

THE INGHAM-HUXLEY ZERO-DENSITY THEOREM

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Introduction

We show that for $\sigma \geq \frac{1}{2}$, $T \geq 2$, the number of zeros (counting multiplicity) $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\sigma \leq \beta \leq 1$ and $0 < \gamma \leq T$ is

$$N(\sigma, T) \ll T^{A(1-\sigma)} \log^c T$$

where

$$A = \frac{12}{5}.$$

As far as I am aware, this is still the smallest known (constant) allowable value of A . The ‘density hypothesis’ is that one may take $A = 2$. This would follow from the Lindelöf hypothesis (see for example Titchmarsh) or from Montgomery’s large values conjecture (see for example Ben Green’s notes [Grn]).

A particular consequence of the zero-density theorem is

$$\pi(x + x^{1-\frac{1}{A}+\epsilon}) - \pi(x) \sim \frac{x^{1-\frac{1}{A}+\epsilon}}{\log x}.$$

So we have this with $1 - \frac{1}{A} = \frac{7}{12}$. (It is conjectured to hold with $A = 1$ but even on RH we can’t get any better than $A = 2$. The first result like this was Hoheisel’s $A = 33000$.) The proof of this uses the Landau formula (truncated explicit formula for $\psi(x)$) and also requires a zero-free region slightly stronger than Weyl’s (which in turn is slightly stronger than de la Vallée-Poussin’s). As far as I can tell all known proofs of a sufficiently strong zero-free region are at least as hard as Vinogradov-Korobov.

The original zero-density theorem was $N(\sigma, T) \ll_{\sigma} T$ ($\sigma > \frac{1}{2}$), which was proved by Bohr and Landau (see Edwards or Titchmarsh). I think for certain bogus ‘zeta functions’ (not possessing Euler products and not satisfying RH) it has been shown that you can’t do much (any?) better than this.

By following the same methods as used here one can make use of results like

$$\int_T^{T+T^{\frac{2}{3}}} |\zeta(\frac{1}{2} + it)|^4 dt \ll T^{\frac{2}{3}+\epsilon}$$

(Iwaniec – deep; previously Heath-Brown had this with $\frac{7}{8}$ in place of $\frac{2}{3}$) to bound $N(\sigma, T + \Delta) - N(\sigma, T)$. Backlund showed that the Lindelöf hypothesis is equivalent to this quantity (with $\Delta = 1$) being $o(\log T)$ for $\sigma \geq \text{const} > 1$ (see Edwards or Titchmarsh).

Ingham published his result (giving $A = 3$) in 1940 [Ing2], Huxley in 1972. Huxley obtained his result (actually generalised to L -functions) whilst preparing his book [Hux1]. His corresponding paper [Hux2] gave a stronger result than the book when $\sigma > \frac{3}{4}$.

The basic prerequisites for the present exposition are $N(T+1) - N(T) \ll \log T$, the simplest van der Corput type exponential sum estimates and the approximate functional equation. Some boring details are omitted, including the smallness of various quantities using crude bounds for the gamma function. We shan't concern ourselves with trying to make c as small as possible.

Large values theorems for Dirichlet polynomials

First we derive a bound of a type due to Davenport using a couple of lemmas of Gallagher [Gall] (mean values and mean-to-max). Better constants could be obtained by the methods of Montgomery and Vaughan or Selberg. Suppose $m \neq n \Rightarrow |\lambda_m - \lambda_n| \geq \delta$. For $x \leq \frac{1}{2}$, $|\sin \pi x| \geq 2|x|$ so $\text{sinc } x \geq 2/\pi$. Write $\Lambda(x) = \max(0, 1 - |x|)$.

$$\begin{aligned} \int_{U-T/2}^{U+T/2} \left| \sum_{n=1}^N a_n e(\lambda_n t) \right|^2 dt &\leq \frac{\pi^2}{4} \int_{-\infty}^{\infty} \left| \sum_{n=1}^N a_n e(\lambda_n(U+t)) \right|^2 \text{sinc}^2 \frac{t}{T} dt \\ &= \frac{\pi^2}{4} \sum_{m,n} a_m \bar{a}_n e((\lambda_m - \lambda_n)U) \int_{-\infty}^{\infty} e((\lambda_m - \lambda_n)t) \text{sinc}^2 \frac{t}{T} dt \\ &= \frac{\pi^2}{4} \sum_{m,n} a_m \bar{a}_n e((\lambda_m - \lambda_n)U) T \Lambda((\lambda_m - \lambda_n)T). \end{aligned}$$

