The logarithmic mean

Notes by G.J.O. Jameson

The *logarithmic mean* of two distinct positive numbers a, b is

$$L(a,b) = \frac{b-a}{\log b - \log a}.$$

An immediate observation is:

PROPOSITION 1. *If $a < b$, then $a < L(a,b) < b$.*

Proof. Since $\frac{d}{dx} \log x = \frac{1}{x}$, the mean-value theorem gives

$$\frac{1}{L(a,b)} = \frac{\log b - \log a}{b-a} = \frac{1}{c},$$

where $a < c < b$, hence $L(a,b) = c$. $\square$

Hence $L(a,b) \to a$ as $b \to a$, and it is natural to complete the definition by putting $L(a,a) = a$. Note also that $L(1,b) = (b-1)/\log b$.

Clearly, as with other means, we have $L(\lambda a, \lambda b) = \lambda L(a,b)$ for $\lambda > 0$. In particular, $L(a,b) = aL(1, \frac{b}{a}) = abL(\frac{1}{a}, \frac{1}{b})$.

We mention three integral expressions. First, since $\frac{d}{dt}(a^{1-t}b^t) = a^{1-t}b^t(\log b - \log a)$, we have

$$\int_0^1 a^{1-t}b^t \, dt = \frac{1}{\log b - \log a} \left[a^{1-t}b^t\right]_0^1 = \frac{b-a}{\log b - \log a} = L(a,b). \tag{1}$$

Next,

$$\int_0^1 \frac{1}{a(1-t)+bt} \, dt = \frac{1}{b-a} \Big[\log[a(1-t)+bt] \Big]_0^1 = \frac{1}{L(a,b)}. \tag{2}$$

Third,

$$\begin{aligned} \int_0^\infty \frac{1}{(t+a)(t+b)} \, dt &= \frac{1}{b-a} \int_0^\infty \left(\frac{1}{t+a} - \frac{1}{t+b} \right) dt \\ &= \frac{1}{b-a} \left[\log \frac{t+a}{t+b} \right]_0^\infty \\ &= \frac{\log b - \log a}{b-a} \\ &= \frac{1}{L(a,b)}. \end{aligned} \tag{3}$$

We now compare $L(a,b)$ with other means, in particular the *arithmetic mean* $A(a,b) = \frac{1}{2}(a+b)$, the *geometric mean* $G(a,b) = (ab)^{1/2}$ and the *power mean* $M_p(a,b) = [\frac{1}{2}(a^p + b^p)]^{1/p}$ for suitable p. The basic result is:

THEOREM 2. *We have*

$$G(a,b) \le L(a,b) \le A(a,b). \tag{4}$$

Note that the special case $a = 1$ (with b replaced by x) says

$$x^{1/2} \le \frac{x-1}{\log x} \le \tfrac{1}{2}(x+1). \tag{5}$$

Conversely, the substitution $x = b/a$ transforms (5) into (4), so in fact the statements are equivalent. In this way, (4) reduces to inequalities in terms of a single variable x.

Numerous proofs of Theorem 2 have been given. We present four of them. Of these, Proof 1 is the one that signals the method that will be used for further inequalities later.

Henceforth we write just L for $L(a,b)$, and similarly A and G.

Proof 1: substitution [JM]. In (5), substitute $x = e^{2y}$: the statement becomes

$$e^y \le \frac{e^{2y}-1}{2y} \le \tfrac{1}{2}(e^{2y}+1).$$

After division by e^y, this says

$$1 \le \frac{\sinh y}{y} \le \cosh y. \tag{6}$$

So (4) is equivalent to (6), which follows at once from the series expansions

$$\frac{\sinh y}{y} = 1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots, \tag{7}$$

$$\cosh y = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots. \tag{8}$$

Proof 2: mean-value theorem ([Mer], [Mit, p. 273], with slight variations). It is clearly enough to prove (5) for $x > 1$. Rewrite it as

$$\frac{2(x-1)}{x+1} \le \log x \le x^{1/2} - x^{-1/2}.$$

For the left-hand inequality, write

$$f(x) = \log x - \frac{2(x-1)}{x+1} = \log x - 2 + \frac{4}{x+1}.$$

Then $f(1) = 0$ and

$$f'(x) = \frac{1}{x} - \frac{4}{(x+1)^2} = \frac{(x+1)^2 - 4x}{x(x+1)^2} = \frac{(x-1)^2}{x(x+1)^2} > 0$$

for $x > 1$. So $f(x) > 0$ for $x > 1$, by the mean-value theorem.

