Logsine integrals

Notes by G.J.O. Jameson

The basic "logsine" integrals are:

$$\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log \cos\theta \, d\theta = -\frac{\pi}{2} \log 2, \tag{1}$$

$$\int_0^{\pi/2} \log(2\sin\theta) \, d\theta = \int_0^{\pi/2} \log(2\cos\theta) \, d\theta = 0. \tag{2}$$

The equivalence of (1) and (2) is obvious. To prove (1) (following [BM, p. 246]), denote the first integral by I. The substitutions $\theta = \frac{\pi}{2} - \phi$ and $\theta = \pi - \phi$ give

$$\int_0^{\pi/2} \log \cos\theta \, d\theta = \int_{\pi/2}^{\pi} \log \sin\theta \, d\theta = I.$$

Hence $2I = \int_0^{\pi} \log \sin\theta \, d\theta$. Now substituting $\theta = 2\phi$ and using $\sin 2\phi = 2\sin\phi\cos\phi$, we have

$$\begin{aligned} 2I &= 2\int_0^{\pi/2} \log \sin 2\phi \, d\phi \\ &= 2\int_0^{\pi/2} (\log\sin\phi + \log\cos\phi + \log 2) \, d\phi \\ &= 4I + \pi \log 2, \end{aligned}$$

hence (1). Of course, we have also established

$$\int_0^{\pi} \log \sin\theta \, d\theta = -\pi \log 2, \qquad \int_0^{\pi} \log(2\sin\theta) \, d\theta = 0. \tag{3}$$

Also (without knowing the value of I), it is clear that

$$\int_0^{\pi/2} \log \tan\theta \, d\theta = 0. \tag{4}$$

An apparently instant alternative proof of (2) is by termwise integration of the series

$$\log|2\sin\theta| = -\sum_{n=1}^{\infty} \frac{1}{n} \cos 2n\theta \tag{5}$$

(valid for $\theta \neq 2k\pi$), which can be derived from the series $\log(1 - e^{2i\theta}) = -\sum_{n=1}^{\infty} \frac{1}{n} e^{2in\theta}$ together with $|1 - e^{2i\theta}| = 2\sin\theta$. However, justification of termwise integration is not entirely trivial.

Next, we state two results obtained by considering $\pi - \theta$. First, we show

$$\int_0^\pi \theta \log(2\sin\theta)\, d\theta = 0, \qquad \int_0^\pi \theta \log\sin\theta\, d\theta = -\frac{\pi^2}{2}\log 2. \tag{6}$$

Again it is clear that the two statements are equivalent. Denote the first integral by I (we will repeatedly re-use this notation). Substituting $\theta = \pi - \phi$ and applying (3), we see that

$$I = \int_0^\pi (\pi - \phi)\log(2\sin\phi)\, d\phi = 0 - I,$$

hence $I = 0$. Secondly, observe that since $\cos(\pi - \theta) = -\cos\theta$, we have

$$\int_0^\pi \log(1 + \cos\theta)\, d\theta = \int_0^\pi \log(1 - \cos\theta)\, d\theta.$$

Denote this integral by J. Taking the average of the two expressions, we have

$$J = \tfrac{1}{2}\int_0^\pi \log(1 - \cos^2\theta)\, d\theta = \int_0^\pi \log\sin\theta\, d\theta = -\pi\log 2. \tag{7}$$

Later, we will show how to evaluate (6) and (7) on the interval $[0, \frac{\pi}{2}]$.

We now list a number of integrals derived from (1) by substitution or integration by parts. Substituting $x = \sin\theta$ in the following integral, we obtain:

$$\int_0^1 \frac{\log x}{(1 - x^2)^{1/2}}\, dx = \int_0^{\pi/2} \log\sin\theta\, d\theta = -\frac{\pi}{2}\log 2. \tag{8}$$

Similarly, the substitution $x = \tan\theta$ gives

$$\int_0^\infty \frac{\log(1 + x^2)}{1 + x^2}\, dx = \int_0^{\pi/2} \log\sec^2\theta\, d\theta = -2\int_0^{\pi/2} \log\cos\theta\, d\theta = \pi\log 2. \tag{9}$$

