

## Logsine integrals

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The basic “logsine” integrals are:

$$\int_0^{\pi/2} \log \sin \theta \, d\theta = \int_0^{\pi/2} \log \cos \theta \, d\theta = -\frac{\pi}{2} \log 2, \quad (1)$$

$$\int_0^{\pi/2} \log(2 \sin \theta) \, d\theta = \int_0^{\pi/2} \log(2 \cos \theta) \, d\theta = 0. \quad (2)$$

The equivalence of (1) and (2) is obvious. To prove (1) (following [BM, p. 246]), denote the first integral by  $I$ . The substitutions  $\theta = \frac{\pi}{2} - \phi$  and  $\theta = \pi - \phi$  give

$$\int_0^{\pi/2} \log \cos \theta \, d\theta = \int_{\pi/2}^{\pi} \log \sin \theta \, d\theta = I.$$

Hence  $2I = \int_0^{\pi} \log \sin \theta \, d\theta$ . Now substituting  $\theta = 2\phi$  and using  $\sin 2\phi = 2 \sin \phi \cos \phi$ , we have

$$\begin{aligned} 2I &= 2 \int_0^{\pi/2} \log \sin 2\phi \, d\phi \\ &= 2 \int_0^{\pi/2} (\log \sin \phi + \log \cos \phi + \log 2) \, d\phi \\ &= 4I + \pi \log 2, \end{aligned}$$

hence (1). Of course, we have also established

$$\int_0^{\pi} \log \sin \theta \, d\theta = -\pi \log 2, \quad \int_0^{\pi} \log(2 \sin \theta) \, d\theta = 0. \quad (3)$$

Also (without knowing the value of  $I$ ), it is clear that

$$\int_0^{\pi/2} \log \tan \theta \, d\theta = 0. \quad (4)$$

An apparently instant alternative proof of (2) is by termwise integration of the series

$$\log |2 \sin \theta| = - \sum_{n=1}^{\infty} \frac{1}{n} \cos 2n\theta \quad (5)$$

(valid for  $\theta \neq 2k\pi$ ), which can be derived from the series  $\log(1 - e^{2i\theta}) = - \sum_{n=1}^{\infty} \frac{1}{n} e^{2in\theta}$  together with  $|1 - e^{2i\theta}| = 2 \sin \theta$ . However, justification of termwise integration is not entirely trivial.

Next, we state two results obtained by considering  $\pi - \theta$ . First, we show

$$\int_0^\pi \theta \log(2 \sin \theta) d\theta = 0, \quad \int_0^\pi \theta \log \sin \theta d\theta = -\frac{\pi^2}{2} \log 2. \quad (6)$$

Again it is clear that the two statements are equivalent. Denote the first integral by  $I$  (we will repeatedly re-use this notation). Substituting  $\theta = \pi - \phi$  and applying (3), we see that

$$I = \int_0^\pi (\pi - \phi) \log(2 \sin \phi) d\phi = 0 - I,$$

hence  $I = 0$ . Secondly, observe that since  $\cos(\pi - \theta) = -\cos \theta$ , we have

$$\int_0^\pi \log(1 + \cos \theta) d\theta = \int_0^\pi \log(1 - \cos \theta) d\theta.$$

Denote this integral by  $J$ . Taking the average of the two expressions, we have

$$J = \frac{1}{2} \int_0^\pi \log(1 - \cos^2 \theta) d\theta = \int_0^\pi \log \sin \theta d\theta = -\pi \log 2. \quad (7)$$

Later, we will show how to evaluate (6) and (7) on the interval  $[0, \frac{\pi}{2}]$ .

We now list a number of integrals derived from (1) by substitution or integration by parts. Substituting  $x = \sin \theta$  in the following integral, we obtain:

$$\int_0^1 \frac{\log x}{(1 - x^2)^{1/2}} dx = \int_0^{\pi/2} \log \sin \theta d\theta = -\frac{\pi}{2} \log 2. \quad (8)$$

Similarly, the substitution  $x = \tan \theta$  gives

$$\int_0^\infty \frac{\log(1 + x^2)}{1 + x^2} dx = \int_0^{\pi/2} \log \sec^2 \theta d\theta = -2 \int_0^{\pi/2} \log \cos \theta d\theta = \pi \log 2. \quad (9)$$