If $\delta T \geq 1$ then the RHS is $(\pi^2/4)T \sum_n |a_n|^2$. But if $T < 1/\delta$ then we can extend the interval of integration to one of length $1/\delta$. Thus in any case we have

$$\int_{U-T/2}^{U+T/2} \left| \sum_{n=1}^N a_n e(\lambda_n t) \right|^2 dt \leq \frac{\pi^2}{4} \max\left(T, \frac{1}{\delta}\right) \sum_{n=1}^N |a_n|^2.$$

In the case $\lambda_n = (\log n)/(2\pi)$, $e(\lambda_n t) = n^{it}$ we have (for $N \geq 2$) $2\pi\delta = \log(N/(N-1)) \geq 1/N$ so for

$$S(t) = \sum_{n=1}^N a_n n^{it}$$

we get

$$\int_{T_1}^{T_2} |S(t)|^2 dt \ll (T_2 - T_1 + N) \sum_n |a_n|^2.$$

We now deduce a discrete version of this.

$$\int_0^{\frac{1}{2}} (x - \frac{1}{2}) f'(x) dx = \left[(x - \frac{1}{2}) f(x) \right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f = \frac{1}{2} f(0) - \int_0^{\frac{1}{2}} f$$

$$\int_{-\frac{1}{2}}^0 (x + \frac{1}{2})f'(x)dx = [(x + \frac{1}{2})f(x)]_{-\frac{1}{2}}^0 - \int_{-\frac{1}{2}}^0 f = \frac{1}{2}f(0) - \int_{-\frac{1}{2}}^0 f$$

$$f(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f + \int_{-\frac{1}{2}}^0 (x + \frac{1}{2})f'(x)dx + \int_0^{\frac{1}{2}} (x - \frac{1}{2})f'(x)dx$$

$$|f(0)| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f| + \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f'|$$

Suppose $q \neq r \Rightarrow 1 \leq |t_q - t_r| \leq T$, say $T_1 + \frac{1}{2} \leq t_r \leq T_2 - \frac{1}{2}$ with $T_2 - T_1 = T + 1$. Then

$$\sum_r |f(t_r)| \leq \int_{-T_1}^{T_2} |f| + \frac{1}{2} \int_{-T_1}^{T_2} |f'|$$

With $f = S^2$,

$$\begin{aligned} \sum_r |S(t_r)|^2 &\leq \int_{-T_1}^{T_2} |S|^2 + \int_{-T_1}^{T_2} |SS'| \\ &\leq \int_{-T_1}^{T_2} |S|^2 + \left(\int_{-T_1}^{T_2} |S|^2 \right)^{\frac{1}{2}} \left(\int_{-T_1}^{T_2} |S'|^2 \right)^{\frac{1}{2}} \\ &\ll \left(1 + 1^{\frac{1}{2}} (\log^2 N)^{\frac{1}{2}} \right) (T + N) \sum_n |a_n|^2 \\ &= (T + N) (\log eN) \sum_n |a_n|^2 \end{aligned}$$

Results like this are called ‘large values’ theorem. The idea is that we can use such a result to give a bound for R when we assume $S(t_r) \geq V$, i.e. we can count the number of times $S(t)$ is ‘large’. We could have carried on to the discrete case retaining the generality of λ_n . The case of $\lambda_n = n$, $T = 1$ and the t_r ϵ -spaced with ϵ small is the ‘large sieve’, and sometimes people still call it ‘large sieve’ when it is a more general statement.

