A note on series of positive and negative terms

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Introduction

In 1888, the Italian mathematician Ernesto Cesàro published some results about series of the form $\sum_{n=1}^{\infty} \varepsilon_n a_n$, where (a_n) is positive and decreasing and tends to 0, and ε_n is 1 or -1 for each n. His results were reproduced and discussed by G. H. Hardy in the note [1], published in 1912. Very much more recently, some interesting further results on series of this type were presented by J.R. Nurcombe in the *Gazette* note [2].

In this note, we will review these results and identify some questions suggested by them, which we resolve by suitable examples.

We assume throughout that (a_n) is positive and decreasing and tends to 0. Also, we denote by p_n the number of integers $r \le n$ with $\varepsilon_r = 1$, and by q_n the number with $\varepsilon_r = -1$. So $p_n + q_n = n$ and $p_n - q_n = \sum_{r=1}^{n} \varepsilon_r$. By Abel summation, the partial sums can be rewritten in the following form, which is often useful:

$$\sum_{r=1}^{n} \varepsilon_r a_r = \sum_{r=1}^{n-1} (p_r - q_r)(a_r - a_{r+1}) + (p_n - q_n)a_n. \tag{1}$$

Cesàro's first theorem and Kronecker's theorem

One of Cesàro's results, arguably the most basic one, is:

Theorem 1 : If (a_n) is a decreasing sequence with limit 0 and $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent, then $(p_n - q_n)a_n \to 0$ as $n \to \infty$.

According to Hardy, Cesàro did not state the theorem explicitly in this form, but it appears visibly in his working. In [2], Theorem 1 is stated with the unnecessary extra condition that $a_n - a_{n+1}$ is decreasing ("Proposition", p. 116).

An immediate consequence of Theorem 1 is that if $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent (say to S), then we can let $n \to \infty$ in (1), concluding that $\sum_{n=1}^{\infty} (p_n - q_n)(a_n - a_{n+1})$ also equals S.

Now $p_n - q_n = \sum_{r=1}^{n} \varepsilon_r$, and Theorem 1 is actually a special case of the following more general theorem, in which (ε_n) is replaced by an arbitrary sequence (b_n).

Theorem 2: Suppose that (a_n) is a decreasing sequence with limit 0. Let (b_n) be another sequence such that $\sum_{n=1}^{\infty} a_n b_n$ is convergent, and write $B_n = \sum_{r=1}^{n} b_r$. Then $a_n B_n \to 0$ as

$n \to \infty$.

This is a version of a theorem of Kronecker, which seems to have been accorded less attention in the literature than it merits. For example, it is stated in [3, p. 66], but only as an exercise. Proofs can be seen in [1] and [4, p. 73], but both rely on other theorems. So we digress here to supply a self-contained proof for any readers with the appetite for it.

Proof of Theorem 2: Suppose that the result has been proved for the case where $\sum_{n=1}^{\infty} a_n b_n = 0$. Then if $\sum_{n=1}^{\infty} a_n b_n = C$, put $b_1' = b_1 - C/a_1$ and $b_n' = b_n$ for $n \geq 2$, so that $\sum_{n=1}^{\infty} a_n b_n' = 0$. We deduce that $a_n B_n = a_n B_n' + a_n C/a_1 \to 0$ as $n \to \infty$.

So suppose that $\sum_{n=1}^{\infty} a_n b_n = 0$. We also assume that $a_n > 0$ for all n, for if $a_m = 0$ for some m, then $a_n = 0$ for all $n > m$, and the result is trivial. With this assumption, we can put $M_n = 1/a_n$, so that (M_n) increases with n. Also, write $a_r b_r = c_r$ and $C_n = \sum_{r=1}^{n} c_r$, so that $C_n \to 0$ as $n \to \infty$. Then $b_r = c_r M_r$, and by Abel summation,

$$B_n = \sum_{r=1}^{n-1} C_r (M_r - M_{r+1}) + C_n M_n.$$

Now choose $\varepsilon > 0$. There exists n_0 such that $|C_r| \leq \varepsilon$ when $r > n_0$. For $n > n_0$, let

$$S_1 = \sum_{r=1}^{n_0} C_r (M_r - M_{r+1}), \qquad S_2 = \sum_{r=n_0+1}^{n-1} C_r (M_r - M_{r+1}).$$

Then $a_n B_n = a_n S_1 + a_n S_2 + C_n$. Clearly $a_n S_1 \to 0$ as $n \to \infty$, so for some $n_1 \geq n_0$, we have $|a_n S_1| \leq \varepsilon$ for all $n \geq n_1$. Also, since $|M_r - M_{r+1}| = M_{r+1} - M_r$,

$$a_n |S_2| \leq \varepsilon a_n \sum_{r=n_0+1}^{n-1} (M_{r+1} - M_r) = \varepsilon a_n (M_n - M_{n_0+1}) \leq \varepsilon a_n M_n = \varepsilon.$$

Hence $a_n |B_n| \leq 3\varepsilon$ for all $n \geq n_1$, and the theorem is proved.

