

**Half-integer estimates for harmonic sums and the digamma function:
de Temple's method**

Notes by G.J.O. Jameson

Write H_n for the harmonic sum $\sum_{r=1}^n \frac{1}{r}$. The following well-known estimation can be established by Euler-Maclaurin summation, or by the logarithmic and binomial series:

$$H_n - \gamma = \log n + \frac{1}{2n} - \frac{1}{12n^2} + r(n), \quad (1)$$

where γ is Euler's constant and

$$0 \leq r(n) \leq \frac{1}{120n^4}.$$

Since $\log(n + \frac{1}{2}) = \log n + \frac{1}{2n} + O(\frac{1}{n^2})$, it is a natural idea to absorb the term $\frac{1}{2n}$ into the logarithmic term and compare $H_n - \gamma$ directly with $\log(n + \frac{1}{2})$. This was done by De Temple [DT]. His result (reproduced in [BM, p. 181–2]) states that

$$H_n - \gamma = \log(n + \frac{1}{2}) + r_1(n), \quad (2)$$

where

$$\frac{1}{24(n+1)^2} \leq r_1(n) \leq \frac{1}{24n^2}.$$

De Temple also stated without proof the following more accurate estimation:

$$H_n - \gamma = \log(n + \frac{1}{2}) + \frac{1}{24(n + \frac{1}{2})^2} - r_2(n), \quad (3)$$

where

$$\frac{7}{960(n+1)^4} \leq r_2(n) \leq \frac{7}{960n^4}.$$

We will prove both these estimations. Without any more effort, the same method applies to the more general sum

$$H_n(x) = \sum_{r=1}^n \frac{1}{r+x}.$$

With this notation, $H_n = H_n(0)$. We write $C(x) = \lim_{n \rightarrow \infty} [H_n(x) - \log n]$. The existence of the limit $C(x)$ follows from elementary integral estimation of the sum $H_n(x)$, but it will also be established in the proof to follow. Clearly, $C(0) = \gamma$. We will establish a variant of De Temple's estimations applying to $H_n(x) - C(x)$.

In fact, $C(x)$ can be expressed in terms of the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. From Euler's limit formula for the gamma function, we have $\psi(x+1) = \lim_{n \rightarrow \infty} \psi_n(x+1)$, where

$$\psi_n(x+1) = \log n - \sum_{r=1}^n \frac{1}{x+r} = \log n - H_n(x).$$

So in fact $C(x) = -\psi(x+1)$. The resulting estimations will equate to a variant of Stirling's formula.

Note that

$$\log(x + \frac{1}{2}) - \log(x - \frac{1}{2}) = \int_{x-1/2}^{x+1/2} \frac{1}{t} dt.$$

The mid-point approximation to this integral is $1/x$. The proofs are based on estimates (at various levels of accuracy) comparing $\log(x + \frac{1}{2}) - \log(x - \frac{1}{2})$ with $1/x$.

PROPOSITION 1. For $x > \frac{1}{2}$,

$$\log(x + \frac{1}{2}) - \log(x - \frac{1}{2}) = \frac{1}{x} + \delta_1(x), \quad (4)$$

where

$$\frac{1}{12x^3} \leq \delta_1(x) \leq \frac{1}{12(x - \frac{1}{2})^3}.$$

Proof. Let $\delta_1(x)$ be defined by (4). Then $\delta_1(x) \rightarrow 0$ as $x \rightarrow \infty$, so $\delta_1(x) = -\int_x^\infty \delta_1'(t) dt$.

Now

$$\begin{aligned} -\delta_1'(t) &= \frac{1}{t - \frac{1}{2}} - \frac{1}{t + \frac{1}{2}} - \frac{1}{t^2} \\ &= \frac{1}{t^2 - \frac{1}{4}} - \frac{1}{t^2} \\ &= \frac{1}{4t^2(t^2 - \frac{1}{4})} \\ &> \frac{1}{4t^4}. \end{aligned}$$

Hence

$$\delta_1(x) > \int_x^\infty \frac{1}{4t^4} dt = \frac{1}{12x^3}.$$

Also, $t^2(t^2 - \frac{1}{4}) > (t - \frac{1}{2})^4$, so

$$\delta_1(x) < \int_x^\infty \frac{1}{4(t - \frac{1}{2})^4} dt = \frac{1}{12(x - \frac{1}{2})^3}. \quad \square$$

