Beyond the ratio test

G. J. O. Jameson

Introduction

D'Alembert's ratio test, a very basic plank in the theory of infinite series, can be stated as follows:

Suppose that $a_n > 0$ *for all* $n \geq 1$. *Then:*

(i) *if for some* n_0 *and some* $\rho < 1$, *we have* $\dfrac{a_{n+1}}{a_n} \leq \rho$ *for all* $n \geq n_0$, *then* $\sum_{n=1}^{\infty} a_n$ *is convergent;*

(ii) *if for some* n_0, *we have* $\dfrac{a_{n+1}}{a_n} \geq 1$ *for all* $n \geq n_0$, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

A sufficient condition for (i) is that a_{n+1}/a_n tends to a limit $r < 1$ as $n \to \infty$, and a sufficient condition for (ii) is that a_{n+1}/a_n tends to a limit $s > 1$. The ratio test is often presented in this form, which is perfectly suited for the application to power series.

It is then customary to observe that no conclusion about convergence follows from the assumption that $a_{n+1}/a_n \to 1$ as $n \to \infty$, demonstrated by the fact that this condition is satisfied both by the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ and by the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Actually, quite a lot can be said about what happens when a_{n+1}/a_n tends to 1, given a little more information about this ratio. In particular, the classical convergence tests of Kummer, Raabe and Gauss address this case. These tests can be seen in numerous books, for example [Fe], [Hy], [Ph], though they are strangely neglected in some more recent texts. However, they are usually presented in isolation, purely as criteria for convergence or divergence. In fact, the conditions assumed have wider consequences. We aim to describe some of them in this article.

A remark on the familiar proof of the ratio test will help to point the way. In case (i), the hypothesis leads directly to an inequality of the form $a_n \leq C\rho^n$, while in case (ii) it simply implies $a_n \geq a_{n_0}$. Convergence of $\sum_{n=1}^{\infty} a_n$ in case (i) follows from convergence of the geometric series $\sum_{n=1}^{\infty} \rho^n$, while divergence in case (ii) follows from the fact that a_n does not converge to zero. In both cases, the essential conclusion is really an estimation of a_n itself; convergence or divergence of $\sum_{n=1}^{\infty} a_n$ is then a straightforward deduction.

Here we will derive estimations of a_n from assumptions of the type considered, thereby setting the convergence tests in a context of related results. These estimations are of interest in themselves, not only for the purpose of determining convergence or divergence of $\sum_{n=1}^{\infty} a_n$. We describe some applications, for example to generalised binomial coefficients and the Wallis product.

The starting point is the following very simple observation: if a_n, b_n are positive and

$$\frac{a_n}{a_{n+1}} \geq \frac{b_n}{b_{n+1}} \qquad \text{for } n \geq n_0, \tag{1}$$

then

$$\frac{a_n}{b_n} \geq \frac{a_{n+1}}{b_{n+1}} \qquad \text{for } n \geq n_0,$$

so a_n/b_n is decreasing for $n \geq n_0$. Clearly, this implies the inequality $a_n \leq Cb_n$ for $n \geq n_0$, where $C = a_{n_0}/b_{n_0}$.

An opposite inequality can be deduced if we also happen to know that a_n/b_n tends to a limit L as $n \to \infty$: then $a_n \geq Lb_n$, since the terms of a convergent decreasing sequence are not less than the limit.

Of course, the hypothesis in (1) can equally be written as $\dfrac{a_{n+1}}{a_n} \leq \dfrac{b_{n+1}}{b_n}$.

The following is no more than a restatement with a_n and b_n interchanged, but it will be helpful in the applications below to state it explicitly: if

$$\frac{a_n}{a_{n+1}} \leq \frac{b_n}{b_{n+1}} \qquad \text{for } n \geq n_0, \tag{2}$$

then a_n/b_n is increasing for $n \geq n_0$, and hence $a_n \geq Cb_n$ for $n \geq n_0$.