Note: In [3] and [4], Kronecker's theorem is stated in the following equivalent form, which can be recognised in our proof: If $\sum_{n=1}^{\infty} c_n$ is convergent and (M_n) is increasing and tends to infinity, then $\frac{1}{M_n} \sum_{r=1}^{n} c_r M_r \to 0$ as $n \to \infty$.

Before returning to Cesàro, we record two particular cases of Theorem 2, which give an idea of its scope. First, when $b_n = 1$ for all n, it reproduces Abel's theorem that if (a_n) is positive and decreasing and $\sum_{n=1}^{\infty} a_n$ is convergent, then $n a_n \to 0$.

A second application is to series of the form $\sum_{n=1}^{\infty} b_n / n^{\beta}$ (such series are called "Dirichlet series"). Using Abel summation (again), one can show that if $|B_n| \leq C n^{\alpha}$ for all n, then $\sum_{n=1}^{\infty} b_n / n^{\beta}$ is convergent for all $\beta > \alpha$. Theorem 2 gives the following result in the converse direction: if $\sum_{n=1}^{\infty} b_n / n^{\beta}$ is convergent, then $B_n / n^{\beta} \to 0$ as $n \to \infty$.

Converse results

We now record some results in the converse direction to Theorem 1, describing various conditions on a_n, p_n and q_n that ensure convergence of $\sum_{n=1}^{\infty} \varepsilon_n a_n$. Note first that the question is only interesting when $\sum_{n=1}^{\infty} a_n$ is divergent, for if it is convergent, then $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is absolutely convergent for any choice of the ε_n.

We continue to assume that (a_n) is positive, decreasing and has limit 0. Given this, an immediate consequence of (1) (equivalently, of Dirichlet's criterion for convergence) is:

Theorem 3: If $(p_n - q_n)$ is bounded, then $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent.

The note [2] contains two further results giving sufficient conditions. First (with a slight change from the notation of [2]):

Theorem 4 [2, Theorem 2]: Suppose that for some K and some $\alpha < 1$,

$$|p_n - q_n| a_n^{\alpha} \leq K$$

for all n. Then $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent. This occurs, in particular, if for some $\beta > \gamma > 0$, we have $|p_n - q_n| \leq K_1 n^{\gamma}$ and $a_n \leq K_2 / n^{\beta}$ for all n.

Observe that Theorem 4 extends Theorem 3. The other result from [2] is:

Theorem 5 [2, Theorem 3]: Let $P_n = \sum_{r=1}^{n} p_r$, similarly Q_n. Suppose that $P_n - Q_n = O(n)$, $a_n - a_{n+1}$ is decreasing and $(p_n - q_n) a_n \to 0$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent.

These results are restricted converses to Theorem 1, and one might ask whether in fact the converse holds without restriction. In other words, does $(p_n - q_n) a_n \to 0$ imply convergence of $\sum_{n=1}^{\infty} \varepsilon_n a_n$? Can the other conditions be dropped in Theorem 5? However, a simple counter-example, without any negative terms, is enough to show that the answer is no.

Example 1. Let $\varepsilon_n = 1$ for all n and let $a_n = 1/(n \log n)$ for $n \geq 2$ (also, put $a_1 = 1$). Then $p_n - q_n = n$, so

$$(p_n - q_n) a_n = \frac{1}{\log n} \to 0 \quad \text{as } n \to \infty.$$

However, it is well known that $\sum_{n=1}^{\infty} a_n$ is divergent.

Another example serves to show that $\sum_{n=1}^{\infty} \varepsilon_n a_n$ can be divergent when $p_n - q_n$ grows very slowly with n (in contrast to Theorem 3). More exactly, for any $\gamma > 0$, however small, divergence can occur with p_n, q_n satisfying $|p_n - q_n| \leq K n^{\gamma}$ for all n. In fact, suppose that for some $c > 0$, we also have $|p_n - q_n| \geq c n^{\gamma}$ for infinitely many n. If $a_n = 1/n^{\gamma}$, then

$(p_n - q_n)a_n$ does not tend to 0, so by Theorem 1, $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is divergent. This will also show that we cannot replace β by γ in the last statement in Theorem 4. For it to apply, we just need to show that for arbitrarily small $\gamma > 0$, one can choose ε_n so that $p_n - q_n$ satisfies these conditions.

Example 2. Let $\gamma = \frac{1}{r}$ for an integer $r \geq 2$. Let F_k be the block of integers n given by $(k-1)^r < n \leq k^r$. This has an odd number of terms, since one of $k-1$, k is even and the other odd. We define ε_n to be alternately $+1$ and -1 through F_k, so that $\sum_{n \in F_k} \varepsilon_n = 1$. Hence for $n = k^r$, we have $p_n - q_n = k$, and for each $n \in F_k$, $p_n - q_n$ is either $k-1$ or k. For $n \in F_k$, we also have $k - 1 \leq n^{1/r} \leq k$, so

$$\frac{k-1}{k} n^{1/r} \leq p_n - q_n \leq \frac{k}{k-1} n^{1/r}.$$

So in fact $(p_n - q_n)/n^{1/r} \to 1$ as $n \to \infty$.

