

MDIV. Multiple divisor functions

The functions τ_k

For $k \geq 1$, define $\tau_k(n)$ to be the number of (ordered) factorisations of n into k factors, in other words, the number of ordered k -tuples (j_1, j_2, \dots, j_k) with $j_1 j_2 \dots j_k = n$. So $\tau_1(n) = 1$ and τ_2 is the usual divisor function τ . These notes presuppose familiarity with the elementary facts about τ .

For a prime p , we have $\tau_k(p) = k$: the k -tuples are $(1, \dots, p, \dots, 1)$, with a single p .

We also assume familiarity with convolutions. Denote by u the unit function defined by $u(n) = 1$ for all n . Then $\tau = u * u$. Also, $\tau(n)/n = (h * h)(n)$, where $h(n) = \frac{1}{n}$.

MDIV1. We have

$$\tau_k(n) = \sum_{j|n} \tau_{k-1}(j). \tag{1}$$

Hence $\tau_k = \tau_{k-1} * u = u * u * \dots * u$ (in which u is repeated k times).

Proof. For a fixed divisor j of n , the number of k -tuples with $j_1 j_2 \dots j_{k-1} j = n$ is the number of $(k-1)$ -tuples $j_1 j_2 \dots j_{k-1} = n/j$, that is, $\tau_{k-1}(n/j)$. When j runs through the divisors of n , so does n/j . The statement follows. \square

This implies the Dirichlet series identity

$$\sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta(s)^k,$$

but we will not use this identity in these notes.

Similarly, $\frac{\tau_k(n)}{n} = (h * h * \dots * h)(n)$.

MDIV2. The function τ_k is multiplicative. For prime p ,

$$\tau_k(p^a) = \binom{a+k-1}{k-1} = \binom{a+k-1}{a}. \tag{2}$$

So if n has prime factorisation $\prod_{j=1}^m p_j^{a_j}$, then

$$\tau_k(n) = \prod_{j=1}^m \binom{a_j+k-1}{a_j}.$$

Proof. The function τ_k is multiplicative, since it is a convolution of multiplicative functions. Alternatively, a direct proof is as follows. If n has prime factorisation $\prod_{j=1}^m p_j^{a_j}$, then k -fold factorisations of n are obtained by combining k -fold factorisations of each $p_j^{a_j}$.

Now k -fold factorisations of p^a correspond to k -tuples (a_1, a_2, \dots, a_k) with $a_j \geq 0$ for each j and $a_1 + a_2 + \dots + a_k = a$. Consider sequences composed of a 1's and $k - 1$ zeros. Clearly, the number of such sequences is $\binom{a+k-1}{k-1}$. In such a sequence, the zeros divide the 1's into an ordered sequence of k subsets, with a total of a elements. So such sequences correspond to k -tuples (a_1, a_2, \dots, a_k) as above. \square

Alternative proof of (2). In the power series identity

$$(1 - x)^{-k} = (1 + x + x^2 + \dots)^k,$$

the coefficient of x^n is

$$\frac{k(k+1)\dots(k+n-1)}{n!} = \binom{k+n-1}{n}.$$

But the right-hand side shows that this coefficient is the number of non-negative k -tuples (n_1, n_2, \dots, n_k) with $n_1 + n_2 + \dots + n_k = n$. \square

In particular,

$$\begin{aligned}\tau_3(p^a) &= \frac{1}{2}(a+1)(a+2), \\ \tau_4(p^a) &= \frac{1}{6}(a+1)(a+2)(a+3).\end{aligned}$$

Yet another proof of (2) is by induction on a . This is very simple in the case $k = 3$:

$$\tau_2(p^a) = \sum_{b=0}^a \tau_2(p^b) = \sum_{b=0}^a (b+1) = \frac{1}{2}(a+1)(a+2).$$

We now give two inequalities for $\tau_k(n)$.

