The logarithmic mean

Notes by G.J.O. Jameson

The logarithmic mean of two distinct positive numbers $a, b$ is

$$L(a, b) = \frac{b - a}{\log b - \log a}.$$ 

An immediate observation is:

**PROPOSITION 1.** If $a < b$, then $a < L(a, b) < b$.

*Proof.* Since $\frac{d}{dx} \log x = \frac{1}{x}$, the mean-value theorem gives

$$\frac{1}{L(a, b)} = \frac{\log b - \log a}{b - a} = \frac{1}{c},$$

where $a < c < b$, hence $L(a, b) = c$. $\square$

Hence $L(a, b) \to a$ as $b \to a$, and it is natural to complete the definition by putting $L(a, a) = a$. Note also that $L\left(1, \frac{b}{a}\right) = \left(\frac{b}{a} - 1\right) / \log b$.

Clearly, as with other means, we have $L(\lambda a, \lambda b) = \lambda L(a, b)$ for $\lambda > 0$. In particular, $L(a, b) = aL(1, \frac{b}{a}) = abL(\frac{1}{a}, \frac{1}{b})$.

We mention three integral expressions. First, since $\frac{d}{dt}(a^{1-t}b^t) = a^{1-t}b^t(\log b - \log a)$, we have

$$\int_0^1 a^{1-t}b^t \, dt = \frac{1}{\log b - \log a} \left[ a^{1-t}b^t \right]_0^1 = \frac{b - a}{\log b - \log a} = L(a, b).$$

Next,

$$\int_0^1 \frac{1}{a(1-t) + bt} \, dt = \frac{1}{b-a} \left[ \log[a(1-t) + bt] \right]_0^1 = \frac{1}{L(a, b)}.$$ (2)

Third,

$$\int_0^\infty \frac{1}{(t+a)(t+b)} \, dt = \frac{1}{b-a} \int_0^\infty \left( \frac{1}{t+a} - \frac{1}{t+b} \right) \, dt$$

$$= \frac{1}{b-a} \left[ \log \frac{t+a}{t+b} \right]_0^\infty$$

$$= \frac{\log b - \log a}{b-a}$$

$$= \frac{1}{L(a, b)}.$$ (3)

We now compare $L(a, b)$ with other means, in particular the arithmetic mean $A(a, b) = \frac{1}{2}(a + b)$, the geometric mean $G(a, b) = (ab)^{1/2}$ and the power mean $M_p(a, b) = \left[ \frac{1}{2}(a^p + b^p) \right]^{1/p}$ for suitable $p$. The basic result is:
THEOREM 2. We have

\[ G(a, b) \leq L(a, b) \leq A(a, b). \]  

(4)

Note that the special case \[ a = 1 \] (with \[ b \] replaced by \[ x \]) says

\[ x^{1/2} \leq \frac{x - 1}{\log x} \leq \frac{1}{2}(x + 1). \]  

(5)

Conversely, the substitution \( x = b/a \) transforms (5) into (4), so in fact the statements are equivalent. In this way, (4) reduces to inequalities in terms of a single variable \( x \).

Numerous proofs of Theorem 2 have been given. We present four of them. Of these, Proof 1 is the one that signals the method that will be used for further inequalities later.

Henceforth we write just \( L \) for \( L(a, b) \), and similarly \( A \) and \( G \).

Proof 1: substitution [JM]. In (5), substitute \( x = e^{2y} \): the statement becomes

\[ e^y \leq \frac{e^{2y} - 1}{2y} \leq \frac{1}{2}(e^{2y} + 1). \]

After division by \( e^y \), this says

\[ 1 \leq \frac{\sinh y}{y} \leq \cosh y. \]  

(6)

So (4) is equivalent to (6), which follows at once from the series expansions

\[ \frac{\sinh y}{y} = 1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots, \]  

(7)

\[ \cosh y = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots. \]  

(8)