THEOREM 1. Let $n \geq 1$ and $x > -1$. Let $C(x) = \lim_{n \rightarrow \infty} (H_n(x) - \log n)$. Then

$$H_n(x) - C(x) = \log(n + x + \frac{1}{2}) + r_1(n, x), \quad (5)$$

where

$$\frac{1}{24(n + x + 1)^2} \leq r_1(n, x) \leq \frac{1}{24(n + x)^2}.$$

In particular,

$$H_n - \gamma = \log(n + \frac{1}{2}) + r_1(n), \quad (6)$$

where

$$\frac{1}{24(n+1)^2} \leq r_1(n) \leq \frac{1}{24n^2}$$

Also, (5) holds with $n = 0$ for $x > 0$ (with $H_0(x) = 0$), so that

$$\psi(x+1) = \log(x + \frac{1}{2}) + r_1(x), \quad (7)$$

where

$$\frac{1}{24(x+1)^2} \leq r_1(x) \leq \frac{1}{24x^2}.$$

Proof. Apply (4) to $r+x$ for $1 \leq r \leq n$ and add: we obtain

$$\log(n+x+\frac{1}{2}) - \log(x+\frac{1}{2}) = H_n(x) + \sum_{r=1}^n \delta_1(r+x),$$

equivalently

$$H_n(x) - \log(n+x+\frac{1}{2}) = -\log(x+\frac{1}{2}) - \sum_{r=1}^n \delta_1(r+x). \quad (8)$$

Take the limit as $n \rightarrow \infty$. Clearly, $\log(n+x+\frac{1}{2}) - \log n \rightarrow 0$, so we obtain

$$C(x) = -\log(x+\frac{1}{2}) - \sum_{r=1}^{\infty} \delta_1(r+x). \quad (9)$$

Taking the difference between (8) and (9), we obtain

$$H_n(x) - \log(n+x+\frac{1}{2}) - C(x) = \sum_{r=n+1}^{\infty} \delta_1(r+x). \quad (10)$$

Denote this by $r_1(n, x)$. The condition $x > -1$ ensures that the inequality (4) applies to $\delta_1(r+x)$ for $r \geq 2$. By integral estimation, we now have for $n \geq 1$,

$$r_1(n, x) \geq \sum_{r=n+1}^{\infty} \frac{1}{12(r+x)^3} > \int_{n+1}^{\infty} \frac{1}{12(t+x)^3} dt = \frac{1}{24(n+x+1)^2}.$$

At the same time,

$$r_1(n, x) \leq \sum_{r=n+1}^{\infty} \frac{1}{12(r+x-\frac{1}{2})^3}.$$

The function $1/(t+x)^3$ is convex, and convex functions $h(t)$ satisfy $h(y-\frac{1}{2}) \leq \int_{y-1}^y h(t) dt$, hence

$$r_1(n, x) \leq \int_n^{\infty} \frac{1}{12(t+x)^3} dt = \frac{1}{24(n+x)^2}.$$

For $x > 0$, the case $n = 0$ follows similarly from (9) (note that (4) now applies also to $\delta_1(1+x)$). Since $C(x) = -\psi(x+1)$, this equates to (7). \square

Note. In (6) and (7), we have used the notation $r_1(n)$ for $r_1(n, 0)$ and $r_1(x)$ for $r_1(0, x)$.

The case $x = -\frac{1}{2}$ gives the following estimation for sums of odd reciprocals.