MDIV3. *We have $\tau_k(mn) \leq \tau_k(m)\tau_k(n)$ for all $m, n \geq 1$.*

Proof. The statement is trivial if m or n is 1. Since τ_k is multiplicative, it is enough to prove this for $m = p^a$, $n = p^b$, where p is prime and $a \geq 1$ and $b \geq 1$. Now

$$\tau_k(p^a) = \frac{1}{(k-1)!} \prod_{r=1}^{a-1} (a+r),$$

so

$$[(k-1)!]^2 \tau_k(p^a) \tau_k(p^b) = \prod_{r=1}^{a-1} (a+r)(b+r),$$

$$[(k-1)!]^2 \tau_k(p^{a+b}) = (k-1)! \prod_{r=1}^{a+b-1} (a+b+r) = \prod_{r=1}^{a-1} r(a+b+r).$$

The statement follows, since $(a+r)(b+r) = r(a+b+r) + ab > r(a+b+r)$. \square

MDIV4. For all $j, k, n \geq 1$, we have

$$\tau_j(n)\tau_k(n) \leq \tau_{jk}(n).$$

Proof. It is sufficient to prove this for $n = p^a$, where $a \geq 1$. Now

$$\tau_j(p^a) = \frac{1}{a!} \prod_{r=0}^{a-1} (j+r),$$

so

$$(a!)^2 \tau_j(p^a) \tau_k(p^a) = \prod_{r=0}^{a-1} (j+r)(k+r),$$

$$(a!)^2 \tau_{jk}(p^a) = a! \prod_{r=0}^{a-1} (jk+r) = \prod_{r=0}^{a-1} (r+1)(jk+r).$$

The statement follows, since

$$(r+1)(jk+r) - (j+r)(k+r) = rjk - r(j+k) + r = r(j-1)(k-1) \geq 0. \quad \square$$

Recalling that $\tau_2 = \tau$, we deduce at once by induction:

MDIV5 COROLLARY. We have $\tau(n)^k \leq \tau_K(n)$, where $K = 2^k$. \square

Summary of some results assumed

(A1) (Basic integral estimation, [Jam, 1.4.2]): Let $f(t)$ be a decreasing, non-negative function for $t \geq 1$. Write $S(x) = \sum_{n \leq x} f(n)$ and $I(x) = \int_1^x f(t) dt$. Then for all $x \geq 1$,

$$I(x) \leq S(x) \leq I(x) + f(1).$$

(A2) (Variant of integral estimation, proved by a slight extension of the same method [Nath, p. 206–208]): Define $S(x), I(x)$ as in (A1). Suppose that $f(t)$ is non-negative, increasing for $1 \leq t \leq x_0$ and decreasing for $x \geq x_0$, with maximum value $f(x_0) = M$. Then

$$I(x) - M \leq S(x) \leq I(x) + M.$$

(A3) (Abel's summation formula [Jam, Prop. 1.3.6]): Let $A(x) = \sum_{n \leq x} a(n)$ and let f have continuous derivative on $[1, x]$. Then

$$\sum_{n \leq x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t) dt.$$

(A4) (Partial sums of convolutions [Jam, 1.8.4]): If $A(x) = \sum_{n \leq x} a(n)$ and $B(x) = \sum_{n \leq x} b(n)$, then

$$\sum_{n \leq x} (a * b)(n) = \sum_{j \leq x} a(j) B\left(\frac{x}{j}\right) = \sum_{k \leq x} A\left(\frac{x}{k}\right) b(k).$$

Write

$$H(x) = \sum_{n \leq x} \frac{1}{n}.$$

By (A1), applied to $f(t) = 1/t$, we have

$$\log x \leq H(x) \leq \log x + 1. \quad (3)$$

A more accurate estimation of $H(x)$ [Jam, 1.4.11] is:

$$H(x) = \log x + \gamma + q(x), \quad (4)$$

where γ is Euler's constant and $|q(x)| \leq 1/x$.