Proof 2: mean-value theorem ([Mer], [Mit, p. 273], with slight variations). It is clearly enough to prove (5) for \( x > 1 \). Rewrite it as

\[ \frac{2(x - 1)}{x + 1} \leq \log x \leq x^{1/2} - x^{-1/2}. \]

For the left-hand inequality, write

\[ f(x) = \log x - \frac{2(x - 1)}{x + 1} = \log x - 2 + \frac{4}{x + 1}. \]

Then \( f(1) = 0 \) and

\[ f'(x) = \frac{1}{x} - \frac{4}{(x + 1)^2} = \frac{(x + 1)^2 - 4x}{x(x + 1)^2} = \frac{(x - 1)^2}{x(x + 1)^2} > 0 \]
for $x > 1$. So $f(x) > 0$ for $x > 1$, by the mean-value theorem.

For the right-hand inequality, substitute $x = y^2$ and write $g(y) = y - \frac{1}{y} - 2 \log y$. Then $g(1) = 0$ and

$$g'(y) = 1 + \frac{1}{y^2} - \frac{2}{y} = \left(1 - \frac{1}{y}\right)^2 \geq 0. \quad \Box$$

**Proof 3: convexity** [Bu]. Note that

$$\int_{\log a}^{\log b} e^x \, dx = b - a.$$ 

Now $e^x$ is convex, and the integral of a convex function is not greater than its estimate by the trapezium rule: $\int_a^b f \leq \frac{1}{2}(b - a)[f(a) + f(b)]$. This is geometrically obvious, and easily proved analytically. In this case, it says $b - a \leq \frac{1}{2}(a + b)(\log b - \log a)$, which equates to $L \leq A$.

Also, the integral of a convex function is not less than the area below the tangent at the mid-point $c$, in other words, $\int_a^b f \geq (b - a)f(c)$. In our case, the value of $e^x$ at the mid-point is $(ab)^{1/2}$, so this says that $b - a \geq (ab)^{1/2}(\log b - \log a)$, or $L \geq G$.

A slight variant of this method, using $1/x$ instead of $e^x$, is given in [Br].

**Proof 4, using the integral (3)** [Bh]. By the basic inequality $G \leq A$,

$$(t + a)(t + b) = t^2 + (a + b)t + ab \leq t^2 + (a + b)t + \frac{1}{4}(a + b)^2 = (t + A)^2,$$

also

$$(t + a)(t + b) \geq t^2 + 2(ab)^{1/2}t + ab = (t + G)^2.$$ 

Since

$$\int_0^{\infty} \frac{1}{(t + c)^2} \, dt = \frac{1}{c},$$

we deduce from (3) that

$$\frac{1}{A} \leq \frac{1}{L} \leq \frac{1}{G}. \quad \Box$$

We now establish some more intricate inequalities comparing $L$ with combinations of $A$ and $G$. They were all obtained by quite complicated methods in earlier papers, and a somewhat simpler method (using Cauchy’s mean-value theorem) in [Mer]. Here we present a much simpler method, along the lines of Proof 1 above. The technique is well established for the purpose of proving inequalities for means: e.g. [Alz]. For the results given here, it was indicated in [Zhu], without full details, and set out more fully in [JM].
THEOREM 3 [Ca]. We have

\[
L \leq \frac{2}{3}G + \frac{1}{3}A. \tag{9}
\]

Further, \( \frac{1}{3} \) is the smallest \( p \) for which \( L \leq (1 - p)G + pA \) for all choices of \( a, b \).

Proof: We do not need to know the factor \( \frac{1}{3} \) in advance: we can let it emerge from the reasoning. So consider (9) in the form \( L \leq (1 - p)G + pA \), where \( p \) is to be found. As before, it is sufficient to prove the case \( a = 1 \), in other words (with \( b \) replaced by \( x \))

\[
\frac{x - 1}{\log x} \leq (1 - p)x^{1/2} + \frac{p}{2}(x + 1).
\]

The substitution \( x = e^{2y} \) transforms this into

\[
\frac{e^{2y} - 1}{2y} \leq (1 - p)e^y + \frac{p}{2}(e^{2y} + 1),
\]

equivalently,

\[
\frac{\sinh y}{y} \leq (1 - p) + p \cosh y. \tag{10}
\]

