

## Some inequalities for $(a + b)^p$ and $(a + b)^p + (a - b)^p$

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We start from two simple identities:

$$(a + b) + (a - b) = 2a, \tag{1}$$

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2). \tag{2}$$

For any  $p > 0$  and  $0 \leq b \leq a$ , now let

$$G_p(a, b) = (a + b)^p + (a - b)^p. \tag{3}$$

Can we formulate statements about  $G_p(a, b)$  that generalise (1) and (2)? We cannot hope for equalities, but perhaps we can establish inequalities which somehow reproduce (1) when  $p = 1$  and (2) when  $p = 2$ . For (1), this might mean an inequality of the form  $A_p a^p \leq G_p(a, b) \leq B_p a^p$  for certain constants  $A_p$  and  $B_p$ , and for (2) a similar statement with  $a^p$  replaced by  $a^p + b^p$ . However, these are not the only possibilities, as we shall see.

For this investigation we will require corresponding inequalities for the simpler expression  $(a + b)^p$ . These inequalities are of interest in their own right, and are mostly well known, but we repeat them here, both because we need them, and because they will serve to introduce the methods that work for the wider problem.

We begin with the simple observation that  $x^p$  is strictly increasing on  $(0, \infty)$  if  $p > 0$ , and strictly decreasing if  $p < 0$ . We will repeatedly use the following elementary consequences:

(E1) If  $p > 0$ , then  $x^p > 1$  for  $x > 1$  and  $x^p < 1$  for  $0 < x < 1$ ; reverse inequalities apply when  $p < 0$ .

(E2) If  $p > 1$ , then  $x^p > x$  for  $x > 1$  and  $x^p < x$  for  $0 < x < 1$ ; reverse inequalities apply when  $0 < p < 1$ .

We assume familiarity with *convexity*. Recall that a function  $f$  is “convex” on an interval  $I$  if it lies below the straight-line chords between pairs of points of its graph. In other words, if  $x_1, x_2 \in I$  and  $x_\lambda = (1 - \lambda)x_1 + \lambda x_2$ , where  $0 < \lambda < 1$ , then

$$f(x_\lambda) \leq (1 - \lambda)f(x_1) + \lambda f(x_2).$$

Convex functions satisfy *Jensen’s inequality* (which can be proved, for example, by induction): if  $x_j \in I$  and  $\lambda_j \geq 0$  for  $1 \leq j \leq n$ , with  $\sum_{j=1}^n \lambda_j = 1$ , and  $x_\lambda = \sum_{j=1}^n \lambda_j x_j$ ,

then

$$f(x_\lambda) \leq \sum_{j=1}^n \lambda_j f(x_j).$$

We say  $f$  is “concave” if  $-f$  is convex.

If  $f$  has increasing derivative on  $I$ , then it is convex there, hence  $x^p$  is convex on  $(0, \infty)$  if  $p \geq 1$  or  $p \leq 0$  (then  $x^{p-1}$  is decreasing, so  $px^{p-1}$  is increasing), and concave if  $0 \leq p \leq 1$ . This reversal of behaviour at certain values of  $p$  is typical of statements involving  $x^p$ : we shall see many more examples.

First, we compare  $(a + b)^p$  with  $a^p + b^p$ . Actually, this is just as easy for a sum of  $n$  terms, so we state it this way.

**THEOREM 1.** *If  $a_j \geq 0$  ( $1 \leq j \leq n$ ) and  $p \geq 1$ , then*

$$\sum_{j=1}^n a_j^p \leq \left( \sum_{j=1}^n a_j \right)^p \leq n^{p-1} \sum_{j=1}^n a_j^p. \quad (4)$$

*The reverse inequalities hold if  $0 < p \leq 1$ .*

*Proof.* Assume that  $p \geq 1$ . For the left-hand inequality, let  $\sum_{j=1}^n a_j = A$  and  $x_j = a_j/A$ . Then  $0 \leq x_j \leq 1$  and  $\sum_{j=1}^n x_j = 1$ . By (E2), we have  $x_j^p \leq x_j$ . Hence  $\sum_{j=1}^n x_j^p \leq 1$ , so  $\sum_{j=1}^n a_j^p \leq A^p$ .

Meanwhile, by Jensen’s inequality applied to  $x^p$ , with  $\lambda_j = \frac{1}{n}$  for each  $j$ , we have

$$\left( \frac{1}{n} \sum_{j=1}^n a_j \right)^p \leq \frac{1}{n} \sum_{j=1}^n a_j^p,$$

which equates to the right-hand inequality.