We also need the following estimation, derived from (A2). Write $\log^n x$ for $(\log x)^n$. The function $f(t) = (\log^r t)/t$ increases for $1 \leq t \leq e^r$ and decreases for $t \geq e^r$, with $f(e^r) = (re^{-1})^r$, so we deduce

$$\sum_{n \leq x} \frac{\log^r n}{n} = \frac{1}{r+1} \log^{r+1} x + q_r(x), \quad (5)$$

where $|q_r(x)| \leq (re^{-1})^r$. Of course, $q_0(x) = H(x) - \frac{1}{x} = \gamma + O(1/x)$. In fact, $q_r(x) = \gamma_r + O(x^{-1} \log^r x)$, where γ_r is the r th Stieltjes constant, but we won't use this.

Where convenient, we use the notation $f(x) \ll g(x)$ to mean the same as $f(x) = O[g(x)]$.

Summation functions

We write

$$T_k(x) = \sum_{n \leq x} \tau_k(n),$$

also

$$S_k(x) = \sum_{n \leq x} \frac{\tau_k(n)}{n}.$$

Clearly, $T_k(x)$ is the number of k -tuples with $j_1 j_2 \dots j_k \leq x$, and $S_k(x)$ is the sum of the reciprocals of these products.

Note that $T_1(x) = [x]$ and $S_1(x) = H(x)$.

It is elementary that $T_2(x) = \sum_{n \leq x} [x/n]$. This, together with (3), gives the simple estimation

$$x \log x - x \leq T_2(x) \leq x \log x + x. \quad (6)$$

A more accurate estimation of $T_2(x)$ is given by the famous theorem of Dirichlet ([HWr, p. 264], [Jam, Prop. 2.5.1] and many other books):

$$T_2(x) = x \log x + (2\gamma - 1)x + \Delta_2(x), \quad (7)$$

where $\Delta_2(x) = O(x^{1/2})$. The true order of magnitude of $\Delta_2(x)$ has been the subject of a great deal of study. Denote by θ_0 the infimum of numbers θ such that $\Delta(x)$ is $O(x^\theta)$. It was already shown by Voronoi in 1903 that $\theta_0 \leq \frac{1}{3}$ [Ten, sect. 1.6.4]. The current best estimate, due to M. N. Huxley, is $\theta_0 \leq \frac{131}{416} \approx 0.31490$.

We will give estimations of $T_k(x)$ and $S_k(x)$ corresponding to these two levels of accuracy (the more accurate ones are not needed for the later application to $\tau(n)^2$). The starting point is the following:

MDIV6 PROPOSITION. *We have*

$$T_k(x) = \sum_{n \leq x} T_{k-1} \left(\frac{x}{n} \right) \quad (8)$$

$$= \sum_{n \leq x} \tau_{k-1}(n) \left[\frac{x}{n} \right] \quad (9)$$

and

$$S_k(x) = \sum_{n \leq x} \frac{1}{n} S_{k-1} \left(\frac{x}{n} \right) \quad (10)$$

$$= \sum_{n \leq x} \frac{1}{n} \tau_{k-1}(n) H \left(\frac{x}{n} \right). \quad (11)$$

Proof. These identities all follow from (A4). Direct reasoning is also easy, as follows. Fix $n \leq x$. The number of k -tuples $(j_1, j_2, \dots, j_{k-1}, n)$ with $j_1 j_2 \dots j_{k-1} n \leq x$ is the number of $(k-1)$ -tuples with $j_1 j_2 \dots j_{k-1} \leq x/n$, that is, $T_{k-1}(x/n)$. Hence $T_k(x) = \sum_{n \leq x} T_{k-1}(x/n)$. Also, there are $\tau_{k-1}(n)$ ways to express n as $j_1 j_2 \dots j_{k-1}$. There are then $[x/n]$ choices of j_k such that $n j_k \leq x$. Hence $T_k(x) = \sum_{n \leq x} \tau_{k-1}(n) [x/n]$.