Cesàro's second theorem

Cesàro's other main theorem can be stated as follows:

Theorem 6: Suppose that $\sum_{n=1}^{\infty} a_n$ is divergent and $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent. If $\frac{1}{n}(p_n - q_n)$ tends to a limit, then the limit is 0.

The proof is a fairly straightforward application of (1); it can be seen in [3, p. 61]. This is the theorem described by Bromwich as "curious" (he does not mention Theorem 1).

So we now focus on the limit (if any) of $\frac{1}{n}(p_n - q_n)$ instead of $(p_n - q_n)a_n$. It can be expressed in several equivalent ways. Suppose that $\frac{1}{n}(p_n - q_n) \to \ell$ as $n \to \infty$. Then, since $q_n = n - p_n$,

$$\frac{p_n - q_n}{n} = \frac{2p_n}{n} - 1 \to \ell,$$

so $\frac{p_n}{n} \to \frac{1}{2}(1 + \ell)$ and $\frac{q_n}{n} \to \frac{1}{2}(1 - \ell)$. Hence also (if $\ell < 1$),

$$\frac{p_n}{q_n} \to \frac{1+\ell}{1-\ell}.$$

Conversely, if $p_n/q_n \to L$, then

$$\frac{n}{q_n} = \frac{p_n + q_n}{q_n} \to L + 1,$$

so $\frac{q_n}{n} \to 1/(L+1)$ and $\frac{p_n}{n} \to L/(L+1)$.

So Theorem 6 can be stated equivalently by saying that the only possible limit of p_n/n is $\frac{1}{2}$, and the only possible limit of p_n/q_n is 1.

However, with (a_n) and (ε_n) as stated, these sequences do not necessarily converge to any limit. Hardy, if not Cesàro, was surely aware of this, but does not give an example. We now give an example to demonstrate the point.

Example 3. For $k \geq 1$, write

$$E_k = \{n : 2^{k-1} \leq n < 2^k\},$$

a block of 2^{k-1} terms. Note that $E_1 = \{1\}$ and $E_2 = \{2, 3\}$. Let

$$\varepsilon_n = \begin{cases} 1 & \text{for } n \in E_{2k-1}, \\ -1 & \text{for } n \in E_{2k}. \end{cases}$$

At the end of block E_{2k-1}, we have $n = 2^{2k-1} - 1$ and

$$p_n - q_n = 1 - 2 + 2^2 - \cdots + 2^{2k-2} = \tfrac{1}{3}(2^{2k-1} + 1),$$

so $\frac{1}{n}(p_n - q_n)$ is close to $\frac{1}{3}$. Meanwhile, at the end of block E_{2k}, we have $n = 2^{2k} - 1$ and

$$p_n - q_n = 1 - 2 + 2^2 - \cdots - 2^{2k-1} = -\tfrac{1}{3}(2^{2k} - 1),$$

so $\frac{1}{n}(p_n - q_n)$ is close to $-\frac{1}{3}$. Hence $\frac{1}{n}(p_n - q_n)$ does not tend to 0, or any other limit.

We need to choose a_n so that $\sum_{n=1}^{\infty} a_n$ is divergent and $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is convergent. For a positive sequence (α_k) to be chosen, define a_n to be α_k for n in E_{2k-1} and $\frac{1}{2}\alpha_k$ for n in E_{2k}. Then

$$\sum_{n \in E_{2k-1}} a_n = \sum_{n \in E_{2k}} a_n = 2^{2k-2}\alpha_k,$$

so if the series $\sum_{k=1}^{\infty} 2^{2k}\alpha_k$ is divergent, then so is $\sum_{n=1}^{\infty} a_n$. However, the sum of the terms $\varepsilon_n a_n$ for n in $E_{2k-1} \cup E_{2k}$ is 0, and the largest sum part way through is $2^{2k-2}\alpha_k$, occuring at the end of block E_{2k-1}. So if $2^{2k}\alpha_k \to 0$ as $k \to \infty$, then $\sum_{n=1}^{\infty} \varepsilon_n a_n$ converges to 0. Hence the required counter-example is furnished by any (α_k) such that $2^{2k}\alpha_k \to 0$ and $\sum_{k=1}^{\infty} 2^{2k}\alpha_k$ is divergent. For a_n to be decreasing, we also need $\alpha_{k+1} \leq \frac{1}{2}\alpha_k$. An obvious choice is $\alpha_k = 1/(k2^{2k})$.

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References

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Dept. of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK

e-mail: *g.jameson@lancaster.ac.uk*