In both cases, the reverse holds if  $0 < p \leq 1$ . □

Equality holds on the left when only one  $a_j$  is non-zero, and on the right when all the  $a_j$  are equal.

The  $p$ th power mean of a non-negative vector  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  is defined by

$$M_p(\mathbf{a}) = \left( \frac{1}{n} \sum_{j=1}^n a_j^p \right)^{1/p}.$$

In this notation, the right-hand inequality takes the pleasantly simple form  $M_1(\mathbf{a}) \leq M_p(\mathbf{a})$ .

Readers acquainted with Hölder’s inequality will recognise that the right-hand inequality is a special case of it.

We now restrict attention to a sum of two terms. For  $p \geq 1$ , (4) says

$$a^p + b^p \leq (a + b)^p \leq 2^{p-1}(a^p + b^p),$$

and, specialised even further,

$$1 + x^p \leq (1 + x)^p \leq 2^{p-1}(1 + x^p) \quad \text{for } x > 0.$$

However,  $1 + x^p$  is not a very accurate approximation to  $(1 + x)^p$  for  $x$  close to 0. The beginning of the binomial expansion, and equally the tangent to the curve at  $x = 0$ , is  $1 + px$ , so we now consider a second type of inequality addressing this comparison, including the case  $p < 0$ . Corresponding statements for  $(a + b)^p$  will no longer be symmetrical. Our main tool for this is the mean-value theorem. First we state the basic result obtained by applying it to  $x^p$ .

**THEOREM 2.** *Suppose that  $0 < u < v$ . If  $p \geq 1$  or  $p < 0$ , then*

$$pu^{p-1}(v - u) \leq v^p - u^p \leq pv^{p-1}(v - u). \quad (5)$$

*The reverse inequality holds if  $0 < p \leq 1$ .*

*Proof.* Let  $f(t) = t^p$ , so that  $f'(x) = px^{p-1}$ . By the mean-value theorem,  $v^p - u^p = (v - u)f'(\xi)$  for some  $\xi$  between  $u$  and  $v$ . If  $p \geq 1$  or  $p < 0$ , then  $f'(x)$  is increasing, so  $f'(u) \leq f'(\xi) \leq f'(v)$ . Statement (5) follows. If  $0 < p \leq 1$ , then  $f'(x)$  is decreasing, so the inequalities reverse.  $\square$

We record the particular cases wanted for our current purposes:

**THEOREM 3.** *If  $p \geq 1$  or  $p < 0$ , then*

$$(1 + x)^p \geq 1 + px \quad \text{for } x \geq 0, \quad (6)$$

$$(1 - x)^p \geq 1 - px \quad \text{for } 0 \leq x < 1, \quad (7)$$

$$(a + b)^p \geq a^p + pa^{p-1}b \quad \text{for } a, b \geq 0, \quad (8)$$

$$(a - b)^p \geq a^p - pa^{p-1}b \quad \text{for } a > b \geq 0. \quad (9)$$

*Reverse inequalities hold if  $0 < p \leq 1$ .*

*Proof.* The left-hand inequality in (5) gives (6) when we take  $u = 1$  and  $v = 1 + x$ , and (8) when  $u = a$  and  $v = a + b$ . The right-hand inequality gives (7) when  $u = 1 - x$  and  $v = 1$ , and (9) when  $u = a - b$  and  $v = a$ .  $\square$

Of course, the substitution  $x = b/a$  in (6) and (7) gives (8) and (9). Also, (6) and (7) can be combined into the statement that  $(1 + x)^p \geq 1 + px$  for all  $x > -1$ , but it is more helpful for our purposes to state (7) separately, as above.

For  $p > 1$ , the lower bound  $1 + px$  from (6) is clearly stronger than the lower bound  $1 + x^p$  from (4) when  $x^{p-1} < p$ .