We wish this to hold for all \( y > 0 \). Compare power series. We have

\[
(1 - p) + p \cosh y = 1 + p \left( \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots \right).
\]

Statement (10) will be assured if the coefficients are no smaller than the corresponding ones in (7). The \( y^2 \) term requires that \( \frac{p}{2} \geq \frac{1}{6} \), so \( p \geq \frac{1}{3} \). The \( y^{2n} \) term requires \( p/[(2n)!] \geq 1/[(2n + 1)!] \), or \( p \geq 1/(2n + 1) \). So (10) holds with \( p = \frac{1}{3} \).

At the same time, it is clear that (10) will fail for some \( y \) if \( p < \frac{1}{3} \), since \( (1 - p) + p \cosh y = 1 + \frac{p}{2}y^2 + O(y^4) \), while \( (\sinh y/y) = 1 + \frac{1}{6}y^2 + O(y^4) \). \( \square \)

THEOREM 4 [LSh]. We have

\[
L \geq G^{2/3}A^{1/3}. \tag{11}
\]

Further, \( \frac{1}{3} \) is the largest \( p \) for which \( L \geq G^{1-p}A^p \) for all choices of \( a, b \).

Proof. As before, (11) is equivalent to the case \( a = 1 \), that is,

\[
\frac{x - 1}{\log x} \geq x^{1/3} \left[ \frac{1}{2}(x + 1) \right]^{1/3}.
\]

The substitution \( x = e^{2y} \) transforms this into

\[
\frac{e^{2y} - 1}{2y} \geq e^{2y/3} \left( \frac{1}{2}(e^{2y} + 1) \right)^{1/3}.
\]
After division by $e^y$, this is equivalent to

$$\frac{\sinh y}{y} \geq (\cosh y)^{1/3},$$

hence to

$$\left(\frac{\sinh y}{y}\right)^3 \geq \cosh y.$$ (12)

This time, comparison of the coefficients will cost us a little more work. Note that

$$(\sinh y)^3 = \frac{1}{8}(e^y - e^{-y})^3 = \frac{1}{4}\sinh 3y - \frac{3}{4}\sinh y = \sum_{n=0}^{\infty} c_{2n}y^{2n+3},$$

where

$$c_{2n} = \frac{3^{2n+3} - 3}{4(2n + 3)!}.$$ 

We need to know that $c_{2n} \geq 1/[(2n)!]$ for each $n \geq 0$. This equates to saying that $u_n \geq v_n$, where

$$u_n = 3^{2n+3} - 3, \quad v_n = 4(2n + 1)(2n + 2)(2n + 3).$$

To start, we have $u_0 = v_0 = 24$ and $u_1 = v_1 = 240$. For all $n \geq 1$, it is clear that $u_{n+1}/u_n > 9$, while

$$\frac{v_{n+1}}{v_n} = \frac{(2n + 4)(2n + 5)}{(2n + 1)(2n + 2)} \leq \frac{6 \times 7}{3 \times 4} = \frac{7}{2},$$

so indeed $u_n > v_n$, as required.

If $q < 3$, then by (7) and the binomial series, we have $(\sinh y/y)^q = 1 + \frac{2}{3}y^2 + O(y^4) < \cosh y$ for sufficiently small $y$, hence if $p > \frac{1}{3}$, then $\sinh y/y < (\cosh y)^p$ for such $y$, hence $L < G^{1-p}A^p$. (Alternately, as observed in [Jam], this follows from (9) and the fact that if $p > \frac{1}{3}$, then $x^py^{1-p} > \frac{1}{3}x + \frac{2}{3}y$ for $\frac{2}{3}$ just larger than 1.)

We mention two alternative proofs of (12) (which the reader, of course, may ignore).