(10) and (11) are proved in the same way, adding reciprocals of terms instead of counting them. \square

In the case $k = 2$, both (8) and (9) equate to $\sum_{n \leq x} [x/n]$, and both (10) and (11) to $\sum_{n \leq x} \frac{1}{n} H \left(\frac{x}{n} \right)$.

MDIV7 COROLLARY. *We have*

$$xS_{k-1}(x) - T_{k-1}(x) \leq T_k(x) \leq xS_{k-1}(x). \quad (12)$$

Proof. By (9),

$$\sum_{n \leq x} \tau_{k-1}(n) \left(\frac{x}{n} - 1 \right) \leq T_k(x) \leq \sum_{n \leq x} \tau_{k-1}(n) \frac{x}{n}.$$

This equates to (12). □

We shall derive estimations of $S_k(x)$ and $T_k(x)$ (in that order) from estimations of $A_k(x)$, where

$$A_k(x) = \sum_{n \leq x} \frac{1}{n} (\log x - \log n)^k. \quad (13)$$

At the first level of accuracy, we have:

MDIV8. *We have*

$$\frac{1}{k+1} \log^{k+1} x \leq A_k(x) \leq \frac{1}{k+1} \log^{k+1} x + \log^k x. \quad (14)$$

Proof. Let $f(t) = \frac{1}{t} (\log x - \log t)^k$ for $1 \leq t \leq x$ (also $f(t) = 0$ for $t > x$). Then $f(t)$ is decreasing and non-negative, and

$$\int_1^x f(t) dt = \left[-\frac{1}{k+1} (\log x - \log t)^{k+1} \right]_1^x = \frac{\log^{k+1}(x)}{k+1}.$$

The statement follows, by (A1). □

MDIV9 THEOREM. *For all $k \geq 1$,*

$$S_k(x) = \frac{\log^k x}{k!} + O(\log^{k-1} x). \quad (15)$$

Also,

$$S_k(x) \geq \frac{\log^k x}{k!}. \quad (16)$$

Proof. Induction on k . The case $k = 1$ is (3). Assume (15) for k , with error term denoted by $q_k(x)$. Then by (10),

$$S_{k+1}(x) = \sum_{n \leq x} \frac{1}{n} S_k \left(\frac{x}{n} \right) = I(x) + Q(x),$$

where

$$I(x) = \frac{1}{k!} \sum_{n \leq x} \frac{1}{n} \log^k \frac{x}{n} = \frac{1}{k!} A_k(x),$$

$$Q(x) = \sum_{n \leq x} \frac{1}{n} q_k \left(\frac{x}{n} \right) = \sum_{n \leq x} \frac{1}{n} \log^{k-1} \frac{x}{n} = A_{k-1}(x).$$

By (14),

$$I(x) = \frac{1}{(k+1)!} \log^{k+1} x + O(\log^k x),$$

and $Q(x) \ll \log^k x$. Hence (15) holds for $k+1$. Also, $S_1(x) = H(x) > \log x$, and (16) follows easily, using the left-hand side of (14). \square

MDIV10 THEOREM. *For all $k \geq 2$,*

$$T_k(x) = \frac{1}{(k-1)!} x \log^{k-1} x + O(x \log^{k-2} x), \quad (17)$$

Proof. The case $k = 2$ is (6). Assume (17) for k . By (12) and (15), we then have

$$T_{k+1}(x) = x S_k(x) + O[T_k(x)] = \frac{x}{k!} \log^k x + O(x \log^{k-1} x). \quad \square$$

Alternatively, one can prove (17) directly, without reference to $S_k(x)$, in the same way as MDIV9, using (8) instead of (10).

Another alternative is to prove (15) and (17) simultaneously by induction. Instead of using (14), one deduces (15) from (17) by Abel summation; as in MDIV10, the pair of statements for k then implies (17) for $k+1$.

There is no simple lower bound corresponding to (16) for $T_k(x)$; in fact, $T_2(x) < x \log x$ for some values of x , for example on intervals to the left of the integers 1 to 5.