Comparison with $1/(n+c)^p$

By applying (1) and (2) with $b_n = 1/n^p$ (where $p > 0$), we will obtain statements to the effect that $n^p a_n$ is either increasing or decreasing, and hence comparisons of a_n with $1/n^p$. Actually, most of the applications require comparison with $1/(n+c)^p$ instead of $1/n^p$. This makes the statements look slightly more complicated, but generates no extra work in the proofs. The following simple lemma will help to navigate statements presented in this form.

LEMMA 1. *Let $c_2 > c_1$. Then: (i) if $(n+c_1)^p a_n$ is decreasing for $n > -c_1$, then so is $(n+c_2)^p a_n$, and (ii) if $(n+c_2)^p a_n$ is increasing, then so is $(n+c_1)^p a_n$.*

Proof. Note that $(n+c_2)^p a_n = D_n^p (n+c_1)^p a_n$, where

$$D_n = \frac{n+c_2}{n+c_1} = 1 + \frac{c_2 - c_1}{n+c_1}.$$

Clearly, D_n, hence also D_n^p, decreases with n. Statement (i) follows, and (ii) is similar. $\square$

We will make constant use of the well-known inequality given in the next lemma. For $p > 1$, it expresses the fact that the convex function x^p lies above its tangent at $x = 1$ (and the opposite for $0 < p < 1$).

LEMMA 2. *For all* $t > -1$, $(1+t)^p > 1+pt$ *if* $p > 1$, *and* $(1+t)^p < 1+pt$ *if* $0 < p < 1$.

Proof. First, suppose that $t > 0$. By the mean-value theorem, applied to $f(x) = x^p$, we have

$$(1+t)^p - 1 = f(1+t) - f(1) = ptu^{p-1}$$

for some u in $(1, 1+t)$. Clearly, $u^{p-1} > 1$ if $p > 1$, and the reverse holds if $0 < p < 1$. The stated inequalities follow.

Now suppose that $-1 < t < 0$. The stated identity still holds, but now $u < 1$. If $p > 1$, then $u^{p-1} < 1$. Since $t < 0$, we have $ptu^{p-1} > pt$, as required. Again the reverse holds if $0 < p < 1$. □

This Lemma, combined with (1) and (2), delivers our first theorem with no further effort:

THEOREM 1. (i) *If* $0 < p \leq 1$ *and for some* c *(positive or negative) and some* n_0,

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{p}{n+c} \qquad \text{for } n \geq n_0,$$

then $(n+c)^p a_n$ *is decreasing for* $n \geq n_0$, *hence for some* C, *we have* $a_n \leq C/(n+c)^p$ *for all* $n \geq n_0$.

(ii) *If* $p \geq 1$ *and*

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{p}{n+c} \qquad \text{for } n \geq n_0,$$

then $(n+c)^p a_n$ *is increasing for* $n \geq n_0$, *hence* $a_n \geq C/(n+c)^p$ *for* $n \geq n_0$ *and some* C.

Proof. (i) Let $b_n = 1/(n+c)^p$. By Lemma 2,

$$\frac{b_n}{b_{n+1}} = \frac{(n+1+c)^p}{(n+c)^p} = \left(1 + \frac{1}{n+c}\right)^p \leq 1 + \frac{p}{n+c} \leq \frac{a_n}{a_{n+1}}$$

for $n \geq n_0$, hence $a_n/b_n = (n+c)^p a_n$ is decreasing.

The proof of (ii) is the same with the inequalities reversed. □

From now on, to avoid tedious repetition, we take the qualification "for $n \geq n_0$" as understood in further statements of this sort. Also, we will not explicitly state the consequent inequalities for a_n.

A few remarks on Theorem 1 are in order. Firstly, we have not excluded negative c, but in this case we must have $n_0 > -c$, so that $n + c > 0$ for the n considered.

Secondly, if $p \neq 1$, then we can conclude that $(n + c)^p a_n$ is *strictly* decreasing or increasing, since strict inequality holds in Lemma 2.

Thirdly, the two cases coincide when $p = 1$ and $a_n/a_{n+1} = 1 + 1/(n + c)$: clearly, this implies that $(n + c)a_n$ is constant.