Now integrate by parts in (1). Since  $\frac{d}{d\theta} \log \sin \theta = \cot \theta$ , we obtain

$$\int_0^{\pi/2} \log \sin \theta d\theta = \left[ \theta \log \sin \theta \right]_0^{\pi/2} - \int_0^{\pi/2} \theta \cot \theta d\theta = - \int_0^{\pi/2} \theta \cot \theta d\theta,$$

hence

$$\int_0^{\pi/2} \theta \cot \theta d\theta = \frac{\pi}{2} \log 2. \quad (10)$$

The substitution  $x = \sin \theta$  now gives

$$\int_0^1 \frac{\sin^{-1} x}{x} dx = \int_0^{\pi/2} \frac{\theta}{\sin \theta} \cos \theta d\theta = \frac{\pi}{2} \log 2. \quad (11)$$

Integrating by parts again in (10) and using  $\frac{d}{d\theta} \cot \theta = -1/\sin^2 \theta$ , we obtain

$$\frac{\pi}{2} \log 2 = \left[ \frac{1}{2} \theta^2 \cot \theta \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\frac{1}{2} \theta^2}{\sin^2 \theta} d\theta.$$

The first term is zero, hence

$$\int_0^{\pi/2} \frac{\theta^2}{\sin^2 \theta} d\theta = \pi \log 2. \quad (12)$$

*Integrals evaluated using Catalan's constant*

*Catalan's constant*, named after E. C. Catalan (1814–1894) and usually denoted by  $G$ , is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots$$

The numerical value is  $G \approx 0.9159656$ . It is not known whether  $G$  is irrational: this remains a stubbornly unsolved problem.

By termwise integration of the series

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1},$$

we obtain at once

$$\int_0^1 \frac{\tan^{-1} x}{x} dx = G, \quad (13)$$

Termwise integration (for those who care) is easily justified. Write

$$s_{2n}(x) = \sum_{r=0}^n (-1)^r \frac{x^{2r}}{2r+1}.$$

Since the series is alternating, with terms decreasing in magnitude, we have  $\tan^{-1} x/x = s_{2n}(x) + r_{2n}(x)$ , where  $|r_{2n}(x)| \leq x^{2n+2}/(2n+3)$ , so that  $\int_0^1 r_{2n}(x) dx \rightarrow 0$  as  $n \rightarrow \infty$ .

Now integrating by parts in (13), we obtain

$$G = \left[ \tan^{-1} x \log x \right]_0^1 - \int_0^1 \frac{\log x}{1+x^2} dx.$$

The first term is zero, since  $\tan^{-1} x \log x \sim x \log x \rightarrow 0$  as  $x \rightarrow 0^+$ , hence

$$\int_0^1 \frac{\log x}{1+x^2} dx = -G. \quad (14)$$

Now substituting  $x = \tan \theta$  in (14), we obtain the version relevant to logsine integrals:

$$-G = \int_0^{\pi/4} \frac{\log \tan \theta}{\sec^2 \theta} \sec^2 \theta d\theta = \int_0^{\pi/4} \log \tan \theta d\theta. \quad (15)$$

The substitution  $x = \tan \theta$  in (13), or integration by parts in (15), gives

$$\int_0^{\pi/4} \frac{\theta}{\sin \theta \cos \theta} d\theta = G, \quad \int_0^{\pi/2} \frac{\theta}{\sin \theta} d\theta = 2G. \quad (16)$$

Numerous further integrals can be expressed in terms of  $G$ : see [Br] and [JL].

Returning to logsine integrals, let

$$I_S = \int_0^{\pi/4} \log \sin \theta \, d\theta, \quad I_C = \int_0^{\pi/4} \log \cos \theta \, d\theta.$$

Substituting  $\theta = \frac{\pi}{2} - \phi$ , we have  $I_C = \int_{\pi/4}^{\pi/2} \log \sin \theta \, d\theta$ . So by (1),  $I_S + I_C = \int_0^{\pi/2} \log \sin \theta \, d\theta = -\frac{\pi}{2} \log 2$ . Meanwhile by (15),  $I_S - I_C = \int_0^{\pi/4} \log \tan \theta \, d\theta = -G$ . So we conclude

$$I_S = -\frac{1}{2}G - \frac{\pi}{4} \log 2, \quad I_C = \frac{1}{2}G - \frac{\pi}{4} \log 2. \quad (17)$$

Again there is a neat restatement:

$$\int_0^{\pi/4} \log(2 \sin \theta) \, d\theta = -\frac{1}{2}G, \quad \int_0^{\pi/4} \log(2 \cos \theta) \, d\theta = \frac{1}{2}G.$$

However, (17) will be more useful in the ensuing applications. As with (1), a number of integrals can be derived from it by substitution or parts.