By MDIV5 and (17), we have:

MDIV11 PROPOSITION. *Write $2^k = K$. For some constant C_k ,*

$$\sum_{n \leq x} \tau(n)^k \leq \frac{1}{(K-1)!} x \log^{K-1} x + C_k x \log^{K-2} x. \quad \square \quad (18)$$

We now move to the second level of accuracy. We consider $A_1(x)$ separately, since it is much simpler than the general case.

MDIV12 LEMMA. *We have*

$$A_1(x) = \frac{1}{2} \log^2 x + \gamma \log x + O(1). \quad (19)$$

Proof. By (4) and (5), we have

$$\begin{aligned}
A_1(x) &= H(x) \log x - \sum_{n \leq x} \frac{\log n}{n} \\
&= \log x [\log x + \gamma + O(1/x)] - \frac{1}{2} \log^2 x - q_1(x) \\
&= \frac{1}{2} \log^2 x + \gamma \log x + O(1). \quad \square
\end{aligned}$$

Since $q_1(x) = \gamma_1 + O(\log x/x)$, the term $O(1)$ in (19) can be replaced by $-\gamma_1 + O(\log x/x)$ for still greater accuracy.

We derive the estimates for $S_2(x)$ and $T_3(x)$.

MDIV13 PROPOSITION. *We have*

$$S_2(x) = \frac{1}{2} \log^2 x + 2\gamma \log x + O(1). \quad (20)$$

Proof. With $q(x)$ as in (4),

$$\begin{aligned}
S_2(x) &= \sum_{n \leq x} \frac{1}{n} H\left(\frac{x}{n}\right) = \sum_{n \leq x} \frac{1}{n} \left[\log x - \log n + \gamma + q\left(\frac{x}{n}\right) \right] \\
&= A_1(x) + \gamma H(x) + q_1(x),
\end{aligned}$$

where $q_1(x) = \sum_{n \leq x} q(x/n)$. Since $|q(x/n)| \leq n/x$, we have $|q_1(x)| \leq 1$. Also, $\gamma H(x) = \gamma \log x + O(1)$. With (19), we obtain (20). \square

With more effort, one can establish the following estimate, at a third level of accuracy:

$$S_2(x) = \frac{1}{2} \log^2 x + 2\gamma \log x + c_0 + O(x^{-1/2} \log x),$$

where $c_0 = \gamma^2 - 2\gamma_1$. See DIVSUMS.

MDIV14 PROPOSITION. *We have*

$$T_3(x) = \frac{1}{2} x \log^2 x + (3\gamma - 1)x \log x + O(x). \quad (21)$$

Proof. By (8) and (7),

$$\begin{aligned}
T_3(x) &= \sum_{n \leq x} T_2\left(\frac{x}{n}\right) \\
&= \sum_{n \leq x} \frac{x}{n} (\log x - \log n) + \sum_{n \leq x} (2\gamma - 1) \frac{x}{n} + r(x) \\
&= xA_1(x) + (2\gamma - 1)xH(x) + r(x),
\end{aligned}$$

where

$$r(x) = \sum_{n \leq x} \Delta_2 \left(\frac{x}{n} \right) \ll \sum_{n \leq x} \frac{x^{1/2}}{n^{1/2}}.$$

By integral estimation, $\sum_{n \leq x} (1/n^{1/2}) \leq 2x^{1/2}$, so $r(x) = O(x)$. (This estimate would not be improved by using $\Delta(x) = O(x^\theta)$.) Also, by (16),

$$xA_1(x) = \frac{1}{2}x \log^2 x + \gamma x \log x + O(x),$$

and by (4),

$$(2\gamma - 1)xH(x) = (2\gamma - 1)x \log x + O(x). \quad \square$$

We now turn to the general case. The estimation of $A_k(x)$ requires rather more work.

