag{2}$$

Proof: Denote the integral by I. Substituting $x = -y$, we have

$$I = \int_r^{-r} \frac{f(-y)}{1+e^{-ay}}(-1)\,dy = \int_{-r}^{r} \frac{f(y)}{1+e^{-ay}}\,dy.$$

Writing x for y, combining with the original expression and applying (1) with $u = e^{ax}$, we have

$$2I = \int_{-r}^{r} f(x)\left(\frac{1}{1+e^{ax}} + \frac{1}{1+e^{-ax}}\right)dx = \int_{-r}^{r} f(x)\,dx = 2\int_0^r f(x)\,dx. \qquad \square$$

The factor $e^{ax}+1$ seemingly just disappears! Particular cases can now be written down at will, for example (still for any a):

$$\int_{-r}^{r} \frac{1}{e^{ax}+1}\,dx = \int_0^r 1\,dx = r, \tag{3}$$

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^{ax}+1}\, dx = \int_0^{\pi/2} \cos x\, dx = 1, \tag{4}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(e^{ax}+1)}\, dx = \int_0^{\infty} \frac{1}{x^2+1}\, dx = \frac{\pi}{2}. \tag{5}$$

The reader might enjoy writing out further examples.

It might seem that there is no possibility of replacing $e^{ax}+1$ by $e^{ax}-1$, because this becomes 0 at 0. However, there is one quite important special case in which this does make sense. Consider the function

$$h(x) = \frac{1}{x} - \frac{1}{e^x - 1},$$

of interest, as some readers will know, because of its connection to the Bernoulli numbers. By expressing $h(x)$ as a single fraction and inserting the power series for e^x, one finds that $h(x) \to \frac{1}{2}$ as $x \to 0$, so $h(x)$ becomes continuous at 0 with this value assigned there. By another application of (1), we see that $h(x) + h(-x) = 1$, and hence, by the same reasoning,

$$\int_{-r}^{r} h(x)\, dx = r. \tag{6}$$

For the second type of integral, we substitute $x = 1/y$ instead of $x = -y$. The result can be stated as follows:

THEOREM 2. *Let g be a function satisfying $g(1/x) = g(x)$ for $x > 0$, and let $f(x) = g(x)/x$. Then for any s and any $r > 1$,*

$$\int_{1/r}^{r} \frac{f(x)}{x^s+1}\, dx = \tfrac{1}{2} \int_{1/r}^{r} f(x)\, dx = \int_1^r f(x)\, dx. \tag{7}$$

Proof: Denote the integral by I. The condition on g equates to $f(1/x) = x^2 f(x)$. Substituting $x = 1/y$ and applying this, we obtain

$$I = \int_{1/r}^{r} \frac{f(1/y)}{1+y^{-s}} \frac{1}{y^2}\, dy = \int_{1/r}^{r} \frac{f(y)}{1+y^{-s}}\, dy.$$

Combining with the original integral and applying (1) with $u = x^s$, we have

$$2I = \int_{1/r}^{r} f(x) \left(\frac{1}{1+x^s} + \frac{1}{1+x^{-s}} \right) dx = \int_{1/r}^{r} f(x)\, dx.$$

The same substitution shows that $\int_{1/r}^{1} f(x)dx = \int_1^r f(x)dx$, so that $\int_{1/r}^{r} f(x)dx = 2\int_1^r f(x)dx$. $\square$

Alternatively, Theorem 2 can be deduced from Theorem 1 after substituting $x = e^y$, but this is not really any shorter than the direct proof just given.

Further examples of integrals (all arguably "remarkable") are delivered by different choices of $g(x)$. The simplest one is $g(x) = 1$, giving $f(x) = 1/x$, hence (still for any s and any $r > 1$)

$$\int_{1/r}^{r} \frac{1}{x(x^s+1)}\,dx = \int_1^r \frac{1}{x}\,dx = \ln r. \tag{8}$$

The case $g(x) = (\ln x)^2$ gives

$$\int_{1/r}^{r} \frac{(\ln x)^2}{x(x^s+1)}\,dx = \int_1^r \frac{(\ln x)^2}{x}\,dx = \tfrac{1}{3}(\ln r)^3. \tag{9}$$

The case $g(x) = 1/(x + \frac{1}{x})$, with $r \to \infty$, gives the example mentioned from [1]: for any s, we have

$$\int_0^\infty \frac{1}{(x^2+1)(x^s+1)}\,dx = \int_1^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{4} \tag{10}$$

Note that the special cases $s = 2$, 1, -1 and -2 equate to the integrals

$$\int_0^\infty \frac{1}{(x^2+1)^2}\,dx, \quad \int_0^\infty \frac{1}{(x+1)(x^2+1)}\,dx, \quad \int_0^\infty \frac{x}{(x+1)(x^2+1)}\,dx, \quad \int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx.$$

We have evaluated all of these simultaneously, with a minimum of effort!