Corresponding to (9), the substitution  $x = \tan \theta$  gives

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2} \, dx = \int_0^{\pi/4} \log \sec^2 \theta \, d\theta = -2 \int_0^{\pi/4} \log \cos \theta \, d\theta = \frac{\pi}{2} \log 2 - G. \quad (18)$$

With (9), this gives

$$\int_1^\infty \frac{\log(1+x^2)}{1+x^2} \, dx = \frac{\pi}{2} \log 2 + G.$$

Corresponding to (10), integration by parts in (17) gives

$$\begin{aligned} \int_0^{\pi/4} \log \cos \theta \, d\theta &= \left[ \theta \log \cos \theta \right]_0^{\pi/4} + \int_0^{\pi/4} \theta \frac{\sin \theta}{\cos \theta} \, d\theta \\ &= -\frac{\pi}{8} \log 2 + \int_0^{\pi/4} \theta \tan \theta \, d\theta, \end{aligned}$$

so that

$$\int_0^{\pi/4} \theta \tan \theta \, d\theta = \frac{1}{2}G - \frac{\pi}{8} \log 2, \quad (19)$$

Similarly,  $\int_0^{\pi/4} \theta \cot \theta \, d\theta = \frac{1}{2}G + \frac{\pi}{8} \log 2$ . Corresponding to (12), with  $\cos \theta$  instead of  $\sin \theta$ , we deduce

$$\int_0^{\pi/4} \theta^2 \sec^2 \theta \, d\theta = \left[ \theta^2 \tan \theta \right]_0^{\pi/4} - 2 \int_0^{\pi/4} \theta \tan \theta \, d\theta = \frac{\pi^2}{16} - G + \frac{\pi}{4} \log 2. \quad (20)$$

The substitution  $x = \tan \theta$  gives  $\int_0^1 (\tan^{-1} x)^2 \, dx = \int_0^{\pi/4} \theta^2 \sec^2 \theta \, d\theta$ , so we can state also

$$\int_0^1 (\tan^{-1} x)^2 \, dx = \frac{\pi^2}{16} - G + \frac{\pi}{4} \log 2. \quad (21)$$

Readers who are so inclined will easily be able to formulate the statements analogous to (8), (11) and (12).

Next, write

$$I = \int_0^{\pi/2} \log(1 + \sin \theta) d\theta = \int_0^{\pi/2} \log(1 + \cos \theta) d\theta.$$

By (17) and the identity  $1 + \cos \theta = 2 \cos^2 \frac{1}{2}\theta$ , we obtain

$$\begin{aligned} I &= \int_0^{\pi/2} (\log 2 + 2 \log \cos \frac{1}{2}\theta) d\theta \\ &= \frac{\pi}{2} \log 2 + 4 \int_0^{\pi/4} \log \cos \phi d\phi \\ &= \frac{\pi}{2} \log 2 + 2G - \pi \log 2 \\ &= 2G - \frac{\pi}{2} \log 2. \end{aligned} \tag{22}$$

Combining (22) and (1), we have the pleasingly simple result

$$\int_0^{\pi/2} \log(1 + \operatorname{cosec} \theta) d\theta = \int_0^{\pi/2} (\log(1 + \sin \theta) - \log \sin \theta) d\theta = 2G, \tag{23}$$

and of course the same applies with  $\operatorname{cosec} \theta$  replaced by  $\sec \theta$ .