**Alternative proof 1.** Consider

$$\left(\frac{\sinh y}{y}\right)^3 = \left(1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots\right)^3.$$

The constant term is 1, and the $y^2$ term is $3y^2/3! = \frac{1}{2}y^2$. Using only the term $y^2/3!$, we see that the coefficient of $y^4$ is greater than $\frac{3}{2} = \frac{1}{2}$; this is larger than the $1/4!$ occurring in the series for $\cosh y$. Now consider $y^{2n}$, where $2n \geq 6$. The combination

$$\frac{y^2}{3!} \frac{y^{2n-2}}{(2n-1)!} = \frac{y^{2n}}{6(2n-1)!}$$
occurs six times, corresponding to ways of selecting these two terms and 1 from the three brackets. So the coefficient of $y^{2n}$ is at least $1/[(2n - 1)!]$, greater than $1/[(2n)!]$.

Alternative proof 2. The following method is given in [Mit, p. 270], where (12) is stated without any mention of the equivalence with (11). Let $f(y) = y - \sinh y (\cosh y)^{-1/3}$. After what seems like fortuitous cancellation, one finds that $f''(y) = -\frac{4}{9}(\sinh y)^3(\cosh y)^{-7/3}$, so that $f''(y) < 0$ for $y > 0$. By two applications of the mean-value theorem, one deduces that $f(y) < 0$ for $y > 0$.

THEOREM 5 [Lin], [Bu]. We have $L \leq M_{1/3}$.

Proof. After replacing $a$ and $b$ by 1 and $x$, and substituting $x = e^{2y}$, we see that the statement is equivalent to

$$\frac{\sinh y}{y} \leq (\cosh \frac{1}{3}y)^3 = \frac{1}{4} \cosh y + \frac{3}{4} \cosh \frac{1}{3}y.$$  \hfill (13)

The series expression for the right-hand side is $\sum_{n=0}^{\infty} d_{2n} y^{2n}$, where

$$d_{2n} = \frac{1}{(2n)!} \left( \frac{1}{4} + \frac{3}{4.3^{2n}} \right).$$

Comparing with (7), we need to know that $d_{2n} \geq 1/[(2n + 1)!]$ for each $n$. This equates to

$$\frac{1}{4} + \frac{3}{4.3^{2n}} \geq \frac{1}{2n + 1}.$$  \hfill (14)

This is much easier than the previous proof! When $n = 0$, both sides are 1, and when $n = 1$, both sides are $\frac{1}{3}$. For $n \geq 2$, the required inequality holds because $\frac{1}{2n+1} \leq \frac{1}{3}$.

It is well known that $M_p$ increases with $p$. By the binomial series, $(\cosh py)^{1/p} = 1 + \frac{p}{2}y^2 + O(p^4)$, so $\frac{1}{3}$ is the smallest $p$ for which $L \leq M_p$ for all $a$, $b$.

We describe some further inequalities for means of assorted types delivered with minimal effort by this method. Firstly, the following comparison between the upper bounds in Theorems 3 and 5:

THEOREM 6. We have

$$M_{1/3} \leq \frac{2}{3} G + \frac{1}{3} A.$$  \hfill (14)

(This statement does not involve $L$, but of course, together with Theorem 5, it implies Theorem 3.)

Proof. Again replacing $a$ and $b$ by 1 and $x$, and substituting $x = e^{2y}$, we see that (14) is equivalent to

$$(\cosh \frac{1}{3}y)^3 \leq \frac{2}{3} + \frac{1}{3} \cosh y.$$
Now \((\cosh \frac{1}{3}y)^3 = \frac{1}{3} \cosh y + \frac{3}{4} \cosh \frac{1}{3}y\), so the statement is equivalent to
\[
9 \cosh \frac{1}{3}y \leq 8 + \cosh y.
\]
The constant term is 9 on both sides, and for \(n \geq 1\), the coefficient of \(y^{2n}\) on the left-hand side is
\[
\frac{1}{3^{2n-2}(2n)!},
\]
which is not greater than \(1/(2n)!\). \(\Box\)