For the right-hand inequality, substitute $x = y^2$ and write $g(y) = y - \frac{1}{y} - 2\log y$. Then $g(1) = 0$ and

$$g'(y) = 1 + \frac{1}{y^2} - \frac{2}{y} = \left(1 - \frac{1}{y}\right)^2 \geq 0. \qquad \square$$

Proof 3: convexity [Bu]. Note that

$$\int_{\log a}^{\log b} e^x \, dx = b - a.$$

Now e^x is convex, and the integral of a convex function is not greater than its estimate by the trapezium rule: $\int_a^b f \leq \frac{1}{2}(b-a)[f(a) + f(b)]$. This is geometrically obvious, and easily proved analytically. In this case, it says $b - a \leq \frac{1}{2}(a+b)(\log b - \log a)$, which equates to $L \leq A$.

Also, the integral of a convex function is not less than the area below the tangent at the mid-point c, in other words, $\int_a^b f \geq (b-a)f(c)$. In our case, the value of e^x at the mid-point is $(ab)^{1/2}$, so this says that $b - a \geq (ab)^{1/2}(\log b - \log a)$, or $L \geq G$.

A slight variant of this method, using $1/x$ instead of e^x, is given in [Br]. $\square$

Proof 4, using the integral (3) [Bh]. By the basic inequality $G \leq A$,

$$(t+a)(t+b) = t^2 + (a+b)t + ab \leq t^2 + (a+b)t + \tfrac{1}{4}(a+b)^2 = (t+A)^2,$$

also

$$(t+a)(t+b) \geq t^2 + 2(ab)^{1/2}t + ab = (t+G)^2.$$

Since

$$\int_0^\infty \frac{1}{(t+c)^2} \, dt = \frac{1}{c},$$

we deduce from (3) that

$$\frac{1}{A} \leq \frac{1}{L} \leq \frac{1}{G}. \qquad \square$$

We now establish some more intricate inequalities comparing L with combinations of A and G. They were all obtained by quite complicated methods in earlier papers, and a somewhat simpler method (using Cauchy's mean-value theorem) in [Mer]. Here we present a much simpler method, along the lines of Proof 1 above. The technique is well established for the purpose of proving inequalities for means: e.g. [Alz]. For the results given here, it was indicated in [Zhu], without full details, and set out more fully in [JM].

THEOREM 3 [Ca]. *We have*

$$L \leq \tfrac{2}{3}G + \tfrac{1}{3}A. \tag{9}$$

Further, $\frac{1}{3}$ *is the smallest* p *for which* $L \leq (1-p)G + pA$ *for all choices of* a, b.

Proof: We do not need to know the factor $\frac{1}{3}$ in advance: we can let it emerge from the reasoning. So consider (9) in the form $L \leq (1-p)G + pA$, where p is to be found. As before, it is sufficient to prove the case $a = 1$, in other words (with b replaced by x)

$$\frac{x-1}{\log x} \leq (1-p)x^{1/2} + \frac{p}{2}(x+1).$$

The substitution $x = e^{2y}$ tranforms this into

$$\frac{e^{2y}-1}{2y} \leq (1-p)e^{y} + \frac{p}{2}(e^{2y}+1),$$

equivalently,

$$\frac{\sinh y}{y} \leq (1-p) + p\cosh y. \tag{10}$$

We wish this to hold for all $y > 0$. Compare power series. We have

$$(1-p) + p\cosh y = 1 + p\left(\frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right).$$

Statement (10) will be assured if the coefficients are no smaller than the corresponding ones in (7). The y^2 term requires that $\frac{p}{2} \geq \frac{1}{6}$, so $p \geq \frac{1}{3}$. The y^{2n} term requires $p/[(2n)!)] \geq 1/[(2n+1)!]$, or $p \geq 1/(2n+1)$. So (10) holds with $p = \frac{1}{3}$.