Now integrate by parts in (1). Since $\frac{d}{d\theta}\log\sin\theta = \cot\theta$, we obtain

$$\int_0^{\pi/2} \log\sin\theta\, d\theta = \Big[\theta\log\sin\theta\Big]_0^{\pi/2} - \int_0^{\pi/2} \theta\cot\theta\, d\theta = -\int_0^{\pi/2} \theta\cot\theta\, d\theta,$$

hence

$$\int_0^{\pi/2} \theta\cot\theta\, d\theta = \frac{\pi}{2}\log 2. \tag{10}$$

The substitution $x = \sin\theta$ now gives

$$\int_0^1 \frac{\sin^{-1} x}{x}\, dx = \int_0^{\pi/2} \frac{\theta}{\sin\theta}\cos\theta\, d\theta = \frac{\pi}{2}\log 2. \tag{11}$$

Integrating by parts again in (10) and using $\frac{d}{d\theta}\cot\theta = -1/\sin^2\theta$, we obtain

$$\frac{\pi}{2}\log 2 = \Big[\tfrac{1}{2}\theta^2\cot\theta\Big]_0^{\pi/2} + \int_0^{\pi/2} \frac{\frac{1}{2}\theta^2}{\sin^2\theta}\, d\theta.$$

The first term is zero, hence

$$\int_0^{\pi/2} \frac{\theta^2}{\sin^2\theta}\, d\theta = \pi \log 2. \tag{12}$$

Integrals evaluated using Catalan's constant

Catalan's constant, named after E. C. Catalan (1814–1894) and usually denoted by G, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots.$$

The numerical value is $G \approx 0.9159656$. It is not known whether G is irrational: this remains a stubbornly unsolved problem.

By termwise integration of the series

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1},$$

we obtain at once

$$\int_0^1 \frac{\tan^{-1} x}{x}\, dx = G, \tag{13}$$

Termwise integration (for those who care) is easily justified. Write

$$s_{2n}(x) = \sum_{r=0}^{n} (-1)^n \frac{x^{2r}}{2r+1}.$$

Since the series is alternating, with terms decreasing in magnitude, we have $\tan^{-1} x/x = s_{2n}(x) + r_{2n}(x)$, where $|r_{2n}(x)| \le x^{2n+2}/(2n+3)$, so that $\int_0^1 r_{2n}(x)\, dx \to 0$ as $n \to \infty$.

Now integrating by parts in (13), we obtain

$$G = \Big[\tan^{-1} x \log x\Big]_0^1 - \int_0^1 \frac{\log x}{1+x^2}\, dx.$$

The first term is zero, since $\tan^{-1} x \log x \sim x \log x \to 0$ as $x \to 0^+$, hence

$$\int_0^1 \frac{\log x}{1+x^2}\, dx = -G. \tag{14}$$

Now substituting $x = \tan\theta$ in (14), we obtain the version relevant to logsine integrals:

$$-G = \int_0^{\pi/4} \frac{\log\tan\theta}{\sec^2\theta} \sec^2\theta\, d\theta = \int_0^{\pi/4} \log\tan\theta\, d\theta. \tag{15}$$

The substitution $x = \tan\theta$ in (13), or integration by parts in (15), gives

$$\int_0^{\pi/4} \frac{\theta}{\sin\theta\cos\theta}\, d\theta = G, \qquad \int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta = 2G. \tag{16}$$

Numerous further integrals can be expressed in terms of G: see [Br] and [JL].

Returning to logsine integrals, let

$$I_S = \int_0^{\pi/4} \log \sin\theta \, d\theta, \qquad I_C = \int_0^{\pi/4} \log\cos\theta \, d\theta.$$

Substituting $\theta = \frac{\pi}{2} - \phi$, we have $I_C = \int_{\pi/4}^{\pi/2} \log\sin\theta \; d\theta$. So by (1), $I_S + I_C = \int_0^{\pi/2} \log\sin\theta \; d\theta = -\frac{\pi}{2}\log 2$. Meanwhile by (15), $I_S - I_C = \int_0^{\pi/4} \log\tan\theta \; d\theta = -G$. So we conclude

$$I_S = -\tfrac{1}{2}G - \frac{\pi}{4}\log 2, \qquad I_C = \tfrac{1}{2}G - \frac{\pi}{4}\log 2. \tag{17}$$

Again there is a neat restatement:

$$\int_0^{\pi/4} \log(2\sin\theta) \, d\theta = -\tfrac{1}{2}G, \qquad \int_0^{\pi/4} \log(2\cos\theta) \, d\theta = \tfrac{1}{2}G.$$

However, (17) will be more useful in the ensuing applications. As with (1), a number of integrals can be derived from it by substitution or parts.