Writing  $(1 + \sin \theta)(1 - \sin \theta) = \cos^2 \theta$ , we deduce further

$$\begin{aligned} \int_0^{\pi/2} \log(1 - \sin \theta) d\theta &= 2 \int_0^{\pi/2} \log \cos \theta d\theta - \int_0^{\pi/2} \log(1 + \sin \theta) d\theta \\ &= -2G - \frac{\pi}{2} \log 2. \end{aligned} \tag{24}$$

Integrating by parts, we have

$$\int_0^{\pi/2} \log(1 + \sin \theta) d\theta = \left[ \theta \log(1 + \sin \theta) \right]_0^{\pi/2} - J = \frac{\pi}{2} \log 2 - J,$$

where

$$J = \int_0^{\pi/2} \frac{\theta \cos \theta}{1 + \sin \theta} d\theta,$$

hence

$$J = \pi \log 2 - 2G. \tag{25}$$

Furthermore, the substitution  $x = \sin \theta$  gives

$$\int_0^1 \frac{\sin^{-1} x}{1 + x} dx = J. \tag{26}$$

A further deduction from (17) is derived using the identity  $\cos \theta + \sin \theta = \sqrt{2} \cos(\theta - \frac{\pi}{4})$ :

$$\begin{aligned}
\int_0^{\pi/2} \log(\cos \theta + \sin \theta) d\theta &= \int_0^{\pi/2} \left( \frac{1}{2} \log 2 + \log \cos(\theta - \frac{\pi}{4}) \right) d\theta \\
&= \frac{\pi}{4} \log 2 + \int_{-\pi/4}^{\pi/4} \log \cos \phi d\phi \\
&= \frac{\pi}{4} \log 2 + G - \frac{\pi}{2} \log 2 \\
&= G - \frac{\pi}{4} \log 2.
\end{aligned} \tag{27}$$

Alternatively, (27) can be derived from (22), using the identity  $(\cos \theta + \sin \theta)^2 = 1 + \sin 2\theta$ .

By (27) and (1),

$$\begin{aligned}
\int_0^{\pi/2} \log(1 + \tan \theta) d\theta &= \int_0^{\pi/2} \left( \log(\cos \theta + \sin \theta) - \log \cos \theta \right) d\theta \\
&= G - \frac{\pi}{4} \log 2 + \frac{\pi}{2} \log 2 \\
&= G + \frac{\pi}{4} \log 2,
\end{aligned} \tag{28}$$

and hence also, with the substitution  $x = \tan \theta$ ,

$$\int_0^{\infty} \frac{\log(1+x)}{1+x^2} dx = G + \frac{\pi}{4} \log 2, \tag{29}$$

which is proved by a longer method in [Br].

Of course,  $\tan \theta$  can be replaced by  $\cot \theta$  in (28). With (22) and (23), this means that we have obtained the values of  $\int_0^{\pi/2} \log[1 + f(\theta)] d\theta$  for all six trigonometric functions.

Yet further integrals can be derived by integrating by parts in (27) and (28) (rather better with  $\cot \theta$ ): we leave it to the reader to explore this..

### *Integrals involving $\zeta(3)$*

To obtain a companion to (6) for the interval  $[0, \frac{\pi}{2}]$  we allow termwise integration of the series (5); this method follows [Br, formula (35)]. Write

$$\lambda(3) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \frac{7}{8} \zeta(3).$$

Now

$$\begin{aligned}
\int_0^{\pi/2} \theta \cos 2n\theta d\theta &= \left[ \frac{\theta}{2n} \sin 2n\theta \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{2n} \sin 2n\theta d\theta \\
&= 0 + \left[ \frac{\cos 2n\theta}{4n^2} \right]_0^{\pi/2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4n^2}(\cos n\pi - 1) \\
&= \begin{cases} 0 & (n \text{ even}), \\ -\frac{1}{2n^2} & (n \text{ odd}). \end{cases}
\end{aligned}$$

Assuming termwise integration valid in (5), we deduce

$$\int_0^{\pi/2} \theta \log(2 \sin \theta) d\theta = \sum_{n=0}^{\infty} \frac{1}{2(2n+1)^3} = \frac{1}{2}\lambda(3). \quad (30)$$

Substituting  $\theta = \frac{\pi}{2} - \phi$  and applying (2), we deduce further

$$\int_0^{\pi/2} \theta \log(2 \cos \theta) d\theta = \int_0^{\pi/2} \left(\frac{\pi}{2} - \phi\right) \log(2 \sin \phi) d\phi = -\frac{1}{2}\lambda(3). \quad (31)$$