MDIV15 LEMMA. *We have*

$$\sum_{r=0}^k \frac{(-1)^r}{r+1} \binom{k}{r} = \frac{1}{k+1}. \quad (22)$$

Proof. Clearly,

$$\frac{k+1}{r+1} \binom{k}{r} = \binom{k+1}{r+1}.$$

Hence

$$\begin{aligned} (k+1) \sum_{r=0}^k \frac{(-1)^r}{r+1} \binom{k}{r} &= \sum_{r=0}^k (-1)^r \binom{k+1}{r+1} \\ &= - \sum_{s=1}^k (-1)^s \binom{k+1}{s} \\ &= 1 - (1-1)^{k+1} \\ &= 1. \quad \square \end{aligned}$$

MDIV16 PROPOSITION. *We have*

$$A_k(x) = \frac{1}{k+1} \log^{k+1} x + \gamma \log^k x + O(\log^{k-1} x). \quad (23)$$

Proof. By the binomial theorem,

$$\begin{aligned} A_k(x) &= \sum_{n \leq x} \sum_{r=0}^k (-1)^r \binom{k}{r} \log^{k-r} x \log^r n \\ &= \sum_{r=0}^k (-1)^r \binom{k}{r} \log^{k-r} x \sum_{n \leq x} \frac{1}{n} \log^r n. \end{aligned}$$

Using (5) to substitute for $\sum_{n \leq x} \frac{1}{n} \log^r n$, we have $A_k(x) = B_k(x) + Q_k(x)$, where

$$B_k(x) = \sum_{r=0}^k (-1)^r \binom{k}{r} \log^{k-r} x \frac{\log^{r+1} x}{r+1},$$

$$Q_k(x) = \sum_{r=0}^k (-1)^r \binom{k}{r} \log^{k-r} x q_r(x).$$

By (18),

$$B_k(x) = \log^{k+1} x \sum_{r=0}^k \frac{(-1)^r}{r+1} \binom{k}{r} = \frac{1}{k+1} \log^{k+1} x.$$

Also

$$Q_k(x) = q_0(x) \log^k x + O(\log^{k-1} x) = \gamma \log^k x + O(\log^{k-1} x). \quad \square$$

Remark. This proof is given in [Nath, Theorem 6.6, p. 209], but only in order to state (14), which, as we have seen, can be proved much more simply.

MDIV17 THEOREM. For all $k \geq 2$,

$$S_k(x) = \frac{1}{k!} \log^k x + c_k \log^{k-1} x + O(\log^{k-2} x), \quad (24)$$

where

$$c_k = \frac{k\gamma}{(k-1)!}.$$

Proof. Induction on k . The case $k = 2$ is (21). Assume (24) for $k >$, with error term denoted by $\delta_k(x)$. By (10), we then have $S_{k+1}(x) = I_1(x) + I_2(x) + P(x)$, where

$$I_1(x) = \frac{1}{k!} \sum_{n \leq x} \frac{1}{n} \log^k \frac{x}{n} = \frac{1}{k!} A_k(x),$$

$$I_2(x) = c_k \sum_{n \leq x} \frac{1}{n} \log^{k-1} \frac{x}{n} = c_k A_{k-1}(x),$$

$$P(x) = \sum_{k \leq x} \frac{1}{n} \delta_k \left(\frac{x}{n} \right) \ll \sum_{k \leq x} \frac{1}{n} \log^{k-2} \frac{x}{n} = A_{k-2}(x).$$

By (23),

$$I_1(x) = \frac{1}{(k+1)!} \log^{k+1} x + \frac{\gamma}{k!} \log^k x + O(\log^{k-1} x).$$

By (14),

$$I_2(x) = \frac{c_k}{k} \log^k x + O(\log^{k-1} x),$$

and $P(x) \ll \log^{k-1} x$. Together, these estimations give the statement for $k + 1$, with

$$c_{k+1} = \frac{c_k}{k} + \frac{\gamma}{k!}.$$

By (20), $c_2 = 2\gamma$, and it is easily verified by induction that $c_k = k\gamma/(k-1)!$. □

This time, we cannot simply use (12) to deduce the corresponding result for $T_k(x)$. Instead, we prove it separately, in similar style.