Corresponding to (9), the substitution $x = \tan\theta$ gives

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2} \, dx = \int_0^{\pi/4} \log\sec^2\theta \, d\theta = -2\int_0^{\pi/4} \log\cos\theta \, d\theta = \frac{\pi}{2}\log 2 - G. \tag{18}$$

With (9), this gives

$$\int_1^{\infty} \frac{\log(1+x^2)}{1+x^2} \, dx = \frac{\pi}{2}\log 2 + G.$$

Corresponding to (10), integration by parts in (17) gives

$$\begin{aligned} \int_0^{\pi/4} \log\cos\theta \, d\theta &= \Big[\theta\log\cos\theta\Big]_0^{\pi/4} + \int_0^{\pi/4} \theta\frac{\sin\theta}{\cos\theta} \, d\theta \\ &= -\frac{\pi}{8}\log 2 + \int_0^{\pi/4} \theta\tan\theta \, d\theta, \end{aligned}$$

so that

$$\int_0^{\pi/4} \theta\tan\theta \, d\theta = \tfrac{1}{2}G - \frac{\pi}{8}\log 2, \tag{19}$$

Similarly, $\int_0^{\pi/4} \theta\cot\theta \, d\theta = \frac{1}{2}G + \frac{\pi}{8}\log 2$. Corresponding to (12), with $\cos\theta$ instead of $\sin\theta$, we deduce

$$\int_0^{\pi/4} \theta^2\sec^2\theta \, d\theta = \Big[\theta^2\tan\theta\Big]_0^{\pi/4} - 2\int_0^{\pi/4} \theta\tan\theta \, d\theta = \frac{\pi^2}{16} - G + \frac{\pi}{4}\log 2. \tag{20}$$

The substitution $x = \tan\theta$ gives $\int_0^1 (\tan^{-1} x)^2 \, dx = \int_0^{\pi/4} \theta^2\sec^2\theta \, d\theta$, so we can state also

$$\int_0^1 (\tan^{-1} x)^2 \, dx = \frac{\pi^2}{16} - G + \frac{\pi}{4}\log 2. \tag{21}$$

Readers who are so inclined will easily be able to formulate the statements analogous to (8), (9) and (12).

Next, write

$$I = \int_0^{\pi/2} \log(1 + \sin\theta)\, d\theta = \int_0^{\pi/2} \log(1 + \cos\theta)\, d\theta.$$

By (17) and the identity $1 + \cos\theta = 2\cos^2 \frac{1}{2}\theta$, we obtain

$$\begin{aligned} I &= \int_0^{\pi/2} (\log 2 + 2 \log\cos \tfrac{1}{2}\theta)\, d\theta \\ &= \frac{\pi}{2}\log 2 + 4 \int_0^{\pi/4} \log\cos\phi\, d\phi \\ &= \frac{\pi}{2}\log 2 + 2G - \pi\log 2 \\ &= 2G - \frac{\pi}{2}\log 2. \end{aligned} \tag{22}$$

Combining (22) and (1), we have the pleasingly simple result

$$\int_0^{\pi/2} \log(1 + \operatorname{cosec}\theta)\, d\theta = \int_0^{\pi/2} \Big(\log(1 + \sin\theta) - \log\sin\theta \Big)\, d\theta = 2G, \tag{23}$$

and of course the same applies with $\operatorname{cosec}\theta$ replaced by $\sec\theta$.

Writing $(1 + \sin\theta)(1 - \sin\theta) = \cos^2\theta$, we deduce further

$$\begin{aligned} \int_0^{\pi/2} \log(1 - \sin\theta)\, d\theta &= 2\int_0^{\pi/2} \log\cos\theta\, d\theta - \int_0^{\pi/2} \log(1 + \sin\theta)\, d\theta \\ &= -2G - \frac{\pi}{2}\log 2. \end{aligned} \tag{24}$$