In all of these examples, the evaluation of $\int_1^r f(x)\,dx$ was immediate. When it is not immediate, it may be achieved more easily by exploiting the fact that the value of original integral is independent of s. To illustrate this, take

$$g(x) = \frac{x^4 + 1/x^4}{x + 1/x} = \frac{x^8+1}{x^3(x^2+1)},$$

so that the integral is

$$I(s) = \int_{1/r}^{r} \frac{x^8+1}{x^4(x^2+1)(x^s+1)}\,dx.$$

Since $I(s)$ is independent of s, we can take $s = 8$ to conclude that

$$I(s) = \int_{1/r}^{r} \frac{1}{x^4(x^2+1)}\,dx.$$

The integrand is

$$\frac{1}{x^2}\left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) = \frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{x^2+1},$$

leading to the value

$$I(s) = \tfrac{1}{3}\left(r^3 - \frac{1}{r^3}\right) - \left(r - \frac{1}{r}\right) + 2\tan^{-1} r - \frac{\pi}{2}. \tag{11}$$

This has averted most of the algebra required for a direct evaluation of $\int_1^r f(x)\,dx$.

Again, readers are invited to explore further examples of their own.

We digress here to mention a simple companion statement to Theorem 2, though it is not actually an application of (1). Suppose that $g(1/x) = -g(x)$ for $x > 0$ and $f(x) = g(x)/x$. Let $I = \int_{1/r}^{r} f(x)\,dx$. Then $f(1/x) = -x^2 f(x)$, and substituting $x = 1/y$ as before, we obtain

$$I = \int_{1/r}^{r} \frac{f(1/y)}{y^2}\,dy = -\int_{1/r}^{r} f(y)\,dy = -I,$$

so $I = 0$. For example, the choices $(\ln x)/(x + \frac{1}{x})$ and $(\ln x)/(x + \frac{1}{x})^2$ for $g(x)$ give

$$\int_0^\infty \frac{\ln x}{x^2+1}\,dx = \int_0^\infty \frac{x \ln x}{(x^2+1)^2}\,dx = 0, \tag{12}$$

and with the substitution $x = ay$, the first of these leads to

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln a}{2a}. \tag{13}$$

For our last example, we outline the derivation of the "master formula" of [2, p. 250]. The starting point is the integral

$$I(a,p) = \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^p dx. \tag{14}$$

The integrand is $1/k(x)^p$, where

$$k(x) = x^2 + 2a + \frac{1}{x^2} = \left(x - \frac{1}{x} \right)^2 + c,$$

with $c = 2a + 2$. To ensure that $k(x) > 0$ for all x, we require that $a > -1$. Also, for convergence of the integral, we require that $p > \frac{1}{2}$. Substituting $x = 1/y$, we have

$$I(a,p) = \int_0^\infty \frac{1}{[(y - \frac{1}{y})^2 + c]^p} \frac{1}{y^2}\,dy. \tag{15}$$

Adding the original integral, as before, we have

$$2I(a,p) = \int_0^\infty \frac{1}{[(x - \frac{1}{x})^2 + c]^p} \left(1 + \frac{1}{x^2} \right) dx. \tag{16}$$

Substituting $x - \frac{1}{x} = t$ and then $t = c^{1/2}u$, we now have

$$2I(a,p) = \int_{-\infty}^{\infty} \frac{1}{(t^2+c)^p}\,dt = \frac{1}{c^{p-1/2}} \int_{-\infty}^{\infty} \frac{1}{(u^2+1)^p}\,du. \tag{17}$$

In the case $p = 1$, this gives at once

$$I(a,1) = \frac{\pi}{2c^{1/2}} \tag{18}$$

and it is worth noting that by (15), $I(a,1)$ also equals $\int_0^\infty [1/(x^4+2ax^2+1)]\,dx$ (this case was presented in elegant style in [3]). For general p, standard substitutions in (17) lead to the following evaluation in terms of beta integrals:

$$I(a,p) = \frac{1}{2c^{p-1/2}} B(\tfrac{1}{2}, p-\tfrac{1}{2}). \tag{19}$$

Now let

$$f_p(x) = \left(\frac{x^2}{x^4+2ax^2+1}\right)^p \left(1+\frac{1}{x^2}\right)$$

and $g_p(x) = xf_p(x)$. Clearly, $g_p(1/x) = g_p(x)$. So by (16) and Theorem 2, we have, for any s,

$$\int_0^\infty \frac{f_p(x)}{x^s+1}\,dx = I(a,p) = \frac{1}{2c^{p-1/2}} B(\tfrac{1}{2}, p-\tfrac{1}{2}). \tag{20}$$

This is the "master formula". At the cost of some complication, it incorporates a very wide repertoire of special cases, a large selection of which is presented in [2]. When $p=1$, it simplifies to the rather more pleasant statement

$$\int_0^\infty \frac{x^2+1}{(x^4+2ax^2+1)(x^s+1)}\,dx = \frac{\pi}{2c^{1/2}}. \tag{21}$$

The case $a=1$ reproduces (10). Another example was provided to us by Nick Lord: take $a=\frac{1}{2}$ and $s=6$. Since $x^6+1=(x^2+1)(x^4-x^2+1)$, we obtain

$$\int_0^\infty \frac{1}{x^8+x^4+1}\,dx = \frac{\pi}{2\sqrt{3}}. \tag{22}$$

References

1. J. Edwards, *A Treatise on Integral Calculus*, Macmillan (1922).
2. George Boros and Victor Moll, *Irresistible Integrals*, Cambridge Univ. Press (2004).
3. Michael D. Hirschhorn, An interesting integral, *Math. Gazette* **95** (2011), 90–91.

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