At the same time, it is clear that (10) will fail for some y if $p < \frac{1}{3}$, since $(1-p) + p\cosh y = 1 + \frac{p}{2}y^2 + O(y^4)$, while $(\sinh y/y) = 1 + \frac{1}{6}y^2 + O(y^4)$. □

THEOREM 4 [LSh]. *We have*

$$L \geq G^{2/3}A^{1/3}. \tag{11}$$

Further, $\frac{1}{3}$ *is the largest* p *for which* $L \geq G^{1-p}A^{p}$ *for all choices of* a, b.

Proof. As before, (11) is equivalent to the case $a = 1$, that is,

$$\frac{x-1}{\log x} \geq x^{1/3}[\tfrac{1}{2}(x+1)]^{1/3}.$$

The substitution $x = e^{2y}$ transforms this into

$$\frac{e^{2y}-1}{2y} \geq e^{2y/3}\left(\tfrac{1}{2}(e^{2y}+1)\right)^{1/3}.$$

After division by e^y, this is equivalent to

$$\frac{\sinh y}{y} \geq (\cosh y)^{1/3},$$

hence to

$$\left(\frac{\sinh y}{y}\right)^3 \geq \cosh y. \tag{12}$$

This time, comparison of the coefficients will cost us a little more work. Note that

$$\begin{aligned}(\sinh y)^3 &= \tfrac{1}{8}(e^y - e^{-y})^3 \\ &= \tfrac{1}{4}\sinh 3y - \tfrac{3}{4}\sinh y \\ &= \sum_{n=0}^{\infty} c_{2n} y^{2n+3},\end{aligned}$$

where

$$c_{2n} = \frac{3^{2n+3}-3}{4(2n+3)!}.$$

We need to know that $c_{2n} \geq 1/[(2n)!]$ for each $n \geq 0$. This equates to saying that $u_n \geq v_n$, where

$$u_n = 3^{2n+3} - 3, \qquad v_n = 4(2n+1)(2n+2)(2n+3).$$

To start, we have $u_0 = v_0 = 24$ and $u_1 = v_1 = 240$. For all $n \geq 1$, it is clear that $u_{n+1}/u_n > 9$, while

$$\frac{v_{n+1}}{v_n} = \frac{(2n+4)(2n+5)}{(2n+1)(2n+2)} \leq \frac{6 \times 7}{3 \times 4} = \frac{7}{2},$$

so indeed $u_n > v_n$, as required.

If $q < 3$, then by (7) and the binomial series, we have $(\sinh y/y)^q = 1 + \frac{q}{6}y^2 + O(y^4) < \cosh y$ for sufficiently small y, hence if $p > \frac{1}{3}$, then $\sinh y/y < (\cosh y)^p$ for such y, hence $L < G^{1-p}A^p$. (Alternatively, as observed in [Jam], this follows from (9) and the fact that if $p > \frac{1}{3}$, then $x^p y^{1-p} > \frac{1}{3}x + \frac{2}{3}y$ for $\frac{x}{y}$ just larger than 1.) □

We mention two alternative proofs of (12) (which the reader, of course, may ignore).

Alternative proof 1. Consider

$$\left(\frac{\sinh y}{y}\right)^3 = \left(1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots\right)^3.$$

The constant term is 1, and the y^2 term is $3y^2/3! = \frac{1}{2}y^2$. Using only the term $y^2/3!$, we see that the coefficient of y^4 is greater than $\frac{3}{6^2} = \frac{1}{12}$: this is larger than the 1/4! occurring in the series for $\cosh y$. Now consider y^{2n}, where $2n \geq 6$. The combination

$$\frac{y^2}{3!}\frac{y^{2n-2}}{(2n-1)!} = \frac{y^{2n}}{6(2n-1)!}$$

occurs six times, corresponding to ways of selecting these two terms and 1 from the three brackets. So the coefficient of y^{2n} is at least $1/[(2n-1)!]$, greater than $1/[(2n)!]$.