Integrating by parts, we have

$$\int_0^{\pi/2} \log(1 + \sin\theta)\, d\theta = \Big[\theta \log(1 + \sin\theta)\Big]_0^{\pi/2} - J = \frac{\pi}{2}\log 2 - J,$$

where

$$J = \int_0^{\pi/2} \frac{\theta\cos\theta}{1 + \sin\theta}\, d\theta,$$

hence

$$J = \pi\log 2 - 2G. \tag{25}$$

Furthermore, the substitution $x = \sin\theta$ gives

$$\int_0^1 \frac{\sin^{-1} x}{1 + x}\, dx = J. \tag{26}$$

A further deduction from (17) is derived using the identity $\cos\theta+\sin\theta = \sqrt{2}\cos(\theta-\frac{\pi}{4})$:

$$\begin{aligned}
\int_0^{\pi/2} \log(\cos\theta+\sin\theta)\,d\theta &= \int_0^{\pi/2} \left(\tfrac{1}{2}\log 2 + \log\cos(\theta-\tfrac{\pi}{4})\right)d\theta \\
&= \frac{\pi}{4}\log 2 + \int_{-\pi/4}^{\pi/4} \log\cos\phi\,d\phi \\
&= \frac{\pi}{4}\log 2 + G - \frac{\pi}{2}\log 2 \\
&= G - \frac{\pi}{4}\log 2. \qquad (27)
\end{aligned}$$

Alternatively, (27) can be derived from (22), using the identity $(\cos\theta+\sin\theta)^2 = 1+\sin 2\theta$.

By (27) and (1),

$$\begin{aligned}
\int_0^{\pi/2} \log(1+\tan\theta)\,d\theta &= \int_0^{\pi/2} \left(\log(\cos\theta+\sin\theta) - \log\cos\theta\right)d\theta \\
&= G - \frac{\pi}{4}\log 2 + \frac{\pi}{2}\log 2 \\
&= G + \frac{\pi}{4}\log 2, \qquad (28)
\end{aligned}$$

and hence also, with the substitution $x=\tan\theta$,

$$\int_0^\infty \frac{\log(1+x)}{1+x^2}\,dx = G + \frac{\pi}{4}\log 2, \qquad (29)$$

which is proved by a longer method in [Br].

Of course, $\tan\theta$ can be replaced by $\cot\theta$ in (28). With (22) and (23), this means that we have obtained the values of $\int_0^{\pi/2}\log[1+f(\theta)]\,d\theta$ for all six trigonometric functions.

Yet further integrals can be derived by integrating by parts in (27) and (28) (rather better with $\cot\theta$): we leave it to the reader to explore this..

Integrals involving $\zeta(3)$

To obtain a companion to (6) for the interval $[0,\frac{\pi}{2}]$ we allow termwise integration of the series (5); this method follows [Br, formula (35)]. Write

$$\lambda(3) = \sum_{n=0}^\infty \frac{1}{(2n+1)^3} = \tfrac{7}{8}\zeta(3).$$

Now

$$\begin{aligned}
\int_0^{\pi/2} \theta\cos 2n\theta\,d\theta &= \left[\frac{\theta}{2n}\sin 2n\theta\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{1}{2n}\sin 2n\theta\,d\theta \\
&= 0 + \left[\frac{\cos 2n\theta}{4n^2}\right]_0^{\pi/2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4n^2}(\cos n\pi - 1) \\
&= \begin{cases} 0 & (n \text{ even}), \\ -\frac{1}{2n^2} & (n \text{ odd}). \end{cases}
\end{aligned}
$$

Assuming termwise integration valid in (5), we deduce

$$
\int_0^{\pi/2} \theta \log(2\sin\theta)\, d\theta = \sum_{n=0}^{\infty} \frac{1}{2(2n+1)^3} = \tfrac{1}{2}\lambda(3). \tag{30}
$$

Substituting $\theta = \frac{\pi}{2} - \phi$ and applying (2), we deduce further

$$
\int_0^{\pi/2} \theta \log(2\cos\theta)\, d\theta = \int_0^{\pi/2} \left(\frac{\pi}{2} - \phi\right) \log(2\sin\phi)\, d\phi = -\tfrac{1}{2}\lambda(3). \tag{31}
$$