COROLLARY 1.1. *Let $U_n = \sum_{r=1}^n \frac{1}{2r-1}$. Then*

$$U_n - \frac{1}{2}\gamma - \log 2 = \frac{1}{2} \log n + \rho_1(n), \quad (11)$$

where

$$\frac{1}{48(n + \frac{1}{2})^2} \leq \rho_1(n) \leq \frac{1}{48(n - \frac{1}{2})^2}.$$

Proof. Note that $2U_n = H_n(-\frac{1}{2})$. Also, $2U_n = 2H_{2n} - H_n$; it follows easily that $2U_n - \log n \rightarrow \gamma + 2 \log 2$ as $n \rightarrow \infty$. The statement is the case $x = -\frac{1}{2}$ in Theorem 1. \square

Integration of (7) delivers the following variant of Stirling's formula:

COROLLARY 1.2. *For a certain constant c , we have*

$$\log \Gamma(x) = (x - \frac{1}{2}) \log(x - \frac{1}{2}) - x + c - P_1(x), \quad (12)$$

where

$$\frac{1}{24x} \leq P_1(x) \leq \frac{1}{24(x-1)}.$$

Proof. Write (7) in the form $\psi(t) = \log(t - \frac{1}{2}) + r_1(t - 1)$ and integrate on $[\frac{3}{2}, x]$ to obtain

$$\log \Gamma(x) - \log \Gamma(\frac{3}{2}) = (x - \frac{1}{2}) \log(x - \frac{1}{2}) - x + \frac{3}{2} + \int_{3/2}^x r_1(t - 1) dt.$$

Let $I = \int_{3/2}^{\infty} r_1(t - 1) dt$. Then (12) holds with $c = \log \Gamma(\frac{3}{2}) + \frac{3}{2} + I$ and

$$P_1(x) = \int_x^{\infty} r_1(t - 1) dt \leq \int_x^{\infty} \frac{1}{24(t-1)^2} dt = \frac{1}{24(x-1)},$$

and similarly $P_1(x) \geq 1/24x$. \square

As in most proofs of Stirling's formula, one now needs to invoke the Wallis product (or some equivalent) to evaluate c .

We now improve upon the estimate in Proposition 1, and derive the generalised version of (3).

PROPOSITION 2. *For $x > \frac{1}{2}$,*

$$\log(x + \frac{1}{2}) - \log(x - \frac{1}{2}) = \frac{1}{x} + \frac{1}{24} \left(\frac{1}{(x - \frac{1}{2})^2} - \frac{1}{(x + \frac{1}{2})^2} \right) - \delta_2(x), \quad (13)$$

where

$$\frac{7}{240x^5} \leq \delta_2(x) \leq \frac{7}{240(x - \frac{1}{2})^5}.$$

Proof. Use (13) as the definition of $\delta_2(x)$. Using the expression for $\delta_2'(t)$ from Proposition 1, we obtain

$$\begin{aligned} -\delta_2'(t) &= -\frac{1}{4t^2(t^2 - \frac{1}{4})} + \frac{1}{12(t - \frac{1}{2})^3} - \frac{1}{12(t + \frac{1}{2})^3} \\ &= \frac{G(t)}{12t^2(t^2 - \frac{1}{4})^3}, \end{aligned}$$

where

$$\begin{aligned} G(t) &= -3(t^2 - \frac{1}{4})^2 + t^2 \left((t^2 + \frac{1}{2})^3 - (t^2 - \frac{1}{2})^3 \right) \\ &= -3(t^4 - \frac{1}{2}t^2 + \frac{1}{16}) + t^2(3t^2 + \frac{1}{4}) \\ &= \frac{7}{4}t^2 - \frac{3}{16} \\ &= \frac{7}{4}(t^2 - \frac{3}{28}). \end{aligned}$$

Hence

$$-\delta_2'(t) > \frac{7}{48t^2(t^2 - \frac{1}{4})^2} > \frac{7}{48t^6}$$

and

$$\delta_2(x) = -\int_x^\infty \delta_2'(t) dt > \int_x^\infty \frac{7}{48t^6} dt = \frac{7}{240x^5}.$$