Alternative proof 2. The following method is given in [Mit, p. 270], where (12) is stated without any mention of the equivalence with (11). Let $f(y) = y - \sinh y(\cosh y)^{-1/3}$. After what seems like fortuitous cancellation, one finds that $f''(y) = -\frac{4}{9}(\sinh y)^3(\cosh y)^{-7/3}$, so that $f''(y) < 0$ for $y > 0$. By two applications of the mean-value theorem, one deduces that $f(y) < 0$ for $y > 0$.

THEOREM 5 [Lin], [Bu] . *We have* $L \le M_{1/3}$.

Proof. After replacing a and b by 1 and x, and substituting $x = e^{2y}$, we see that the statement is equivalent to

$$\frac{\sinh y}{y} \le (\cosh \tfrac{1}{3}y)^3 = \tfrac{1}{4}\cosh y + \tfrac{3}{4}\cosh \tfrac{1}{3}y. \tag{13}$$

The series expression for the right-hand side is $\sum_{n=0}^{\infty} d_{2n}y^{2n}$, where

$$d_{2n} = \frac{1}{(2n)!}\left(\frac{1}{4} + \frac{3}{4.3^{2n}}\right).$$

Comparing with (7), we need to know that $d_{2n} \ge 1/[(2n+1)!]$ for each n. This equates to

$$\frac{1}{4} + \frac{3}{4.3^{2n}} \ge \frac{1}{2n+1}.$$

This is much easier than the previous proof! When $n = 0$, both sides are 1, and when $n = 1$, both sides are $\frac{1}{3}$. For $n \ge 2$, the required inequality holds because $\frac{1}{2n+1} \le \frac{1}{5}$. □

It is well known that M_p increases with p. By the binomial series, $(\cosh py)^{1/p} = 1 + \frac{p}{2}y^2 + O(p^4)$, so $\frac{1}{3}$ is the smallest p for which $L \le M_p$ for all a, b.

We describe some further inequalities for means of assorted types delivered with minimal effort by this method. Firstly, the following comparison between the upper bounds in Theorems 3 and 5:

THEOREM 6. *We have*

$$M_{1/3} \le \tfrac{2}{3}G + \tfrac{1}{3}A. \tag{14}$$

(This statement does not involve L, but of course, together with Theorem 5, it implies Theorem 3.)

Proof. Again replacing a and b by 1 and x, and substituting $x = e^{2y}$, we see that (14) is equivalent to

$$(\cosh \tfrac{1}{3}y)^3 \le \tfrac{2}{3} + \tfrac{1}{3}\cosh y.$$

Now $(\cosh \frac{1}{3}y)^3 = \frac{1}{4}\cosh y + \frac{3}{4}\cosh \frac{1}{3}y$, so the statement is equivalent to

$$9\cosh \tfrac{1}{3}y \le 8 + \cosh y.$$

The constant term is 9 on both sides, and for $n \ge 1$, the coefficient of y^{2n} on the left-hand side is

$$\frac{1}{3^{2n-2}(2n)!},$$

which is not greater than $1/[(2n)!]$. ☐

Remark. This raises the obvious question of how M_p compares with $(1-p)G + pA$ for p in general. Now $M_1 = A$ and, on writing it out, one sees that $M_{1/2} = \frac{1}{2}G + \frac{1}{2}A$. A proof like the one just given (with a little more work) shows that $M_{2/3} \ge \frac{1}{3}G + \frac{2}{3}A$. It seems natural to conjecture that $M_p \le (1-p)G + pA$ for $0 < p \le \frac{1}{2}$ and that the opposite holds for $\frac{1}{2} < p \le 1$. However, a word of caution is in order. All these means, unlike L, make sense for three or more numbers. Simple examples show that for three numbers, $M_{1/2}$ can be either greater or less than $\frac{1}{2}G + \frac{1}{2}A$, and equally, $M_{1/3}$ can be greater or less than $\frac{2}{3}G + \frac{1}{3}A$.

For the next result, we will work from the other end. Start from the obvious inequality $\cosh y \ge 1 + \frac{1}{2}y^2$. How does this translate into a statement about means when the steps above are applied in the reverse order? Multiplication by e^y transforms it into

$$\tfrac{1}{2}(e^{2y} + 1) \ge (1 + \tfrac{1}{2}y^2)e^y.$$

With the substitution $x = e^{2y}$, this becomes

$$\tfrac{1}{2}(x+1) \ge \Big(1 + \tfrac{1}{8}(\log x)^2\Big)x^{1/2}.$$

so, finally substituting $x = b/a$, we conclude

$$A \ge \Big(1 + \tfrac{1}{8}(\log b - \log a)^2\Big)G. \tag{15}$$

This is an enhanced version of the basic inequality $A \ge G$. It was proved, by a more elaborate method, in [ZJ]. Clearly, it could be enhanced further, at the cost of greater complication, by incorporating further terms of the cosh series.