Fourthly, applying (for example) (i) to $1/a_n$, we obtain: if $0 < p \leq 1$ and $a_{n+1}/a_n \geq 1 + \frac{p}{n+c}$, then $a_n/(n + c)^p$ is increasing.

There are two missing cases in Theorem 1. What conditions will ensure that $(n+c)^p a_n$ is increasing when $p \leq 1$, or decreasing when $p \geq 1$? The following companion theorem provides the answers. A formulation that neatly mirrors Theorem 1 is obtained by stating the conditions in terms of a_n/a_{n-1}.

THEOREM 2. (i) *If* $0 < p \leq 1$ *and*

$$\frac{a_n}{a_{n-1}} \geq 1 - \frac{p}{n + c},$$

then $(n + c)^p a_n$ *is increasing.*

(ii) *If* $p \geq 1$ *and*

$$\frac{a_n}{a_{n-1}} \leq 1 - \frac{p}{n + c},$$

then $(n + c)^p a_n$ *is decreasing.*

Proof. (i) Again, let $b_n = 1/(n + c)^p$ and apply Lemma 2 (the case $t < 0$) to obtain

$$\frac{b_n}{b_{n-1}} = \frac{(n - 1 + c)^p}{(n + c)^p} = \left(1 - \frac{1}{n + c}\right)^p \leq 1 - \frac{p}{n + c} \leq \frac{a_n}{a_{n-1}}.$$

By (2), $a_n/b_n = (n + c)^p a_n$ is increasing.

The proof of (ii) is the same with the inequalities reversed. □

We have suppressed the qualification "for $n \geq n_0$", but clearly we require at least that $n > p - c$, so that $\frac{p}{n+c} < 1$.

When $p = 1$, Theorem 2 actually says the same as Theorem 1.

While this formulation indeed mirrors the statement of Theorem 1, the hypotheses are not the exact reversal of the corresponding ones in Theorem 1. In fact, taking (for the moment) $c = 0$, the reversal of the hypothesis in Theorem 1(i) is $a_n/a_{n+1} \leq 1 + \frac{p}{n}$. This is equivalent to

$$\frac{a_n}{a_{n-1}} \geq \frac{n-1}{n-1+p} = 1 - \frac{p}{n+p-1},$$

so we can read off the following conclusions from Theorem 2:

COROLLARY 2.1. (i) *If* $0 < p \leq 1$ *and*

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{p}{n},$$

then $(n + p - 1)^p a_n$ *is increasing.*

(ii) *If* $p \geq 1$ *and*

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{p}{n},$$

then $(n + p - 1)^p a_n$ *is decreasing.* □

It is now abundantly clear why we need statements in terms of $(n + c)^p$, not just n^p. In Corollary 2.1 itself, we could of course now substitute $n + c$ for n.

Together, Corollary 2.1 and Theorem 1 give complete coverage of the outcomes when a_n/a_{n+1} is compared with $1 + \frac{p}{n}$. Consider now the case where $a_n/a_{n+1} = 1 + \frac{p}{n}$ for all n (given a_1, this defines a_n; the case $p = 2$ is given specifically by $a_n = 1/[n(n + 1)]$). Then both results apply: for example, if $p > 1$, then $n^p a_n$ is increasing and $(n + p - 1)^p a_n$ is decreasing, so a_n is sandwiched between a_1/n^p and $p^p a_1/(n + p - 1)^p$.

At the sacrifice of some accuracy, we can actually combine Theorems 1 and 2 to formulate a pair of statements that do not switch at $p = 1$:

THEOREM 3. (i) *If* $p > 0$ *and*

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n+c},$$

then $(n + c)^p a_n$ *is decreasing.*

(ii) *If* $p > 0$ *and*

$$\frac{a_{n-1}}{a_n} \leq 1 + \frac{p}{n+c},$$

then $(n + c)^p a_n$ *is increasing.*

Proof. (i) If $0 < p \leq 1$, then $a_n/a_{n+1} \geq 1 + p/(n + c)$, since $1/(1 - x) > 1 + x$ for $0 < x \leq 1$, and Theorem 1(i) applies.