**Remark.** This raises the obvious question of how \(M_p\) compares with \((1 - p)G + pA\) for \(p\) in general. Now \(M_1 = A\) and, on writing it out, one sees that \(M_{1/2} = \frac{1}{2}G + \frac{1}{2}A\). A proof like the one just given (with a little more work) shows that \(M_{2/3} \geq \frac{1}{3}G + \frac{2}{3}A\). It seems natural to conjecture that \(M_p \leq (1 - p)G + pA\) for \(0 < p \leq \frac{1}{2}\) and that the opposite holds for \(\frac{1}{2} < p \leq 1\). However, a word of caution is in order. All these means, unlike \(L\), make sense for three or more numbers. Simple examples show that for three numbers, \(M_{1/2}\) can be either greater or less than \(\frac{1}{2}G + \frac{1}{2}A\), and equally, \(M_{1/3}\) can be greater or less than \(\frac{2}{3}G + \frac{1}{3}A\).

For the next result, we will work from the other end. Start from the obvious inequality \(\cosh y \geq 1 + \frac{1}{2}y^2\). How does this translate into a statement about means when the steps above are applied in the reverse order? Multiplication by \(e^y\) transforms it into
\[
\frac{1}{2}(e^{2y} + 1) \geq (1 + \frac{1}{2}y^2)e^y.
\]
With the substitution \(x = e^{2y}\), this becomes
\[
\frac{1}{2}(x + 1) \geq \left(1 + \frac{1}{8}(\log x)^2\right)x^{1/2}.
\]
so, finally substituting \(x = b/a\), we conclude
\[
A \geq \left(1 + \frac{1}{8}(\log b - \log a)^2\right)G. \quad (15)
\]
This is an enhanced version of the basic inequality \(A \geq G\). It was proved, by a more elaborate method, in [ZJ]. Clearly, it could be enhanced further, at the cost of greater complication, by incorporating further terms of the \(\cosh\) series.

In the same way, the inequality \(\frac{1}{y}\sinh y \geq 1 + \frac{1}{8}y^2\) translates into
\[
L \geq \left(1 + \frac{1}{24}(\log b - \log a)^2\right)G, \quad (16)
\]
bringing us back to the logarithmic mean.

7
A scale of inequalities and an iteration for $L(a, b)$

The following results are from [Ca]. Note first that by Proposition 1, applied to $a^t$ and $b^t$ (with $a < b$ and $t > 0$),

$$a^t \leq \frac{b^t - a^t}{t(\log b - \log a)} \leq b^t.$$ 

Now $a^t$ and $b^t$ tend to 1 as $t \to 0^+$, so $\frac{1}{2}(b^t - a^t)$ tends to $\log b - \log a$, hence

$$L(a, b) = \lim_{t \to 0^+} \frac{(b - a)t}{b^t - a^t}.$$ 

Now apply Theorem 2 to $a^t$ and $b^t$: we obtain

$$\left(ab\right)^{t/2} \leq \frac{b^t - a^t}{t(\log b - \log a)} \leq \frac{1}{2}(b^t + a^t),$$

hence

$$G_t(a, b) \leq L(a, b) \leq A_t(a, b),$$

where

$$G_t(a, b) = t(b - a)(ab)^{t/2},$$

$$A_t(a, b) = \frac{1}{2}t(b - a)\frac{b^t + a^t}{b^t - a^t}. $$

Write just $G_t$, $A_t$. Clearly, $G_1 = G$ and $A_1 = A$. It is easily checked that $G_{-t} = G_t$ and $A_{-t} = A_t$. Also, using Theorem 3 instead of Theorem 2, we see, for example, that $L \leq \frac{2}{3}G_t + \frac{1}{3}A_t$.

Observe next that

$$A_t^2 - G_t^2 = t^2(b - a)^2 \frac{1}{4}(b^t + a^t)^2 - b^t a^t \frac{1}{4}(b^t - a^t)^2 = \frac{1}{3}t^2(b - a)^2,$$

hence (since $A_t + G_t \geq 2L$),

$$A_t - G_t = \frac{t^2(b - a)^2}{4(A_t + G_t)} \leq \frac{t^2(b - a)^2}{8L},$$

so this bound applies to $A_t - L$ and $L - G_t$.