Also

$$-\delta_2'(t) < \frac{7}{48(t^2 - \frac{1}{4})^3} < \frac{7}{48(t - \frac{1}{2})^6},$$

so that

$$\delta_2(x) < \frac{7}{240(x - \frac{1}{2})^5}. \quad \square$$

THEOREM 2. *With the notation of Theorem 1, we have for $n \geq 1$ and $x > -1$,*

$$H_n(x) - C(x) = \log(n + x + \frac{1}{2}) + \frac{1}{24(n + x + \frac{1}{2})^2} - r_2(n, x), \quad (14)$$

where

$$\frac{7}{960(n + x + 1)^4} \leq r_2(n, x) \leq \frac{7}{960(n + x)^4}.$$

In particular,

$$H_n - \gamma = \log(n + \frac{1}{2}) + \frac{1}{24(n + \frac{1}{2})^2} - r_2(n), \quad (15)$$

where

$$\frac{7}{960(n + 1)^4} \leq r_2(n) \leq \frac{7}{960n^4}.$$

Also, for $x > 0$,

$$\psi(x+1) = \log(x + \frac{1}{2}) + \frac{1}{24(x + \frac{1}{2})^2} - r_2(x), \quad (16)$$

where

$$\frac{7}{960(x+1)^4} \leq r_2(x) \leq \frac{7}{960x^4}.$$

Proof. Modify the proof of Theorem 1 as follows. Applying (13) to $r+x$ for $1 \leq r \leq n$ and adding, we obtain

$$H_n(x) - \log(n+x+\frac{1}{2}) = -\log(x+\frac{1}{2}) - \frac{1}{24(x+\frac{1}{2})^2} + \frac{1}{24(n+x+\frac{1}{2})^2} + \sum_{r=1}^n \delta_2(r+x).$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$C(x) = -\log(x+\frac{1}{2}) - \frac{1}{24(x+\frac{1}{2})^2} + \sum_{r=1}^{\infty} \delta_2(r+x).$$

and taking the difference, we have

$$H_n(x) - \log(n+x+\frac{1}{2}) - C(x) = \frac{1}{24(n+x+\frac{1}{2})^2} - r_2(n,x),$$

where

$$r_2(n,x) = \sum_{r=n+1}^{\infty} \delta_2(r+x).$$

By (13) and integral estimation as before, we obtain the stated bounds for $r_2(n,x)$. \square

COROLLARY 2.1. *Let $U_n = \sum_{r=1}^n \frac{1}{2r-1}$. Then*

$$U_n - \frac{1}{2}\gamma - \log 2 = \frac{1}{2} \log n + \frac{1}{48n^2} - \rho_2(n), \quad (17)$$

where

$$\frac{7}{1920(n+\frac{1}{2})^4} \leq \rho_2(n) \leq \frac{7}{1920(n-\frac{1}{2})^4}. \quad \square$$

Note that (17) is considerably sharper than the estimate of the same type found by applying (1) to H_{2n} and H_n .

Reasoning as in Corollary 1.2, with (7) replaced by (16), we obtain:

COROLLARY 2.2. *With c as in Corollary 1.2, we have*

$$\log \Gamma(x) = (x - \frac{1}{2}) \log(x - \frac{1}{2}) - x + c - \frac{1}{24(x - \frac{1}{2})} + P_2(x), \quad (18)$$

where

$$\frac{7}{2880x^3} \leq P_2(x) \leq \frac{7}{2880(x-1)^3}. \quad \square$$

We derive a lower bound of a rather different type, which implies one given in [Ch]. For simplicity, we only state it for H_n instead of $H_n(x)$.

COROLLARY 2.3. *For all $n \geq 1$,*

$$H_n - \gamma \geq \log\left(n + \frac{1}{2}\right) + \frac{1}{24\left(n + \frac{1}{2} + \frac{1}{4n}\right)^2}. \quad (19)$$

Proof. By elementary algebra, for $a, b > 0$,

$$\frac{1}{a^2} - \frac{1}{(a+b)^2} > \frac{2b}{(a+b)^3}.$$

Hence for $n \geq 2$ (with $c \leq 1$ to be chosen),

$$\frac{1}{\left(n + \frac{1}{2}\right)^2} - \frac{1}{\left(n + \frac{1}{2} + \frac{c}{n}\right)^2} > \frac{2c}{n(n+1)^3}.$$

To derive (19) from (15), we require this to be not less than $7/(40n^4)$, which equates to $[(n+1)/n]^3 \leq \frac{80}{7}c$. With $c = \frac{1}{4}$, this holds for $n \geq 3$, and one can check that (19) also holds for $n = 1$ and $n = 2$. \square

Chen actually gives the lower bound $1/[24(n+a)^2]$, where $a \approx 0.55106$, and upper bound $1/[24(n + \frac{1}{2})^2]$.