If $p > 1$, then

$$\frac{a_n}{a_{n-1}} \leq 1 - \frac{p}{n-1+c} < 1 - \frac{p}{n+c},$$

and Theorem 2(ii) applies.

The proof of (ii) is similar, using Theorem 1(ii) and Theorem 2(i). □

Convergence tests: Raabe and Gauss

Joseph Raabe (1801–1859) was professor of mathematics at Zürich from 1833 onwards. His convergence test is probably his best known contribution to the subject. One way to state it is as follows, with assumptions of exactly the type we have been considering.

THEOREM 4 (Raabe's test). (i) *If* $a_n > 0$ *for all* n *and for some* $p > 1$, *we have*

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{p}{n}$$

for all large enough n, *then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

(ii) *If*

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n}$$

for all large enough n *and some* $c \geq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

If we take it as known that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, and that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ (both of which can be proved by comparison with the integral $\int_1^{\infty} \frac{1}{t^p}\, dt$, or by the condensation test), then Raabe's test is an immediate consequence of Theorems 1 and 2:

Proof of Theorem 4. (i) By Corollary 2.1(ii), the hypothesis implies that $(n+p-1)^p a_n$ is decreasing, so for some C, we have

$$a_n \leq \frac{C}{(n+p-1)^p} \leq \frac{C}{n^p}$$

for large enough n. By the comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) By Theorem 1(ii), this implies $a_n \geq \frac{C}{n}$ for large enough n and some C. □

Conversely, if Theorem 4(i) has been proved by another method (we indicate later how this can be done), then it provides an alternative proof of convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Indeed, if $a_n = \frac{1}{n^p}$ and $p > 1$, then, as before,

$$\frac{a_n}{a_{n+1}} = \left(1 + \frac{1}{n}\right)^p \geq 1 + \frac{p}{n}.$$

As with the ordinary ratio test, we can deduce a second version in terms of limits:

COROLLARY 4.1. *Suppose that*

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \to \sigma \quad \textit{as } n \to \infty.$$

Then $\sum_{n=1}^{\infty} a_n$ *is convergent if* $\sigma > 1$ *and divergent if* $\sigma < 1$.

Proof. If $\sigma > 1$, choose p such that $\sigma > p > 1$. Then for large enough n, we have $n(a_n/a_{n+1} - 1) \geq p$, so $a_n/a_{n+1} \geq 1 + \frac{p}{n}$. Similarly, if $\sigma < 1$, then the hypothesis of (ii) holds for large enough n. □

A further consequence is the following result, a version of what is known as Gauss's test. Gauss established it by different methods in 1812, well before Raabe's working days.

COROLLARY 4.2. *Suppose that*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\sigma}{n} + r_n,$$

where $\sigma > 1$ *and* $nr_n \to 0$ *as* $n \to \infty$. *Then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Proof. This follows at once from Corollary 4.1, since

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \sigma + nr_n \to \sigma \quad \text{as } n \to \infty. \qquad \square$$

A corresponding criterion for divergence will emerge shortly.

Comparison with $1/[n(\log n)^p]$

We now describe, more briefly, variants of Theorems 1 and 2 giving comparisons with $1/[n(\log n)^p]$. We restrict ourselves to the cases when $p \geq 1$. We need the elementary inequalities

$$\frac{1}{n+1} \leq \log(n+1) - \log n \leq \frac{1}{n}, \tag{3}$$

easily deduced from the fact that $\log(n+1) - \log n = \int_n^{n+1} \frac{1}{t}\, dt$.