Now suppose the t_r are as above and $s_r = \sigma_r + t_r$ with $0 \leq \sigma_r \leq \frac{1}{2}$. For $1 \leq U \leq x \leq 2U$, write

$$A(x, t) = \sum_{U < n \leq x} a_n n^{-it}$$

Then

$$\begin{aligned} \sum_{U < n \leq 2U} a_n n^{-\sigma-it} &= \int_{U^+}^{2U^+} x^{-\sigma} dA(x, t) \\ &= (2U)^{-\sigma} A(2U, t) + \int_U^{2U} \sigma x^{-\sigma-1} A(x, t) dx \end{aligned}$$

$$\left| \sum_{U < n \leq 2U} a_n n^{-\sigma-it} \right| \leq |A(2U, t)| + \frac{1}{2U} \int_U^{2U} |A(x, t)| dx$$

$$\begin{aligned}
\sum_r \left| \sum_{U < n \leq 2U} a_n n^{-s_r} \right| &\leq \sum_r |A(2U, t_r)| + \frac{1}{2U} \int_U^{2U} \sum_r |A(x, t_r)| dx \\
&\ll \max_{U < x \leq 2U} \sum_r |A(x, t_r)| \\
&\leq \max_{U < x \leq 2U} \left(R \sum_r |A(x, t_r)|^2 \right)^{\frac{1}{2}} \\
&\ll \left(R(T+U)(\log 2U) \sum_{U < n \leq 2U} |a_n|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

The situation has been simplified by assuming an upper bound on the σ_r and by the range being $u \sim U$ instead of $u \leq U$.

Now we derive a bound of a different shape (depending in an essential way on R) which we will call the Halász-Montgomery-Huxley large values theorem (HMH). This proof is essentially due to Graham [Gra] (for the bound on the sum over r) and Huxley (for the bound on R). By duality,

$$\sum_r \left| \sum_n a_n n^{-s_r} \right|^2 \leq \Delta \sum_n |a_n|^2$$

is equivalent to

$$\sum_n \left| \sum_r y_r n^{-s_r} \right|^2 \leq \Delta \sum_r |y_r|^2,$$

the LHS of which is

$$\begin{aligned}
\sum_{q,r} y_q \bar{y}_r \sum_n n^{-\sigma_q - \sigma_r - it_q + it_r} &\leq \sum_{q,r} \frac{1}{2} (|y_q|^2 + |y_r|^2) \left| \sum_n n^{-\sigma_q - \sigma_r - it_q + it_r} \right| \\
&= \sum_r |y_r|^2 \sum_q \left| \sum_n n^{-\sigma_q - \sigma_r - it_q + it_r} \right| \\
&\leq \left(\sum_r |y_r|^2 \right) \max_r \sum_q \left| \sum_n n^{-\sigma_q - \sigma_r - it_q + it_r} \right|.
\end{aligned}$$

Now for $t \geq 1$, $1 \leq U \leq x \leq 2U$ we have

$$\sum_{U < n \leq x} n^{it} \ll \begin{cases} \frac{U}{t^{\frac{1}{2}}} & (\text{for } U \geq Ct \text{ where } C > 1/(2\pi), \text{ by Kusmin-Landau}) \\ t^{\frac{1}{2}} & (\text{for } U \ll t, \text{ by the exponent pair } (\frac{1}{2}, \frac{1}{2})). \end{cases}$$

The same bound holds for $\sum_n n^{-s}$ (where $0 \leq \sigma \leq 1$) by partial summation ($a_n = 1$ in an earlier identity). Then by summing over $U = N/2, N/4, N/8, \dots$ we see that

$$\sum_{n=1}^N n^{-s} \ll t^{\frac{1}{2}} \log 2N + \left(\frac{N}{t} \text{ if } N \gg t \right).$$

Thus we may take

$$\Delta \asymp N \text{ (from } q = r) + RT^{\frac{1}{2}} \log 2N + \max_r \sum_{\substack{q \\ 1 \leq |t_q - t_r| \ll N}} \frac{N}{|t_q - t_r|} \ll (N + RT^{\frac{1}{2}})(\log 2N).$$