MDIV18 THEOREM. For $k \geq 3$,

$$T_k(x) = \frac{x}{(k-1)!} \log^{k-1} x + d_k x \log^{k-2} x + O(x \log^{k-3} x). \quad (25)$$

where

$$d_k = \frac{k\gamma - 1}{(k-2)!}.$$

Proof. Induction on k . The case $k = 3$ is (21). Assume (25) for k , with error term denoted by $\Delta_k(x)$. By (8), we then have $T_{k+1}(x) = J_1(x) + J_2(x) + Q(x)$, where

$$J_1(x) = \frac{x}{(k-1)!} \sum_{n \leq x} \frac{1}{n} \log^{k-1} \frac{x}{n} = \frac{x}{(k-1)!} A_{k-1}(x),$$

$$J_2(x) = d_k x \sum_{n \leq x} \frac{1}{n} \log^{k-2} \frac{x}{n} = d_k x A_{k-2}(x),$$

$$Q(x) = \sum_{n \leq x} \Delta_k \left(\frac{x}{n} \right) \ll \sum_{n \leq x} \frac{x}{n} \log^{k-3} \frac{x}{n} = x A_{k-3}(x).$$

By (23),

$$J_1(x) = \frac{1}{k!} x \log^k x + \frac{\gamma}{(k-1)!} x \log^{k-1} x + O(x \log^{k-2} x).$$

By (14),

$$J_2(x) = \frac{c_k}{k-1} x \log^{k-1} x + O(x \log^{k-2} x)$$

and $Q(x) \ll x \log^{k-2} x$. Together, these estimations give the statement for $k+1$, with

$$d_{k+1} = \frac{d_k}{k-1} + \frac{\gamma}{(k-1)!}.$$

By (21), $d_3 = 3\gamma - 1$, and it is easily verified by induction that $d_k = (k\gamma - 1)/(k-2)!$.

An alternative for (24) is to deduce it from (25) by Abel summation, but this is not really any shorter than the direct proof given.

Summation of $\tau(n)^2$

By MDIV9, we have

$$\sum_{n \leq x} \tau(n)^2 \leq \frac{1}{6} x \log^3 x + r(x),$$

where $r(x) \ll x \log^2 x$. This, of course, is only an upper bound estimate. We now derive an asymptotic estimate. It was originally obtained by Ramanujan.

We denote the Möbius function by $\mu(n)$, and write $M_2(x) = \sum_{n \leq x} \mu(n)^2$. We now use the well-known estimate

$$M_2(x) = \frac{6}{\pi^2} x + q(x), \quad (26)$$

where $q(x) = O(x^{1/2})$ [HWI, p. 270], [Jam, 2.5.5].

MDIV19. *We have* $\tau^2 = \tau_3 * \mu^2$.

Proof. Both sides are multiplicative, so it is sufficient to consider the values at a prime power p^k . We have

$$(\tau_3 * \mu^2)(p^k) = \tau_3(p^k) + \tau_3(p^{k-1}) = \frac{1}{2}(k+1)(k+2) + \frac{1}{2}k(k+1) = (k+1)^2 = \tau(p^k)^2. \quad \square$$

MDIV20 LEMMA. *We have*

$$\sum_{n \leq x} \frac{\tau_3(n)}{n^{1/2}} \ll x^{1/2} \log^2 x. \quad (27)$$

Proof. By Abel summation,

$$\sum_{n \leq x} \frac{\tau_3(n)}{n^{1/2}} = \frac{T_3(x)}{x^{1/2}} + \frac{1}{2} \int_1^x \frac{T_3(t)}{t^{3/2}} dt.$$

Using only the fact that $T_3(x) \ll x \log^2 x$, we have

$$\frac{T_3(x)}{x^{1/2}} \ll x^{1/2} \log^2 x$$

and

$$\int_1^x \frac{T_3(t)}{t^{3/2}} dt \leq \log^2 x \int_1^x \frac{1}{t^{1/2}} dt < 2x^{1/2} \log^2 x. \quad \square$$

From (15), one can show that the sum in (27) is actually $x^{1/2} \log^2 x + O(x^{1/2} \log x)$.