THEOREM 5. *Let* $p \geq 1$. *If*

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{p}{n \log n}, \tag{4}$$

then $n(\log n)^p a_n$ *is increasing, so* $a_n \geq C/[n(\log n)^p]$. *Opposite conclusions apply if*

$$\frac{a_n}{a_{n-1}} \leq 1 - \frac{1}{n} - \frac{p}{n \log n}. \tag{5}$$

Proof. Let $b_n = 1/[n \log n)^p]$. For the first statement, we have

$$\frac{b_n}{b_{n+1}} = \frac{n+1}{n}\left(\frac{\log(n+1)}{\log n}\right)^p.$$

By (3),

$$\frac{\log(n+1)}{\log n} \geq 1 + \frac{1}{(n+1)\log n},$$

so by Lemma 2,

$$\frac{b_n}{b_{n+1}} \geq \frac{n+1}{n} \left(1 + \frac{p}{(n+1)\log n}\right) = 1 + \frac{1}{n} + \frac{p}{n \log n}.$$

By (2), a_n/b_n is increasing. The proof of the second statement is similar, with n replaced by $n-1$ in (3). □

If we assume it known that $\sum_{n=2}^{\infty} 1/[n(\log n)^p]$ is convergent for $p > 1$ and divergent for $p = 1$ (again, these facts can be proved by the integral test or the condensation test), then we can deduce the following convergence test:

COROLLARY 5.1. *If (4) holds (for large enough n) with $p = 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent. If (5) holds with $p > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.* □

The divergence criterion here can be regarded as a companion to Corollary 3.2, thereby constituting the divergence part of Gauss's test.

Some applications

We describe some applications of Theorems 1 and 2 to particular cases. The main one is to expressions of the form

$$K_n(x,y) = \frac{x(x+1)\dots(x+n-1)}{y(y+1)\dots(y+n-1)},$$

where x, y are positive. A particular case is

$$K_n(x,1) = \frac{x(x+1)\dots(x+n-1)}{n!},$$

which we denote by just $K_n(x)$. This equates to $\binom{n+x-1}{n}$, and to the coefficient of t^n in the binomial series for $(1-t)^{-x}$. Note that $K_n(1) = 1$ and $K_n(x,y) = K_n(x)/K_n(y)$.

Some special values, with fractions removed, are

$$K_n(\tfrac{1}{2}) = \frac{1.3.\dots(2n-1)}{2.4.\dots(2n)},$$

$$K_n(\tfrac{1}{4}, \tfrac{3}{4}) = \frac{1.5.\dots(4n-3)}{3.7.\dots(4n-1)}.$$

The order of magnitude of $K_n(x,y)$, or even $K_n(\frac{1}{2})$, is hardly transparent from these expressions. However, very effective estimations are provided by Theorems 1 and 2. We describe them for the case where $y > x$; estimations for the case $y < x$ can then be read off from the fact that $K_n(y,x) = 1/K_n(x,y)$. We will see that the statements reverse when $y - x = 1$: note that in this case, $K_n(x,y) = x/(n+x)$, so $(n+x)K_n(x)$ is constant.

THEOREM 6. *Let $0 < y - x \leq 1$. Then $(n+x)^{y-x}K_n(x,y)$ decreases with n for all $n \geq 1$, and $(n+y-1)^{y-x}K_n(x,y)$ increases. Hence, for certain constants C_1, C_2 (depending on x and y),*

$$\frac{C_1}{(n+y-1)^{y-x}} \leq K_n(x,y) \leq \frac{C_2}{(n+x)^{y-x}} \tag{6}$$

for all $n \geq 1$. Further, $n^{y-x}K_n(x,y)$ tends to a limit $L(x,y)$ as $n \to \infty$, and

$$\frac{L(x,y)}{(n+x)^{y-x}} \leq K_n(x,y) \leq \frac{L(x,y)}{(n+y-1)^{y-x}}. \tag{7}$$

All these statements reverse when $y - x > 1$.