Now suppose that

$$\left| \sum_n a_n n^{-s_r} \right| \geq V$$

for each r . Write $L = \log 2N$ and $G = \sum_n |a_n|^2$. Then (for some constant C)

$$RV^2 \leq C(N + RT^{\frac{1}{2}})LG,$$

$$R(V^2 - CT^{\frac{1}{2}}GL) \leq CGNL.$$

If $V^2 \leq 2CT^{\frac{1}{2}}GL$ then the LHS is $\geq V^2/2$ so $R \ll GNV^{-2}L$. If however $V^2 > 2CT^{\frac{1}{2}}GL$, then define T_0 by $V^2 = 2CT_0^{\frac{1}{2}}GL$ and split the interval for the t_r into $\leq (1+T/T_0)$ subintervals of length $\leq T_0$. Applying the above to each subinterval and adding up then gives (in all cases)

$$\begin{aligned} R &\ll \left(1 + \frac{T}{T_0}\right) GNV^{-2}L \\ &\asymp \left(1 + T \frac{G^2 L^2}{V^4}\right) GNV^{-2}L \\ &= GNV^{-2}L + G^3 NTV^{-6}L^3. \end{aligned}$$

If the range for n is $U < n \leq 2U$ instead of $n \leq N$ then the above is simplified and we get $\sum_r |\sum_n a_n n^{-s_r}|^2 \ll (UL + RT^{\frac{1}{2}})G$ and $R \ll GNV^{-2}L + G^3 NTV^{-6}L$, where L now means $\log 2U$. (In the application this has not been taken into account yet.)

Montgomery's conjecture (modified to cope with Bourgain's counterexample to the original version) is

$$\sum_r \left| \sum_n a_n n^{-s_r} \right|^2 \ll (N + R)NT^\epsilon \max_n |a_n|^2.$$

Discrete fourth power moment

We prove this by an averaging technique from [Hux1]. See also my notes [Jam]. Let $C = 1/(2\pi)$. Then for $t \asymp T$, $xy = Ct$, $x \asymp y \asymp \sqrt{T}$, the approximate functional equation says

$$\zeta\left(\frac{1}{2} + it\right) = S(t, x) + \chi\left(\frac{1}{2} + it\right) \overline{S(t, y)} + O(T^{-\frac{1}{4}})$$

where

$$S(t, x) = \sum_{n \leq x} n^{-\frac{1}{2}-it}.$$

(Actually the very crude error term $O((\log T)^{\frac{5}{4}})$ in place of $O(T^{-\frac{1}{4}})$ would suffice.) So

$$\zeta(\frac{1}{2} + it)^4 \ll |S(t, x)|^4 + |S(t, y)|^4 + T^{-1},$$

$$\begin{aligned} \zeta(\frac{1}{2} + it)^4 &\ll \frac{1}{\sqrt{T}} \int_{\sqrt{T}}^{2\sqrt{T}} (|S(t, x)|^4 + |S(t, Ct/x)|^4) dx + T^{-1} \\ &\ll \frac{1}{\sqrt{T}} \int_{A\sqrt{T}}^{B\sqrt{T}} |S(t, x)|^4 dx + T^{-1} \end{aligned}$$

for suitable $B > A > 0$, since

$$\begin{aligned} \int_{\sqrt{T}}^{2\sqrt{T}} |S(t, Ct/x)|^4 dx &= \int_{Ct/(2\sqrt{T})}^{Ct/\sqrt{T}} |S(t, y)|^4 \frac{Ct}{y^2} dy \\ &\ll \int_{C\sqrt{T}/2}^{2C\sqrt{T}} |S(t, y)|^4 dy. \end{aligned}$$

So for a 1-spaced set of values $t_1, \dots, t_R \asymp T$ (necessarily with $R \ll T$) we have

$$\sum_r |\zeta(\frac{1}{2} + it_r)|^4 \ll \frac{1}{\sqrt{T}} \int_{A\sqrt{T}}^{B\sqrt{T}} \sum_r |S(t_r, x)|^4 dx + 1$$

Now

$$S(t, x)^2 = \sum_{n \leq x^2} a(n, x) n^{-\frac{1}{2}-it}$$

where

$$a(n, x) = \sum_{\substack{de=n \\ d, e \leq x}} 1 \leq d(n)$$

so

$$\begin{aligned} \sum_r |S(t_r, x)|^4 &\ll (T + x^2)(\log T) \sum_{n \leq x^2} \frac{d(n)^2}{n} \\ &\ll T \log^5 T \end{aligned}$$

and hence

$$\sum_r |\zeta(\frac{1}{2} + it_r)|^4 \ll T \log^5 T$$

By summing over $T, T/2, T/4, \dots$ the same result holds for $0 < t_1, \dots, t_R \leq T$.