An obvious consequence of (8) is:

**COROLLARY.** *If  $b > 0$ , then  $(a + b)^p - a^p$  tends to infinity as  $a \rightarrow \infty$  when  $p > 1$ , and to 0 when  $0 < p < 1$ .  $\square$*

There are various ways in which one can formulate companion inequalities in the opposite direction. One is to regard (6) as a comparison between  $(1 + x)^p - 1$  and  $x$  and ask if we find a constant  $c$  such that  $(1 + x)^p - 1 \leq cx$  for  $x$  in a chosen interval. Clearly, if  $p > 1$ , then there is no such constant applying for all  $x > 0$ . However, a simple application of convexity identifies such a constant for  $x$  in  $[0, 1]$ :

**THEOREM 4.** *For  $p \geq 1$  and  $p < 0$ , we have:*

$$(1 + x)^p \leq 1 + (2^p - 1)x \quad \text{for } 0 \leq x \leq 1, \quad (10)$$

$$(a + b)^p \leq a^p + (2^p - 1)a^{p-1}b \quad \text{for } 0 \leq b \leq a. \quad (11)$$

*Reverse inequalities hold if  $0 < p \leq 1$ .*

*Proof.* Let  $f(x) = (1+x)^p$ . If  $p \geq 1$  or  $p < 0$ , then  $f$  is convex. Since  $x = (1-x).0 + x.1$ , we have

$$(1 + x)^p = f(x) \leq (1 - x)f(0) + xf(1) = (1 - x) + 2^p x.$$

The reverse holds for  $0 < p < 1$ . Equality holds when  $x = 1$ . Substitute  $x = b/a$  to obtain (11).  $\square$

*Example 1.* For  $0 \leq x \leq 1$ , we have  $1 + cx \leq (1 + x)^{1/2} \leq 1 + \frac{1}{2}x$ , where  $c = \sqrt{2} - 1$ . One can verify this directly by squaring and doing some algebra.

For  $1 - x$ , with  $0 \leq x \leq 1$ , the required converse is already given by (E2): we have  $(1 - x)^p \leq 1 - x$  for  $p \geq 1$ , and the reverse for  $0 \leq p \leq 1$ .

For greater accuracy, and for a different type of companion inequality, we can go on to the next term in the binomial expansion,  $\frac{1}{2}p(p - 1)x^2$ , applying the appropriate form of Taylor's theorem. The result is an inequality that reverses three times.

THEOREM 5. For  $x > 0$ , we have

$$(1+x)^p \geq 1+px + \frac{1}{2}p(p-1)x^2 \quad (12)$$

for  $p \geq 2$  and  $0 \leq p \leq 1$ . The reverse holds when  $1 \leq p \leq 2$  and when  $p < 0$ . All the inequalities reverse for  $-1 < x < 0$ .

*Proof.* Let  $f(x) = (1+x)^p$ . Since  $f(0) = 1$ ,  $f'(0) = p$  and  $f''(0) = p(p-1)$ , the first three terms of the Taylor series for  $f(x)$  are  $1+px + \frac{1}{2}p(p-1)x^2$ . Denote this by  $S_2(x)$ . By the Lagrange form of Taylor's theorem, for all  $x > -1$ ,

$$(1+x)^p = S_2(x) + \frac{1}{6}x^3 f^{(3)}(\xi) = S_2(x) + \frac{1}{6}c_p x^3 (1+\xi)^{p-3}$$

for some  $\xi$  between 0 and  $x$ , where  $c_p = p(p-1)(p-2)$ . Now  $(1+\xi)^{p-3} > 0$ , so if  $x > 0$ , then  $(1+x)^p \geq S_2(x)$  provided that  $c_p \geq 0$ . The cases in which this occurs are  $p \geq 2$  and  $0 \leq p \leq 1$ . Clearly, if  $-1 < x < 0$ , then all the inequalities reverse.  $\square$

Of course, equality holds at  $p = 0, 1, 2$ , the points of reversal. When  $0 < x < 1$  and  $p > -1$ , the same conclusions can be deduced from the ultimately alternating nature of the binomial series, but consideration of the various separate cases is quite laborious.

Statement (12) provides a companion inequality to (6) in each of the three cases when  $p < 2$ , and strengthens (6) when  $p > 2$ .

*Example 2.* For  $x > 0$ , we have

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq (1+x)^{1/2} \leq 1 + \frac{1}{2}x,$$

$$1 - \frac{1}{2}x \leq (1+x)^{-1/2} \leq 1 - \frac{1}{2}x + \frac{3}{8}x^2.$$

We mention that inequalities of this sort were applied extensively in [1].

*Example 3.* Taking  $x = 1$ , we see that  $2^p$  compares with  $1 + \frac{1}{2}p(p+1)$  in the way stated in Theorem 5; this is hardly trivial! Note that for  $1 \leq p \leq 2$  and  $0 \leq x \leq 1$ , (12) implies that  $(1+x)^p \leq 1 + \frac{1}{2}p(p+1)x$ , another inequality of the same type as (10). However, by the inequality just stated, the bound given by (10) is better.