Proof. Write just K_n for $K_n(x,y)$. We have

$$\frac{K_n}{K_{n+1}} = \frac{n+y}{n+x} = 1 + \frac{y-x}{n+x}.$$

By Theorem 1, $(n+x)^{y-x}K_n$ decreases with n if $y - x \leq 1$ and increases if $y - x > 1$. Also,

$$\frac{K_n}{K_{n-1}} = \frac{n+x-1}{n+y-1} = 1 - \frac{y-x}{n+y-1},$$

so by Theorem 2, $(n+y-1)^{y-x}K_n$ increases with n if $y-x \leq 1$ and decreases if $y-x > 1$. The inequalities (6) follow, with C_1, C_2 (derived from the term $n = 1$) given by $C_1 = y^{y-x}(x/y)$ and $C_2 = (x+1)^{y-x}(x/y)$.

When $0 < y - x \leq 1$, so that $y - 1 \leq x$, we have $(n+y-1)^{y-x}K_n \leq (n+x)^{y-x}K_n$, so the pair of sequences form a sandwich, showing that $(n+y-1)^{y-x}K_n$ is bounded above. Hence it tends to a limit, $L(x,y)$. Now $(n+c)^{y-x}/n^{y-x} \to 1$ as $n \to \infty$ for any c, so it follows that $n^{y-x}K_n$ and $(n+x)^{y-x}K_n$ also tend to $L(x,y)$. The inequalities (7) now follow from the fact that if an increasing sequence (a_n) tends to a limit L, then $a_n \leq L$ for each n. □

It must be conceded that (6) and (7) look a bit too technical to be memorable. However, they record correctly what follows from Theorems 1 and 2, which in turn were obtained very simply. For the purpose of describing the order of magnitude of $K_n(x,y)$, the following simpler variant, again derived at the cost of some loss of accuracy, is adequate. We just state the case where $y - x \leq 1$.

COROLLARY 6.1. *Let $0 < y - x \leq 1$. Then there are constants C_1', C_2' such that for all $n \geq 1$,*

$$\frac{C_1'}{n^{y-x}} \leq K_n(x,y) \leq \frac{C_2'}{n^{y-x}}. \tag{8}$$

Proof. Since $1/(n+c_1) > 1/(n+c_2)$ when $c_1 < c_2$, (8) holds with $C_2' = C_2$, and with $C_1' = C_1$ if $y < 1$. If $y \geq 1$, then, by Theorem 5 and Lemma 1, $n^{y-x}K_n(x,y)$ increases with n, so (8) holds with $C_1' = K_1(x,y) = x/y$. □

We restate Theorem 6 for $K_n(x) = K_n(x,1)$. There is some simplification, and we incorporate the values of C_1 and C_2.

COROLLARY 6.2. *Let $0 < x \leq 1$. Then $n^{1-x}K_n(x)$ increases with n and $(n+x)^{1-x}K_n(x)$ decreases. Hence*

$$\frac{x}{n^{1-x}} \leq K_n(x) \leq (1+x)^{1-x}\frac{x}{(n+x)^{1-x}} \tag{9}$$

for all $n \geq 1$. Further, $n^{1-x}K_n(x)$ tends to a limit $L(x)$, and

$$\frac{L(x)}{(n+x)^{1-x}} \leq K_n(x) \leq \frac{L(x)}{n^{1-x}}. \qquad \square \tag{10}$$

From the two statements, it is clear that $L(x,y) = L(x)/L(y)$. Furthermore, $L(x)$ can be identified: the limit of $1/[n^{1-x}K_n(x)]$ is exactly Euler's limit formula for the gamma function, so $L(x) = 1/\Gamma(x)$ and $L(x,y) = \Gamma(y)/\Gamma(x)$. An account of Euler's formula, including essentially this proof that it converges, was given in the *Gazette* article [Jam].

So (10) can be rewritten as a pair of inequalities for $\Gamma(x)$. Also, since $\Gamma(x+1) = x\Gamma(x)$, we can express $K_n(x)$ as $\Gamma(n+x)/[n!\Gamma(x)]$, and (10) then becomes

$$\frac{n!}{(n+x)^{1-x}} \leq \Gamma(n+x) \leq \frac{n!}{n^{1-x}}.$$

Inequalities of this sort are explored further in [Jam].