In the same way, the inequality $\frac{1}{y}\sinh y \ge 1 + \frac{1}{6}y^2$ translates into

$$L \ge \Big(1 + \tfrac{1}{24}(\log b - \log a)^2\Big)G, \tag{16}$$

bringing us back to the logarithmic mean.

A scale of inequalities and an iteration for $L(a,b)$

The following results are from [Ca]. Note first that by Proposition 1, applied to a^t and b^t (with $a < b$ and $t > 0$),

$$a^t \le \frac{b^t - a^t}{t(\log b - \log a)} \le b^t.$$

Now a^t and b^t tend to 1 as $t \to 0^+$, so $\frac{1}{t}(b^t - a^t)$ tends to $\log b - \log a$, hence

$$L(a,b) = \lim_{t \to 0^+} \frac{(b-a)t}{b^t - a^t}$$

.

Now apply Theorem 2 to a^t and b^t: we obtain

$$(ab)^{t/2} \le \frac{b^t - a^t}{t(\log b - \log a)} \le \tfrac{1}{2}(b^t + a^t), \tag{17}$$

hence

$$G_t(a,b) \le L(a,b) \le A_t(a,b), \tag{18}$$

where

$$G_t(a,b) = t(b-a)\frac{(ab)^{t/2}}{b^t - a^t}, \tag{19}$$

$$A_t(a,b) = \tfrac{1}{2}t(b-a)\frac{b^t + a^t}{b^t - a^t}. \tag{20}$$

Write just G_t, A_t. Clearly, $G_1 = G$ and $A_1 = A$. It is easily checked that $G_{-t} = G_t$ and $A_{-t} = A_t$. Also, using Theorem 3 instead of Theorem 2, we see, for example, that $L \le \frac{2}{3}G_t + \frac{1}{3}A_t$.

Observe next that

$$A_t^2 - G_t^2 = t^2(b-a)^2 \frac{\frac{1}{4}(b^t + a^t)^2 - b^t a^t}{(b^t - a^t)^2} = \tfrac{1}{4}t^2(b-a)^2, \tag{21}$$

hence (since $A_t + G_t \ge 2L$),

$$A_t - G_t = \frac{t^2(b-a)^2}{4(A_t + G_t)} \le \frac{t^2(b-a)^2}{8L}, \tag{22}$$

so this bound applies to $A_t - L$ and $L - G_t$.

It is not important for our purposes, but [Ca] also shows that G_t decreases with t for $t > 0$, and A_t increases, indicating a proof using differentiation. Our substitution method provides a very simple proof for G_t. As usual, it is enough to consider the case $a = 1$. Putting $b = e^{2y}$, we then have

$$G_t = t(e^{2y} - 1)\frac{e^{ty}}{e^{ty} - e^{-ty}} = \frac{te^y \sinh y}{\sinh ty}.$$

By the series (7), with y replaced by ty, it is clear that $\frac{1}{ty}\sinh ty$ increases with t for $t > 0$, so G_t decreases. For A_t, we need to show that $t\cosh ty/(\sinh ty)$ is increasing: after differentiating, we see that this follows from the elementary fact that $\cosh t \sinh t \geq t$.