Zero counting mechanism

$\zeta(\rho) = 0$, $\rho = \beta + i\gamma$ with $\frac{1}{2} < \sigma \leq \beta < 1$, $0 < \gamma \leq T$. Using $N(T+1) - N(T) \ll \log T$, pick a representative set of such ρ with the γ something like $\log^2 T$ -spaced. X, Y (to be chosen) are between fixed powers of T .

$$M_X(s) = \sum_{d \leq X} \mu(d) d^{-s}$$

$$a_n = \sum_{\substack{d|n \\ d \leq X}} \mu(d)$$

$$1 < n \leq X \Rightarrow a_n = 0$$

$$|a_n| \leq d(n)$$

By the theory of Mellin transforms

$$e^{-1/Y} + \sum_{n > X} a_n n^{-\rho} e^{-n/Y} = M_X(1) \Gamma(1-\rho) Y^{1-\rho} + \frac{1}{2\pi i} \int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} \zeta(w+\rho) M_X(w+\rho) \Gamma(w) Y^w dw,$$

where a residue term has vanished since $\zeta(\rho) = 0$. Since $e^{-1/Y}$ is about 1, all ρ (except possibly a few stupid ones) are of class (i) or class (ii) where class (i) zeros satisfy

$$\left| \sum_{X < n \leq Y \log Y} a_n n^{-\rho} e^{-n/Y} \right| \gg 1,$$

and class (ii) zeros satisfy

$$\left| \int_{\frac{1}{2}-\beta-i\log^2 T}^{\frac{1}{2}-\beta+i\log^2 T} \zeta(w+\rho) M_X(w+\rho) \Gamma(w) Y^w dw \right| \gg 1.$$

Put $w = \frac{1}{2} - \beta + i\tau$, $w + \rho = \frac{1}{2} + i\gamma + i\tau$. The integral above has modulus

$$\begin{aligned} &\leq \left(\int_{-\infty}^{\infty} |\Gamma(\frac{1}{2} - \beta + i\tau)| d\tau \right) Y^{\frac{1}{2}-\beta} |\zeta(\frac{1}{2} + it_\rho) M_X(\frac{1}{2} + it_\rho)| \\ &\ll \left(\log \frac{1}{\sigma - \frac{1}{2}} \right) Y^{\frac{1}{2}-\sigma} |\zeta(\frac{1}{2} + it_\rho) M_X(\frac{1}{2} + it_\rho)| \end{aligned}$$

where $t_\rho \in [\gamma - \log^2 T, \gamma + \log^2 T]$ is chosen to maximise $|\zeta(\frac{1}{2} + it_\rho) M_X(\frac{1}{2} + it_\rho)|$, since for $-\frac{1}{2} < \alpha = \frac{1}{2} - \beta < 0$,

$$\int_{-\infty}^{\infty} |\Gamma(\alpha + i\tau)| d\tau \asymp \int_0^1 \frac{d\tau}{|\alpha| + \tau} = \log \left(1 + \frac{1}{|\alpha|} \right) \asymp \log \frac{1}{|\alpha|}.$$

Class (i) – Ingham

Split the range $X < n \leq Y \log Y$ into subranges $n \sim U$ with $U = 2^r X$. Write $L = \log T$. The number of class (i) zeros (not counting multiplicity) is