Combining (30) and (31), we have

$$
\int_0^{\pi/2} \theta \log\tan\theta\, d\theta = \lambda(3). \tag{32}
$$

Also, we deduce

$$
\int_0^{\pi/2} \theta \log\sin\theta\, d\theta = \tfrac{1}{2}\lambda(3) - \frac{\pi^2}{8}\log 2, \qquad \int_0^{\pi/2} \theta \log\cos\theta\, d\theta = -\tfrac{1}{2}\lambda(3) - \frac{\pi^2}{8}\log 2. \tag{33}
$$

We mention two derived integrals, corresponding to (10) and (11). Integrating by parts in (33), we obtain

$$
\int_0^{\pi/2} \theta \log\sin\theta\, d\theta = \left[\tfrac{1}{2}\theta^2 \log\sin\theta\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \theta^2 \cot\theta\, d\theta,
$$

in which the first term is zero, hence

$$
\int_0^{\pi/2} \theta^2 \cot\theta\, d\theta = \frac{\pi^2}{4}\log 2 - \lambda(3). \tag{34}
$$

The substitution $x = \sin\theta$ now gives

$$
\int_0^1 \frac{(\sin^{-1} x)^2}{x}\, dx = \int_0^{\pi/2} \frac{\theta^2}{\sin\theta} \cos\theta\, d\theta = \frac{\pi^2}{4}\log 2 - \lambda(3). \tag{35}
$$

Next, we evaluate $\int_0^{\pi/4} \theta \log\tan\theta\, d\theta$. Recall from (2) and (17) that $\int_{\pi/4}^{\pi/2} \log(2\sin\theta)\, d\theta = \frac{1}{2}G$. Using this and (30), we have

$$
\begin{aligned}
\int_0^{\pi/4} \theta \log\tan\theta\, d\theta &= \int_0^{\pi/4} \theta \log(2\sin\theta)\, d\theta - \int_0^{\pi/4} \theta \log(2\cos\theta)\, d\theta \\
&= \int_0^{\pi/4} \theta \log(2\sin\theta)\, d\theta - \int_{\pi/4}^{\pi/2} \left(\frac{\pi}{2} - \phi\right) \log(2\sin\phi)\, d\phi \\
&= \int_0^{\pi/4} \theta \log(2\sin\theta)\, d\theta - \frac{\pi}{2}\int_{\pi/4}^{\pi/2} \log(2\sin\theta)\, d\theta \\
&= \tfrac{1}{2}\lambda(3) - \tfrac{1}{4}\pi G. \qquad (36)
\end{aligned}
$$

Since $\frac{d}{d\theta} \log \tan \theta = 1/(\sin \theta \cos \theta)$, we can deduce

$$\int_0^{\pi/4} \frac{\theta^2}{\sin \theta \cos \theta} \, d\theta = \Big[\theta^2 \log \tan \theta \Big]_0^{\pi/4} - \int_0^{\pi/4} 2\theta \log \tan \theta \, d\theta = \tfrac{1}{2}\pi G - \lambda(3). \tag{37}$$

Hence also

$$\int_0^{\pi/2} \frac{\theta^2}{\sin \theta} \, d\theta = \int_0^{\pi/4} \frac{4\phi^2}{2 \sin \phi \cos \phi} 2 \, d\phi = 2\pi G - 4\lambda(3). \tag{38}$$

Further, the substitution $x = \tan \theta$ gives

$$\int_0^1 \frac{(\tan^{-1} x)^2}{x} \, dx = \int_0^{\pi/4} \frac{\theta^2}{\tan \theta} \sec^2 \theta \, d\theta = \int_0^{\pi/4} \frac{\theta^2}{\sin \theta \cos \theta} \, d\theta = \tfrac{1}{2}\pi G - \lambda(3). \tag{39}$$

Although $\int_0^{\pi/4} \theta \log(2 \sin \theta) \, d\theta$ appeared in the reasoning for (36), we did not need to know its value. In fact, by a rather more elaborate consideration of the terms arising from the series (5), one can establish

$$\int_0^{\pi/4} \theta \log(2 \sin \theta) \, d\theta = -\tfrac{1}{8}\pi G + \tfrac{5}{16}\lambda(3), \tag{40}$$

$$\int_0^{\pi/4} \theta \log(2 \cos \theta) \, d\theta = \tfrac{1}{8}\pi G - \tfrac{3}{16}\lambda(3). \tag{41}$$

Details are given, for example, in [Ar].

An interesting survey of Euler's results on logsine-type integrals is given in [Lo].

References

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