PROPOSITION 7. *We have*

$$A_{t/2} = \tfrac{1}{2}(A_t + G_t), \tag{23}$$

$$G_{t/2} = (A_{t/2}G_t)^{1/2}. \tag{24}$$

Proof. By (20),

$$\begin{aligned} \frac{1}{t(b-a)}A_{t/2} &= \frac{b^{t/2}+a^{t/2}}{4(b^{t/2}-a^{t/2})} \\ &= \frac{(b^{t/2}+a^{t/2})^2}{4(b^t-a^t)} \\ &= \frac{(b^t+a^t)+2(ab)^{t/2}}{4(b^t-a^t)} \\ &= \frac{1}{2t(b-a)}(A_t+G_t). \end{aligned} \tag{25}$$

Also, by (19) and (25),

$$\begin{aligned} \frac{1}{t(b-a)}G_{t/2} &= \frac{(ab)^{t/4}}{2(b^{t/2}-a^{t/2})} \\ &= (ab)^{t/4}\frac{b^{t/2}+a^{t/2}}{2(b^t-a^t)} \\ &= \frac{1}{t(b-a)}(A_{t/2}G_t)^{1/2}. \qquad \square \end{aligned}$$

Consequently, the numbers $A_{1/2^n}$ and $G_{1/2^n}$ can be generated by the following double iteration. Set $a_0 = A$ and $g_0 = G$. Then, given a_n and g_n, define

$$a_{n+1} = \tfrac{1}{2}(a_n + g_n), \qquad g_{n+1} = (a_{n+1}g_n)^{1/2}. \tag{26}$$

By Proposition 7 and induction, $a_n = A_{1/2^n}$ and $g_n = G_{1/2^n}$ for all $n \geq 1$. So by (18) and (22), we have $g_n \leq L \leq a_n$ and $a_n - g_n \leq (b-a)^2/(L2^{2n+3})$ for all n. Purely from the iteration itself, it is clear that $a_n > g_n$, $a_{n+1} < a_n$ and $g_{n+1} > g_n$ for all n: in fact, given $a_n > g_n$ for a certain n, (26) gives at once $a_n > a_{n+1} > g_n$ and $a_{n+1} > g_{n+1} > g_n$.

This procedure can be compared with the *arithmetic-geometric mean* iteration, which generates sequences (a'_n) and (g'_n) by: $a'_0 = A$, $g'_0 = G$, then

$$a'_{n+1} = \tfrac{1}{2}(a'_n + g'_n), \qquad g'_{n+1} = (a'_n g'_n)^{1/2}.$$

These sequences converge to a common limit, the "arithmetic-geometric mean", which we denote by $AG(a, b)$. Clearly, $a'_1 = a_1$, but $g'_1 > g_1$, so that $a'_n > a_n$ and $g'_n > g_n$ for all $n \geq 2$, hence $AG(a, b) \geq L(a, b)$. A feature of the AGM iteration is that it converges quadratically (so very rapidly). This property is not shared by the iteration (26): in fact, the rate of convergence is as stated above.

References

[Alz] Horst Alzer, Ungleichungen für Mittelwerte, *Arch. Math.* **47** (1986), 422–426.

[Bh] R. Bhatia, The logarithmic mean, *Resonance* **13** (2008), 583–594.

[Br] I. Bruce, The logarithmic mean, *Math. Gazette* **81** (1997), 89–92.

[Bu] Frank Burk, The geometric, logarithmic and arithmetic mean inequality, *American Math. Monthly* **94** (1987), 527–528.

[Ca] B. C. Carlson, The logarithmic mean, *Amer. Math. Monthly* **79** (1972), 615–618.

[Jam] Graham Jameson, Feedback on 101.07, *Math. Gazette* **101** (2017), p. 545.

[JM] Graham Jameson and Peter Mercer, The logarithmic mean revisited, *Amer. Math. Monthly*, to appear (2019).

[LSh] E. B. Leach and M. C. Sholander, Extended mean values II, *J. Math. Anal. Appl.* **92** (1983), 207–223.

[Lin] Tung-Po Lin, The power mean and the logarithmic mean, *Amer. Math. Monthly* **81** (1974), 879–883.

[Mer] Peter R. Mercer, Cauchy's mean value theorem meets the logarithmic mean, *Math. Gazette* **101** (2017), 108–115.

[Mit] D. S. Mitrinovic, *Analytic Inequalities*, Springer (1970).

[Zhu] Ling Zhu, From chains for mean value inequalities to Mitrinovic's problem II, *Int. J. Math. Education in Science and Technology* **36** (2005), 118–125.

[ZJ] Limin Zou and Youyi Jiang, Improved arithmetic-geometric mean inequality and its application, *J. Math. Ineq.* **9** (2015), 107–111.

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