We leave it to the reader to write out the corresponding statement to (12) for  $(a+b)^p$ .

We are now ready to come back to our original problem, to establish inequalities for the expression  $G_p(a, b)$  defined in (3). The corresponding non-homogeneous function is

$$g_p(x) = (1+x)^p + (1-x)^p,$$

defined for  $|x| \leq 1$  (or  $|x| < 1$  when  $p < 0$ ). Since  $g_p(-x) = g_p(x)$ , it is enough to consider  $x \geq 0$ . Clearly,  $g_p(0) = 2$ ,  $g_p(1) = 2^p$  and  $g'_p(0) = 0$ . Identities (1) and (2) equate to  $g_1(x) = 2$  and  $g_2(x) = 2 + 2x^2$ ; note further that  $g_3(x) = 2 + 6x^2$ .

Given our earlier results, the first of our two comparisons is very easy:

**THEOREM 6.** *Let  $p \geq 1$ . Then*

$$2 \leq g_p(x) \leq 2^p \quad \text{for } 0 \leq x \leq 1, \quad (13)$$

$$2a^p \leq G_p(a, b) \leq 2^p a^p \quad \text{for } a \geq b \geq 0. \quad (14)$$

*Reverse inequalities hold for  $0 \leq p \leq 1$ . The left-hand inequalities also hold for  $p < 0$ .*

*Proof.* The results comparing  $G_p(a, b)$  with  $2a^p$  follow from (8) and (9). Also, by (4) we have  $(a+b)^p + (a-b)^p \leq (2a)^p$  for  $p \geq 1$ . Alternatively, one can show by differentiation that  $g_p(x)$  is increasing or decreasing, as appropriate, to obtain (13).  $\square$

The case  $p = 1$  reproduces our original identity (1).

Note that when  $p < 0$ ,  $g_p(x)$  is not bounded above on  $[0, 1)$ , since it tends to infinity when  $x \rightarrow 1^-$ .

We now work towards the second comparison, but we start with a result more analogous to (6) and (8). Since  $g'_p(0) = 0$ , it makes sense to compare  $g_p(x)$  with expressions of the type  $1 + cx^2$ .

**LEMMA 1.** *Let  $0 \leq x \leq 1$ . Then*

$$(1+x)^p - (1-x)^p \geq 2x \quad (15)$$

*for  $p \geq 1$ , and the reverse holds for  $0 < p \leq 1$ .*

*Proof.* By (E1) and (E2), for  $p \geq 1$  we have  $(1+x)^p \geq 1+x$  and  $(1-x)^p \leq 1-x$ , hence (15). The reverse holds if  $0 < p \leq 1$ .  $\square$

**THEOREM 7.** *For  $p \geq 2$ , we have*

$$g_p(x) \geq 2 + px^2 \quad \text{for } 0 \leq x \leq 1, \quad (16)$$

$$G_p(a, b) \geq 2a^p + pa^{p-2}b^2 \quad \text{for } a \geq b \geq 0. \quad (17)$$

*Reverse inequalities hold for  $1 \leq p \leq 2$ .*

*Proof.* Let  $h_p(x) = g_p(x) - px^2$ . Then

$$h'_p(x) = p[(1+x)^{p-1} - (1-x)^{p-1} - 2x].$$

If  $p \geq 2$ , then, by (15),  $h'_p(x) \geq 0$ , so  $h_p(x)$  is increasing, on  $[0, 1]$ . So  $h_p(x) \geq h_p(0) = 2$  for  $0 \leq x \leq 1$ . The reverse holds if  $1 \leq p \leq 2$ . Substitute  $x = b/a$  for (17).  $\square$

Note that we already know from (13) that  $g_p(x) \leq 2$  for  $0 < p \leq 1$  and  $g_p(x) \geq 2$  for  $p < 0$ : nothing would be gained by replacing 2 by  $2 + px^2$  in these cases.

(17) is an effective inequality for  $G_p(a, b)$ , and reproduces our original (2) when  $p = 2$ , but we have yet to establish a result analogous to (4), comparing  $G_p(a, b)$  with  $a^p + b^p$ . To achieve this, we replace (16) by the following weaker variant:

COROLLARY. *Let  $0 \leq x \leq 1$ . Then*

$$g_p(x) \geq 2 + 2x^p \tag{18}$$

for  $p \geq 2$ , and the reverse holds for  $1 \leq p \leq 2$ .