Combining (30) and (31), we have

$$\int_0^{\pi/2} \theta \log \tan \theta d\theta = \lambda(3). \quad (32)$$

Also, we deduce

$$\int_0^{\pi/2} \theta \log \sin \theta d\theta = \frac{1}{2}\lambda(3) - \frac{\pi^2}{8} \log 2, \quad \int_0^{\pi/2} \theta \log \cos \theta d\theta = -\frac{1}{2}\lambda(3) - \frac{\pi^2}{8} \log 2. \quad (33)$$

We mention two derived integrals, corresponding to (10) and (11). Integrating by parts in (33), we obtain

$$\int_0^{\pi/2} \theta \log \sin \theta d\theta = \left[ \frac{1}{2}\theta^2 \log \sin \theta \right]_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \theta^2 \cot \theta d\theta,$$

in which the first term is zero, hence

$$\int_0^{\pi/2} \theta^2 \cot \theta d\theta = \frac{\pi^2}{4} \log 2 - \lambda(3). \quad (34)$$

The substitution  $x = \sin \theta$  now gives

$$\int_0^1 \frac{(\sin^{-1} x)^2}{x} dx = \int_0^{\pi/2} \frac{\theta^2}{\sin \theta} \cos \theta d\theta = \frac{\pi^2}{4} \log 2 - \lambda(3). \quad (35)$$

Next, we evaluate  $\int_0^{\pi/4} \theta \log \tan \theta d\theta$ . Recall from (2) and (17) that  $\int_{\pi/4}^{\pi/2} \log(2 \sin \theta) d\theta = \frac{1}{2}G$ . Using this and (30), we have

$$\begin{aligned}
\int_0^{\pi/4} \theta \log \tan \theta d\theta &= \int_0^{\pi/4} \theta \log(2 \sin \theta) d\theta - \int_0^{\pi/4} \theta \log(2 \cos \theta) d\theta \\
&= \int_0^{\pi/4} \theta \log(2 \sin \theta) d\theta - \int_{\pi/4}^{\pi/2} \left(\frac{\pi}{2} - \phi\right) \log(2 \sin \phi) d\phi \\
&= \int_0^{\pi/2} \theta \log(2 \sin \theta) d\theta - \frac{\pi}{2} \int_{\pi/4}^{\pi/2} \log(2 \sin \theta) d\theta \\
&= \frac{1}{2}\lambda(3) - \frac{1}{4}\pi G. \quad (36)
\end{aligned}$$

Since  $\frac{d}{d\theta} \log \tan \theta = 1/(\sin \theta \cos \theta)$ , we can deduce

$$\int_0^{\pi/4} \frac{\theta^2}{\sin \theta \cos \theta} d\theta = \left[ \theta^2 \log \tan \theta \right]_0^{\pi/4} - \int_0^{\pi/4} 2\theta \log \tan \theta d\theta = \frac{1}{2}\pi G - \lambda(3). \quad (37)$$

Hence also

$$\int_0^{\pi/2} \frac{\theta^2}{\sin \theta} d\theta = \int_0^{\pi/4} \frac{4\phi^2}{2 \sin \phi \cos \phi} 2 d\phi = 2\pi G - 4\lambda(3). \quad (38)$$

Further, the substitution  $x = \tan \theta$  gives

$$\int_0^1 \frac{(\tan^{-1} x)^2}{x} dx = \int_0^{\pi/4} \frac{\theta^2}{\tan \theta} \sec^2 \theta d\theta = \int_0^{\pi/4} \frac{\theta^2}{\sin \theta \cos \theta} d\theta = \frac{1}{2}\pi G - \lambda(3). \quad (39)$$

Although  $\int_0^{\pi/4} \theta \log(2 \sin \theta) d\theta$  appeared in the reasoning for (36), we did not need to know its value. In fact, by a rather more elaborate consideration of the terms arising from the series (5), one can establish

$$\int_0^{\pi/4} \theta \log(2 \sin \theta) d\theta = -\frac{1}{8}\pi G + \frac{5}{16}\lambda(3), \quad (40)$$

$$\int_0^{\pi/4} \theta \log(2 \cos \theta) d\theta = \frac{1}{8}\pi G - \frac{3}{16}\lambda(3). \quad (41)$$

Details are given, for example, in [Ar].

An interesting survey of Euler's results on logsine-type integrals is given in [Lo].

### References

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