MDIV21 THEOREM. *We have*

$$\sum_{n \leq x} \tau(n)^2 = \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x), \quad (28)$$

$$\sum_{n \leq x} \frac{\tau(n)^2}{n} = \frac{1}{4\pi^2} \log^4 x + O(\log^3 x). \quad (29)$$

Proof. By MDIV19 and (A4), with $q(x)$ as in (26), we have

$$\begin{aligned}\sum_{n \leq x} \tau(n)^2 &= \sum_{n \leq x} \tau_3(n) M_2\left(\frac{x}{n}\right) \\ &= \sum_{n \leq x} \tau_3(n) \left(\frac{6}{\pi^2} \frac{x}{n} + q\left(\frac{x}{n}\right) \right).\end{aligned}$$

By (15),

$$\sum_{n \leq x} \frac{x}{n} \tau_3(n) = x S_3(x) = \frac{1}{6} x \log^3 x + O(x \log^2 x),$$

and by (27)

$$\sum_{n \leq x} \tau_3(n) q\left(\frac{x}{n}\right) \ll x^{1/2} \sum_{n \leq x} \frac{\tau_3(n)}{n^{1/2}} \ll x \log^2 x.$$

This establishes (28).

We deduce (29) by Abel summation. Write $\sum_{n \leq x} \tau(n)^2 = A(x)$, and denote the error term in (28) by $r(x)$. By (A3),

$$\sum_{n \leq x} \frac{\tau(n)^2}{n} = J_1 + J_2,$$

where

$$\begin{aligned}J_1 &= \frac{A(x)}{x} \ll \log^3 x, \\ J_2 &= \int_1^x \left(\frac{1}{\pi^2} \frac{\log^3 t}{t} + \frac{r(t)}{t^2} \right) dt = \frac{1}{4\pi^2} \log^4 x + R(x),\end{aligned}$$

where

$$R(x) \ll \int_1^x \frac{\log^2 t}{t} dt = \frac{1}{3} \log^3 x. \quad \square$$

An alternative method [Nath, sect. 7.2] is as follows. Define

$$\mu_S(n) = \begin{cases} \mu(k) & \text{if } n = k^2, \\ 0 & \text{if } n \text{ is not a square.} \end{cases}$$

One shows that $\tau^2 = \mu_S * \tau_4$, either directly or by combining MDIV19 with the easily shown identity $\mu_S * u = \mu^2$. It follows that

$$\sum_{n \leq x} \tau(n)^2 = \sum_{n \leq x} \mu_S(n) T_4\left(\frac{x}{n}\right) = \sum_{k^2 \leq x} \mu(k) T_4\left(\frac{x}{k^2}\right).$$

Now use the estimate (17) for $T_4(x)$ (instead of $M_2(x)$), together with $\sum_{k \leq y} [\mu(k)/k^2] = 6/\pi^2 + O(1/y)$, to deduce (28).

This method (more readily than the first one) can be refined to give a second-level estimate. Using the more accurate estimate (25) for $T_4(x)$, together with the identity

$$\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^2} = \frac{\zeta'(2)}{\zeta(2)^2},$$

one finds that

$$\sum_{n \leq x} \tau(n)^2 = \frac{1}{\pi^2} x \log^3 x + C_2 x \log^2 x + O(x \log x),$$

where

$$C_2 = \frac{4\gamma - 1}{2\zeta(2)} - \frac{\zeta'(2)}{\zeta(2)^2}.$$

References

- [HWr] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press (1979).
- [Jam] G. J. O. Jameson, *The Prime Number Theorem*, Cambridge Univ. Press (2003).
- [Nath] Melvyn B. Nathanson, *Elementary Methods in Number Theory*, Springer (2000).
- [Ten] Gerald Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press (1995).