

An elementary derivation of the Poisson summation formula

Tim Jameson

The traditional method of proving the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx$$

is to sum the Fourier coefficients for the function $F(x) = \sum_{-\infty}^{\infty} f(n+x)$ (for example [1], page 134). Here I derive it by using the Euler summation formula:

$$\sum_{n=M}^N f(n) = \frac{1}{2}f(M) + \frac{1}{2}f(N) + \int_M^N f(x) dx + \int_M^N (x - [x] - \frac{1}{2})f'(x) dx$$

where $[x]$ denotes the greatest integer less than x for non-integer x , and M and N are integers with $M < N$. Euler's formula is easily proved as follows: integration by parts gives

$$\begin{aligned} \int_n^{n+1} (x - n - \frac{1}{2})f'(x) dx &= [(x - n - \frac{1}{2})f(x)]_n^{n+1} - \int_n^{n+1} f(x) dx \\ &= \frac{1}{2}f(n+1) + \frac{1}{2}f(n) - \int_n^{n+1} f(x) dx. \end{aligned}$$

Now add these identities for $M \leq n \leq N-1$. For our purposes we will replace (M, N) with $(-\infty, \infty)$ and require that the integrals and series converge. Notice that the value of $[x]$ for integer x is irrelevant in the above, so that for convenience we may define it to be $x - \frac{1}{2}$ for integer x . We now make use of the well-known sine series

$$x - [x] - \frac{1}{2} = - \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n} \quad \text{for all real } x$$

which is zero when x is an integer - hence the extended definition of $[x]$. Inserting this into the Euler formula and assuming termwise integration is valid, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) dx - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} f'(x) \sin 2\pi n x dx$$

Since we are assuming that $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, integration by parts gives

$$\int_{-\infty}^{\infty} f'(x) \sin 2\pi n x dx = -2\pi n \int_{-\infty}^{\infty} f(x) \cos 2\pi n x dx.$$

Hence

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f(n) &= \int_{-\infty}^{\infty} f(x) dx + 2 \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(x) \cos 2\pi n x dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx \end{aligned}$$

because the sine terms cancel in this last integral.

Reference

1. P.L. Walker, *The Theory of Fourier Series and Integrals*, Wiley (1986).