$$\begin{aligned}
W_1 &\ll \sum_{\rho} \left| \sum_{X < n \leq Y \log Y} a_n n^{-\rho} e^{-n/Y} \right| \\
&\leq \sum_U \sum_{\rho} \left| \sum_{n \sim U} a_n n^{-\rho} e^{-n/Y} \right| \\
&\ll \sum_U \left(W_1(T+U)L \sum_{n \sim U} d(n)^2 n^{-2\sigma} \right)^{\frac{1}{2}} \\
&\ll \sum_U (W_1(T+U)L \cdot UL^3 \cdot U^{-2\sigma})^{\frac{1}{2}} \\
&= \sum_U (W_1(TU^{1-2\sigma} + U^{2-2\sigma})L^4)^{\frac{1}{2}} \\
&= \sum_U (W_1(TX^{1-2\sigma} + (YL)^{2-2\sigma})L^4)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$\begin{aligned}
W_1 &\ll \left(\sum_U 1 \right)^2 (TX^{1-2\sigma} + Y^{2-2\sigma})L^5 \\
&\ll (TX^{1-2\sigma} + Y^{2-2\sigma})L^7
\end{aligned}$$

Class (i) – Huxley ($\sigma \geq \frac{3}{4}$)

Applying HMH with

$$G \ll \sum_{n \sim U} d(n)^2 n^{-2\sigma} \ll U^{1-2\sigma} L^3,$$

$N = 2U$, $V^{-1} \asymp L$ says that the number of ρ with

$$\left| \sum_{n \sim U} a_n n^{-\rho} e^{-n/Y} \right| \gg \frac{1}{L}$$

is

$$\begin{aligned}
&\ll U^{1-2\sigma} L^3 UL^2 L + (U^{1-2\sigma} L^3)^3 UTL^6 L^3 \\
&= U^{2-2\sigma} L^6 + TU^{4-6\sigma} L^{18} \\
&\ll Y^{2-2\sigma} L^{\frac{13}{2}} + TX^{4-6\sigma} L^{18}
\end{aligned}$$

W_1 is $\ll L \times$ this so

$$W_1 \ll (Y^{2-2\sigma} + TX^{4-6\sigma})L^{19}$$

Class (ii) – Ingham

Fix $p, a, b > 0$ (to be chosen) with $(1/a) + (1/b) = 1$. Then $W_2 \ll \sum_{\rho} 1^p$ and Hölder's inequality gives

$$\begin{aligned} \left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{-p} Y^{-p(\frac{1}{2}-\sigma)} W_2 &\ll \sum_{\rho} |\zeta(\frac{1}{2} + it_{\rho}) M_X(\frac{1}{2} + it_{\rho})|^p \\ &\leq \left(\sum_{\rho} |\zeta|^{pa}\right)^{\frac{1}{a}} \left(\sum_{\rho} |M_X|^{pb}\right)^{\frac{1}{b}} \end{aligned}$$

We choose the exponents so that $pa = 4, pb = 2$. Thus

$$\frac{p}{4} + \frac{p}{2} = 1, \quad p = \frac{4}{3}, \quad \frac{1}{a} = \frac{p}{4} = \frac{1}{3}, \quad \frac{1}{b} = \frac{p}{2} = \frac{2}{3}$$

and the above is

$$\begin{aligned} \left(\sum_{\rho} |\zeta|^4\right)^{\frac{1}{3}} \left(\sum_{\rho} |M_X|^2\right)^{\frac{2}{3}} &\ll (TL^5)^{\frac{1}{3}} \left((T+X)L \sum_{d \leq X} \frac{\mu^2(d)}{d}\right)^{\frac{2}{3}} \\ &\ll (T + T^{\frac{1}{3}} X^{\frac{2}{3}}) L^3 \end{aligned}$$

Hence (not counting multiplicity)

$$W_2 \ll \left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{\frac{4}{3}} Y^{\frac{2}{3}(1-2\sigma)} (T + T^{\frac{1}{3}} X^{\frac{2}{3}}) L^3$$