It is not important for our purposes, but [Ca] also shows that $G_t$ decreases with $t$ for $t > 0$, and $A_t$ increases, indicating a proof using differentiation. Our substitution method provides a very simple proof for $G_t$. As usual, it is enough to consider the case $a = 1$. Putting $b = e^{2y}$, we then have

$$G_t = t(e^{2y} - 1) \frac{e^{ty}}{e^{ty} - e^{-ty}} = \frac{te^y \sinh y}{\sinh ty}.$$
By the series (7), with $y$ replaced by $ty$, it is clear that $\frac{1}{ty} \sinh ty$ increases with $t$ for $t > 0$, so $G_t$ decreases. For $A_t$, we need to show that $t \cosh ty / (\sinh ty)$ is increasing: after differentiating, we see that this follows from the elementary fact that $\cosh t \sinh t \geq t$.

**PROPOSITION 7.** We have

$$A_{t/2} = \frac{1}{2} (A_t + G_t),$$  \hspace{1cm} (23)

$$G_{t/2} = (A_{t/2} G_t)^{1/2}. \hspace{1cm} (24)$$

**Proof.** By (20),

$$\frac{1}{t(b-a)} A_{t/2} = \frac{b^{t/2} + a^{t/2}}{4(b^{t/2} - a^{t/2})}$$

$$= \frac{(b^{t/2} + a^{t/2})^2}{4(b^t - a^t)} \hspace{1cm} (25)$$

$$= \frac{(b^t + a^t) + 2(ab)^{t/2}}{4(b^t - a^t)}$$

$$= \frac{1}{2t(b-a)} (A_t + G_t).$$

Also, by (19) and (25),

$$\frac{1}{t(b-a)} G_{t/2} = \frac{(ab)^{t/4}}{2(b^{t/2} - a^{t/2})}$$

$$= \frac{(ab)^{t/4} b^{t/2} + a^{t/2}}{2(b^t - a^t)}$$

$$= \frac{1}{t(b-a)} (A_{t/2} G_t)^{1/2}. \hspace{1cm} \square$$

Consequently, the numbers $A_{1/2^n}$ and $G_{1/2^n}$ can be generated by the following double iteration. Set $a_0 = A$ and $g_0 = G$. Then, given $a_n$ and $g_n$, define

$$a_{n+1} = \frac{1}{2} (a_n + g_n), \hspace{1cm} g_{n+1} = (a_n g_n)^{1/2}. \hspace{1cm} (26)$$

By Proposition 7 and induction, $a_n = A_{1/2^n}$ and $g_n = G_{1/2^n}$ for all $n \geq 1$. So by (18) and (22), we have $g_n \leq L \leq a_n$ and $a_n - g_n \leq (b-a)^2/(L2^{n+3})$ for all $n$. Purely from the iteration itself, it is clear that $a_n > g_n$, $a_{n+1} < a_n$ and $g_{n+1} > g_n$ for all $n$: in fact, given $a_n > g_n$ for a certain $n$, (26) gives at once $a_n > a_{n+1} > g_n$ and $a_{n+1} > g_{n+1} > g_n$.

This procedure can be compared with the *arithmetic-geometric mean* iteration, which generates sequences $(a'_n)$ and $(g'_n)$ by: $a'_0 = A$, $g'_0 = G$, then

$$a'_{n+1} = \frac{1}{2} (a'_n + g'_n), \hspace{1cm} g'_{n+1} = (a'_n g'_n)^{1/2}.$$
These sequences converge to a common limit, the “arithmetic-geometric mean”, which we denote by $AG(a, b)$. Clearly, $a'_1 = a_1$, but $g'_1 > g_1$, so that $a'_n > a_n$ and $g'_n > g_n$ for all $n \geq 2$, hence $AG(a, b) \geq L(a, b)$. A feature of the AGM iteration is that it converges quadratically (so very rapidly). This property is not shared by the iteration (26): in fact, the rate of convergence is as stated above.

References


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