Example 1. $K_n(\frac{1}{2})$. By (9), we have

$$\frac{1}{2n^{1/2}} \leq K_n(\tfrac{1}{2}) \leq \frac{C}{(n+\frac{1}{2})^{1/2}},$$

where $C = \sqrt{3}/(2\sqrt{2})$. Also, by the Wallis product, $L(\frac{1}{2}) = 1/\sqrt{\pi}$ (a good way to prove this, without involving the gamma function, is by considering the integral $\int_0^{\pi/2} \cos^n t\, dt$). With this value inserted, (10) becomes

$$\frac{1}{(n+\frac{1}{2})^{1/2}\sqrt{\pi}} \leq K_n(\tfrac{1}{2}) \leq \frac{1}{n^{1/2}\sqrt{\pi}}.$$

Example 2. $K_n(\frac{1}{4},\frac{3}{4})$. By (6),

$$\frac{C_1}{(n-\frac{1}{4})^{1/2}} \leq K_n(\tfrac{1}{4},\tfrac{3}{4}) \leq \frac{C_2}{(n+\frac{1}{4})^{1/2}},$$

where $C_1 = \sqrt{3}/6$ and $C_2 = \sqrt{5}/6$.

We now describe a rather different application of Theorem 6.

Example 3: a series for γ. In an interesting recent *Gazette* note [Sc], the following series expression is given for Euler's constant: $\gamma = \sum_{n=1}^{\infty} c_n$, where

$$\begin{aligned} c_n &= \frac{(-1)^{n-1}}{n!n} \int_0^1 x(x-1)(x-2)\dots(x-n+1)\,dx \\ &= \frac{1}{n!n} \int_0^1 x(1-x)(2-x)\dots(n-1-x)\,dx. \end{aligned}$$

We will apply Theorem 6 to estimate the magnitude of c_n, and hence the rate of convergence of the series. From the second expression, it is clear that $c_n > 0$. Now

$$K_{n-1}(1-x) = \frac{(1-x)(2-x)\dots(n-1-x)}{(n-1)!},$$

so

$$c_n = \frac{1}{n^2} \int_0^1 x K_{n-1}(1-x)\,dx.$$

By (9), for $0 \le x \le 1$,

$$K_{n-1}(1-x) \le \frac{(2-x)^x(1-x)}{(n-x)^x} \le \frac{2^x(1-x)}{n^x} \le \frac{2(1-x)}{n^x},$$

since $(2-x)/(n-x) \le 2/n$ for $n \ge 2$, so

$$c_n \le \frac{2}{n^2} \int_0^1 \frac{x(1-x)}{n^x}\,dx.$$

To estimate this, recall that $n^{-x} = e^{-x \log n}$. Integrating by parts, we see that

$$\int_0^1 (x - x^2)e^{-ax}\,dx = \frac{1}{a} \int_0^1 (1-2x)e^{-ax}\,dx.$$

Call this $I(a)$. It could be written out exactly, but for our purposes it is enough to observe that $I(a) < \frac{1}{a} \int_0^1 e^{-ax}\,dx < \frac{1}{a^2}$. Applying this with $a = \log n$, we conclude finally that

$$c_n \le \frac{2}{n^2(\log n)^2}.$$

The error remaining after adding n terms is $\sum_{r=n+1}^{\infty} c_r$. Since $\sum_{r=n+1}^{\infty} \frac{1}{r^2} < \sum_{r=n+1}^{\infty} \frac{1}{(r-1)r} = \frac{1}{n}$, the error is less than $2/[n(\log n)^2]$.

Convergence tests revisited

Ernst Kummer (1810–1893) made significant contributions in several areas of mathematics, notably algebra and number theory. With Weierstrass and Kronecker, he was a leader of the very strong mathematics group at Berlin for many years. His convergence test is the most general of the ratio-type tests, implying all the others. His criterion for divergence is simply our (2), together with the requirement that $\sum_{n=1}^{\infty} b_n$ is divergent. However, the criterion for convergence is an enhanced version of our (1), introducing a positive difference between a_n/a_{n+1} and b_n/b_{n+1}. The exact statement is as follows. The proof can be seen, for example, in [Ph, p. 127–129]. It is pleasantly short and simple, so we reproduce it here for comparison.