Class (ii) – Huxley ($\sigma \geq \frac{3}{4}$)

$$|\zeta(\frac{1}{2} + it_{\rho}) M_X(\frac{1}{2} + it_{\rho})| \gg Y^{\sigma - \frac{1}{2}} \Rightarrow |\zeta| \geq U \text{ or } |M_X| \geq V$$

where

$$UV \asymp Y^{\sigma - \frac{1}{2}}$$

The number of ρ with $|\zeta| \geq U$ is

$$\ll TU^{-4} L^5 \asymp TY^{2-4\sigma} V^4 L^5$$

Applying HMM with

$$G \ll \sum_{d \leq X} \frac{\mu^2(d)}{d} \ll L,$$

$N = X$, says that the number of ρ with $|M_X| \geq V$ is

$$\ll LXV^{-2} L + L^3 XTV^{-6} L^3 = XV^{-2} L^2 + TXV^{-6} L^6$$

Suppose we choose V so that $V^4 \ll T$. Then the second term dominates the first term. That is, we ignore the first term and choose V to minimise

$$TY^{2-4\sigma}V^4 + TXV^{-6},$$

that is

$$V^{10} = XY^{4\sigma-2}$$

giving

$$W_2 \ll TX(XY^{4\sigma-2})^{-\frac{3}{5}}L^c = TX^{\frac{2}{5}}Y^{\frac{3}{5}(2-4\sigma)}L^c$$

provided $V^4 \ll T$.

Parameters – Ingham

For σ close to $\frac{1}{2}$ we can just use $N(T) \ll TL$, so we may assume $\sigma \geq \text{const} > \frac{1}{2}$ and can ignore the $(-\log(\sigma - \frac{1}{2}))^{\frac{4}{3}}$ factor. Thus

$$N(\sigma, T) \ll (TX^{1-2\sigma} + Y^{2-2\sigma} + TY^{\frac{2}{3}(1-2\sigma)} + T^{\frac{1}{3}}X^{\frac{2}{3}}Y^{\frac{2}{3}(1-2\sigma)})L^c.$$

First choose Y to minimise the terms not involving X . Thus

$$Y^{2-2\sigma} = TY^{-\frac{2}{3}(2\sigma-1)},$$

$$Y = T^{1/(2-2\sigma+\frac{2}{3}(2\sigma-1))} = T^{3/(2-\sigma)},$$

giving

$$Y^{2-2\sigma} = T^{3(1-\sigma)/(2-\sigma)}.$$

Now we show that we can choose X so that the other terms are no worse than this. We have

$$\begin{aligned} TX^{-(2\sigma-1)} \leq TY^{-\frac{2}{3}(2\sigma-1)} &\Leftrightarrow Y^{\frac{2}{3}(2\sigma-1)} \leq X^{2\sigma-1} \\ &\Leftrightarrow Y^{\frac{2}{3}} \leq X \\ &\Leftrightarrow T^{1/(2-\sigma)} \leq X \end{aligned}$$

and

$$T^{\frac{1}{3}}X^{\frac{2}{3}}Y^{\frac{2}{3}(1-2\sigma)} \leq TY^{\frac{2}{3}(1-2\sigma)} \Leftrightarrow X \leq T,$$

so we are OK provided $1/(2-\sigma) \leq 1$, which is fine because $\sigma \leq 1$. We have now proved Ingham's theorem:

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} \log^c T.$$

Ingham had $c = 5$. Note that $3/(2-\sigma)$ is increasing but $3(1-\sigma)/(2-\sigma)$ is decreasing.

Parameters – Huxley ($\sigma \geq \frac{3}{4}$)

Provided $V^4 \ll T$ (see above), we have

$$N(\sigma, T) \ll (Y^{2-2\sigma} + TX^{4-6\sigma} + TX^{\frac{2}{5}}Y^{\frac{3}{5}(2-4\sigma)})L^c$$

Given Y , we choose X to minimise the last two terms, so

$$\begin{aligned} X^{\frac{2}{5}+6\sigma-4} = Y^{\frac{3}{5}(4\sigma-2)} &\Leftrightarrow X^{30\sigma-18} = Y^{12\sigma-6} \\ &\Leftrightarrow X^{5\sigma-3} = Y^{2\sigma-1}, \end{aligned}$$

giving

$$TX^{4-6\sigma} = TY^{-(6\sigma-4)(2\sigma-1)/(5\sigma-3)},$$

so that we choose

$$\begin{aligned} Y &= T^{1/(2-2\sigma+(2\sigma-1)(6\sigma-4)/(5\sigma-3))} \\ &= T^{(5\sigma-3)/((2-2\sigma)(5\sigma-3)+(2\sigma-1)(6\sigma-4))} \\ &= T^{(5\sigma-3)/(10\sigma-6-10\sigma^2+6\sigma+12\sigma^2-8\sigma-6\sigma+4)} \\ &= T^{(5\sigma-3)/(2\sigma^2+2\sigma-2)}, \end{aligned}$$