Negoi [Neg] demonstrated that the n^{-2} term in (15) can be absorbed into the log term by considering $\log h(n)$, where $h(n) = n + \frac{1}{2} + \frac{1}{24n}$. His proof was by another variation of De Temple's method, entailing a rather laborious calculation. Here we show how to deduce his result (in fact, a slightly stronger one) from Theorem 2.

COROLLARY 2.4. *Let $h(n) = n + \frac{1}{2} + \frac{1}{24n}$. Then*

$$H_n - \gamma = \log h(n) - \rho(n), \quad (20)$$

where

$$\frac{1}{48(n+1)^3} \leq \rho(n) \leq \frac{1}{48\left(n + \frac{1}{2}\right)^3}.$$

(Negoi's statement has $\rho(n) \leq 1/(48n^3)$.)

Proof. Write $n + \frac{1}{2} = m$. By Theorem 2,

$$H_n - \gamma \leq \log m + \frac{1}{24m^2}$$

(we do not need the term $r_2(n)$). Now $m = h(n) - \frac{1}{24n}$, so by the inequality $\log(1-x) < -x$, we have

$$\log m \leq \log h(n) - \frac{1}{24nh(n)}.$$

The stated upper bound in (20) will follow if we can show that

$$\frac{1}{nh(n)} - \frac{1}{m^2} \geq \frac{1}{2m^3},$$

which equates to $2m[m^2 - nh(n)] \geq nh(n)$. One checks that $m^2 - nh(n) = \frac{1}{2}n + \frac{5}{24}$, so that

$$2m[m^2 - nh(n)] = (n + \frac{1}{2})(n + \frac{5}{12}) > nh(n).$$

The lower bound is proved in a similar way, using

$$\log h(n) \leq \log m + \frac{1}{24mn}.$$

Of course, the term $r_2(n)$ must be taken into account. We omit the details. \square

Comparison with $\frac{1}{2} \log(n^2 + n + \frac{1}{3})$

In a development of Negoi's result, Lu [Lu] has demonstrated that approximation to the accuracy $O(n^{-4})$ is obtained by comparing instead with $\frac{1}{2} \log f(n)$, where $f(n) = n^2 + n + \frac{1}{3}$. He showed that

$$H_n - \gamma = \frac{1}{2} \log f(n) - r_3(n), \tag{21}$$

where $n^4 r_3(n) \rightarrow \frac{1}{180}$ as $n \rightarrow \infty$.

The relation of (21) to (15) is reflected by the fact that $\log(n + \frac{1}{2}) = \frac{1}{2} \log(n^2 + n + \frac{1}{4})$. Also, (21) explains the choice of the term $\frac{1}{24n}$ in Negoi's result (20), because this ensures that $h(n)^2 = n^2 + n + \frac{1}{3} + O(\frac{1}{n})$.

By a further application of De Temple's method, we will present a generalisation of Lu's result, applying to $H_n(x) - C(x)$ instead of $H_n - \gamma$, and incorporating explicit upper and lower bounds.

PROPOSITION 3. *Let $f(x) = x^2 + x + \frac{1}{3}$. Then for all $x \geq 1$,*

$$\log f(x) - \log f(x-1) = \frac{2}{x} - \delta_3(x), \tag{22}$$

where

$$\frac{2}{45x^5} < \delta_3(x) < \frac{2}{45(x - \frac{1}{2})^5}.$$

Proof. We start with $f(x) = x^2 + x + c$ and allow the choice of c to emerge from the reasoning. Let $\delta_3(x)$ be defined by (22). Now $f(x)/f(x-1) \rightarrow 1$, and hence $\delta_3(x) \rightarrow 0$, as $x \rightarrow \infty$. So $\delta_3(x) = -\int_x^\infty \delta_3'(t) dt$ for all $x > 0$. Now

$$\begin{aligned} -\delta_3'(t) &= \frac{2t+1}{f(t)} - \frac{2t-1}{f(t-1)} + \frac{2}{t^2} \\ &= \frac{G(t)}{t^2 f(t) f(t-1)}, \end{aligned}$$

where

$$\begin{aligned} G(t) &= t^2(2t+1)(t^2-t+c) - t^2(2t-1)(t^2+t+c) + 2(t^2+t+c)(t^2-t+c) \\ &= -t^2(2t^2-2c) + 2[t^4 + (2c-1)t^2 + c^2] \\ &= 2(3c-1)t^2 + 2c^2. \end{aligned}$$