*Proof.* When  $p \geq 2$ , we have  $x^2 \geq x^p$ , hence  $px^2 \geq 2x^p$ , for  $0 \leq x \leq 1$ . The reverse holds when  $1 \leq p \leq 2$ .  $\square$

The required comparison now follows in elegant style:

THEOREM 8. *Let  $a \geq b > 0$ . Then*

$$2(a^p + b^p) \leq G_p(a, b) \leq 2^{p-1}(a^p + b^p) \tag{19}$$

for  $p \geq 2$ , and the reverse holds for  $1 < p \leq 2$ .

*Proof.* The left-hand inequality is obtained by substituting  $x = b/a$  in (18). In a rather neat way, the left-hand inequality implies the right-hand one: taking  $a = c + d$  and  $b = c - d$ , we obtain (for  $p \geq 2$ )

$$2(c + d)^p + 2(c - d)^p \leq (2c)^p + (2d)^p,$$

which equates to the right-hand inequality.  $\square$

The left-hand inequality is attained when  $b = 0$ , and the right-hand inequality when  $b = a$ .

We could have stated (19) (reversed) for  $0 < p \leq 1$ , but better bounds are given by (14). Conversely, (19) gives better bounds than (14) when  $p > 2$ . When  $1 < p < 2$ , either can be better. For example, if  $b = 0$ , then (14) gives the exact lower bound, while the reversed (19) doesn't, and (19) gives the exact upper bound, while (14) doesn't.

The comparison with  $2 + px^2$  has served us well, but it still leaves unfinished business in the estimation of  $g_p(x)$ . The second term in the combined binomial series for  $g_p(x)$  is

$p(p-1)x^2$ , not  $px^2$ . We can apply Taylor's theorem as before to obtain comparisons involving this term. The next non-zero term is the one involving the fourth derivative, which results in an inequality that reverses four times.

**THEOREM 9.** *For  $0 \leq x < 1$ , we have*

$$g_p(x) \geq 2 + p(p-1)x^2 \tag{20}$$

for  $p \geq 3$ ,  $1 \leq p \leq 2$  and  $p < 0$ . The reverse holds for  $2 \leq p \leq 3$  and  $0 \leq p \leq 1$ .

*Proof.* We have  $g'_p(0) = g_p^{(3)}(0) = 0$ ,  $g''_p(0) = p(p-1)$  and  $g_p^{(4)}(x) = c_p g_{p-4}(x)$ , where  $c_p = p(p-1)(p-2)(p-3)$ . So by the Lagrange form of Taylor's theorem,

$$g_p(x) = 2 + p(p-1)x^2 + \frac{1}{4!}x^4 c_p g_{p-4}(\xi)$$

for some  $\xi$  in  $(0, x)$ . Now  $g_{p-4}(\xi) > 0$ , so  $g_p(x) \geq 2 + p(p-1)x^2$  in the cases when  $c_p \geq 0$ . Clearly, these cases are  $p \geq 3$ ,  $1 \leq p \leq 2$  and  $p < 0$ .  $\square$

Careful scrutiny will show that (20) strengthens the bounds from (16) and (13) in the cases when  $p > 3$  and  $p < 1$ , and provides companion ones when  $1 < p < 3$ .

*Example 4.* Choosing from the repertoire of bounds at our disposal, we can state:

$$2 + \frac{3}{4}x^2 \leq g_{3/2}(x) \leq 2 + \frac{3}{2}x^2, \quad \sqrt{2} \leq g_{1/2}(x) \leq 2 - \frac{1}{4}x^2, \quad g_{-1/2}(x) \geq 2 + \frac{3}{4}x^2.$$

These investigations can be continued in many ways. One can use the Taylor expansion to compare  $(1+x)^p - (1-x)^p$  with  $2px$ : the reader might care to check that the difference is positive or negative according to the sign of  $p(p-1)(p-2)$ . Or one can restore symmetry in (8) and (11), and look for the best constants on both sides when  $(a+b)^p - a^p - b^p$  is compared with  $a^{p-1}b + ab^{p-1}$ . This was done by the author in [2]: it turns out that different pairs of bounds apply on four intervals for  $p$ , separated by the points 1, 2 and 3.

### References

1. G.J.O. Jameson, An approximation to the arithmetic-geometric mean, *Math. Gaz.* **98** (2014).
2. G.J.O. Jameson, Inequalities comparing  $(a+b)^p - a^p - b^p$  and  $a^{p-1}b + ab^{p-1}$ , *Elemente Math.* **68** (2013), 1–6.