THEOREM 7 (Kummer's test). *Suppose that a_n, b_n are positive for all n and for some $\delta > 0$ and some n_0,*

$$\frac{1}{b_n}\frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} \geq \delta \tag{11}$$

for all $n \geq n_0$. Then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let $A_n = \sum_{r=1}^{n} a_r$. It is enough to show that (A_n) is bounded above. We can rewrite (11) (with n replaced by r) as

$$\delta a_{r+1} \leq \frac{a_r}{b_r} - \frac{a_{r+1}}{b_{r+1}}.$$

Adding these inequalities for $n_0 \leq r \leq n-1$, we obtain

$$\delta(A_n - A_{n_0}) = \delta(a_{n_0+1} + a_{n_0+2} + \cdots + a_n) \leq \frac{a_{n_0}}{b_{n_0}} - \frac{a_n}{b_n} < \frac{a_{n_0}}{b_{n_0}}, \tag{12}$$

hence (A_n) is bounded above, as required. □

Again there is a limit version: it is sufficient for convergence if the expression in (11) tends to a positive limit as $n \to \infty$.

Note that the case $b_n = 1$ equates to the ordinary ratio test.

The convergence part of Raabe's test follows easily:

Deduction of Theorem 4(i). Suppose that $a_n/a_{n+1} \geq 1 + \frac{p}{n}$ for large enough n, where $p > 1$. Take $b_n = \frac{1}{n}$ in Theorem 6. Then the left-hand side of (11) is

$$n\frac{a_n}{a_{n+1}} - (n+1) \geq (n+p) - (n+1) = p - 1,$$

so (11) applies with $\delta = p - 1$. □

A further application is a slight variant of the convergence part of Corollary 5.1 (it actually implies this Corollary, but not conversely):

THEOREM 8. *Suppose that for some $p > 1$,*

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{p}{n \log n}$$

for sufficiently large n. Then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof: Apply Theorem 7 with $b_n = 1/(n \log n)$. Denote the left-hand side of (11) by L_n. Then

$$L_n = n \log n \, \frac{a_n}{a_{n+1}} - (n+1)\log(n+1) \geq (n+1)\log n + p - (n+1)\log(n+1).$$

It is clear from (3) that $(n+1)[\log(n+1) - \log n] \to 1$ as $n \to \infty$. Hence $L_n \to p - 1$, and the series is convergent, by the limit version of Theorem 7. □

We saw in Theorem 5 that this condition is satisfied by $a_n = 1/[n(\log n)^p]$.

Note that Kummer's test, unlike our other results, establishes convergence without estimating a_n. However, it does something else: it delivers an upper bound for A_n, and hence for the sum of the series itself. In the case where $n_0 = 1$, (12) says $\delta(A_n - a_1) \leq a_1/b_1$, hence

$$\sum_{n=1}^{\infty} a_n \leq \frac{a_1}{\delta b_1} + a_1, \tag{13}$$

since A_n satisfies this inequality for each n.

For example, if $a_n = 1/n^p$, then $a_n/a_{n+1} \geq 1 + \frac{p}{n}$, so (11) applies with $\delta = p - 1$ (also $a_1 = b_1 = 1$), so (13) gives

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{1}{p-1} + 1.$$

This is the same as the estimate obtained by the integral method.

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References

[Fe] W. L. Ferrar, *A Text-book of Convergence*, Oxford Univ. Press (1938).

[Hy] J. M. Hyslop, *Infinite Series*, Oliver and Boyd (1959).

[Jam] G. J. O. Jameson, A fresh look at Euler's limit formula for the gamma function, *Math. Gazette* **98** (2014), 235–242.

[Ph] E. G. Phillips, *A Course of Analysis*, 2nd ed., Cambridge (1939).

[Sc] J. A. Scott, On series for the Euler constant, *Math. Gazette* **101** (2017), 486–488.

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