giving

$$Y^{2-2\sigma} = T^{(5\sigma-3)(1-\sigma)/(\sigma^2+\sigma-1)}.$$

Now we check

$$\begin{aligned} V^4 \leq T &\Leftrightarrow V^{10} \leq T^{\frac{5}{2}} \\ &\Leftrightarrow XY^{4\sigma-2} \leq T^{\frac{5}{2}} \\ &\Leftrightarrow \frac{(2\sigma-1) + (4\sigma-2)(5\sigma-3)}{2\sigma^2+2\sigma-2} \leq \frac{5}{2} \\ &\Leftrightarrow 2\sigma-1 + (4\sigma-2)(5\sigma-3) \leq 5(\sigma^2+\sigma-1) \\ &\Leftrightarrow 2\sigma-1 + 20\sigma^2 - 12\sigma - 10\sigma + 6 \leq 5(\sigma^2+\sigma-1) \\ &\Leftrightarrow 20\sigma^2 - 20\sigma + 5 \leq 5(\sigma^2+\sigma-1) \\ &\Leftrightarrow 4\sigma^2 - 4\sigma + 1 \leq \sigma^2 + \sigma - 1 \\ &\Leftrightarrow 3\sigma^2 - 5\sigma + 2 \leq 0 \\ &\Leftrightarrow \left(\sigma - \frac{5}{6}\right)^2 \leq \frac{25}{36} - \frac{2}{3} = \frac{1}{36} \\ &\Leftrightarrow \left|\sigma - \frac{5}{6}\right| \leq \frac{1}{6} \\ &\Leftrightarrow \frac{2}{3} \leq \sigma \leq 1, \end{aligned}$$

which is OK. So we have proved (as in Huxley's book [Hux1])

$$N(\sigma, T) \ll T^{(5\sigma-3)(1-\sigma)/(\sigma^2+\sigma-1)} \log^c T \quad \left(\frac{3}{4} \leq \sigma \leq 1\right).$$

We leave showing that $(5\sigma-3)/(\sigma^2+\sigma-1)$ is decreasing on at least the interval in question as another tedious exercise. [Note added by Graham Jameson: this is easy. Write $\sigma^2 + \sigma - 1 = (\sigma - \sigma_1)(\sigma - \sigma_2)$, where $\sigma_1 = \frac{1}{2}(\sqrt{5} - 1) > \frac{3}{5}$ and $\sigma_2 < 0$. Then the expression is $5/(\sigma - \sigma_2) + \delta/(\sigma^2 + \sigma - 1)$, where $\delta > 0$.] We've said all this for $\sigma \geq \frac{3}{4}$, but in fact it would all work for $\sigma \geq \frac{2}{3}$ or something. However we see that Ingham's result is better for $\sigma < \frac{3}{4}$ since $3/(2-\sigma) = (5\sigma-3)/(\sigma^2+\sigma-1) = \frac{12}{5}$ when $\sigma = \frac{3}{4}$, which gives our main statement.

By an elaboration of the argument, Huxley's paper [Hux2] shows that

$$N(\sigma, T) \ll T^{3(1-\sigma)/(3\sigma-1)} \log^c T \quad \left(\frac{3}{4} \leq \sigma \leq 1\right),$$

which is superior for $\sigma > \frac{3}{4}$ and gives (better than) the density hypothesis for $\sigma \geq \frac{5}{6}$. although in fact one obtains a larger value of c here so that for our final result we would want to use the book one.

Further improvements (but only for $\sigma > \frac{3}{4}$) by people like Jutila are demonstrated in Ivić's book. These use things like Heath-Brown's result for the 12^{th} power moment

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{12} dt \ll T^2 \log^{17} T.$$

There are also delicate results (Selberg) for $\sigma = \sigma(T)$ close to $\frac{1}{2}$, whereas for σ near 1 one can use the known zero-free region.

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