To eliminate the t^2 term, we now choose $c = \frac{1}{3}$, so that $G(t) = \frac{2}{9}$ and

$$-\delta_3'(t) = \frac{2}{9t^2 f(t) f(t-1)}.$$

Now $f(t)f(t-1) = t^4 - \frac{1}{3}t^2 + \frac{1}{9} < t^4$ for $t \geq 1$, so

$$\delta_3(x) > \int_x^\infty \frac{2}{9t^6} dt = \frac{2}{45x^5}$$

for $x \geq 1$. On the other hand, $f(t) > f(t-1) > (t - \frac{1}{2})^2$, hence

$$\delta_3(x) < \int_x^\infty \frac{2}{9(t - \frac{1}{2})^6} dt = \frac{2}{45(x - \frac{1}{2})^5}. \quad \square$$

THEOREM 3. Define $H_n(x)$ and $C(x)$ as in Theorem 1. Let $f(t) = t^2 + t + \frac{1}{3}$. Then for $n \geq 1$ and $x > -1$,

$$H_n(x) - C(x) = \frac{1}{2} \log f(n+x) - r_3(n, x), \quad (23)$$

where

$$\frac{1}{180(n+x+1)^4} \leq r_3(n, x) \leq \frac{1}{180(n+x)^4}.$$

In particular,

$$H_n - \gamma = \frac{1}{2} \log f(n) - r_3(n), \quad (24)$$

where

$$\frac{1}{180(n+1)^4} \leq r_3(n) \leq \frac{1}{180n^4}.$$

Also, (23) holds with $n = 0$ for $x > 0$, so that

$$\psi(x+1) = \frac{1}{2} \log f(x) - r_3(x), \quad (25)$$

where

$$\frac{1}{180(x+1)^4} \leq r_3(x) \leq \frac{1}{180x^4}.$$

Proof. Apply the identity (22) to $r+x$ for $1 \leq r \leq n$ and add: we find

$$\log f(n+x) - \log f(x) = 2H_n(x) - \sum_{r=1}^n \delta_3(r+x),$$

equivalently

$$2H_n(x) - \log f(n+x) = -\log f(x) + \sum_{r=1}^n \delta_3(r+x). \quad (26)$$

Now $f(n+x)/n^2 \rightarrow 1$, so $\log f(n+x) - 2\log n \rightarrow 0$, as $n \rightarrow \infty$. Taking the limit in (26), we see that

$$2C(x) = -\log f(x) + \sum_{r=1}^{\infty} \delta_3(r+x). \quad (27)$$

Now taking the difference, we have

$$2H_n(x) - \log f(n+x) - 2C(x) = -2r_3(n, x), \quad (28)$$

where

$$2r_3(n, x) = \sum_{r=n+1}^{\infty} \delta_3(r+x).$$

If $x > -1$, then the inequality (22) applies to $\delta_3(r+x)$ for $r \geq 2$. By integral estimation, we now have for $n \geq 1$,

$$r_3(n, x) \geq \sum_{r=n+1}^{\infty} \frac{1}{45(r+x)^5} > \int_{n+1}^{\infty} \frac{1}{45(t+x)^5} dt = \frac{1}{180(n+x+1)^4}.$$

At the same time,

$$r_3(n, x) \leq \sum_{r=n+1}^{\infty} \frac{1}{45(r - \frac{1}{2} + x)^5}.$$

By convexity of $1/(t+x)^5$, we deduce

$$r_3(n, x) \leq \int_n^{\infty} \frac{1}{45(t+x)^5} dt = \frac{1}{180(n+x)^4}.$$

For $x > 0$, the case $n = 0$ follows similarly from (27): this equates to (25). \square

Note 1. The upper bounds for $\delta_3(x)$ in Proposition 3 and $r_3(n, x)$ in Theorem 1 can be slightly improved. One can verify that $t^2 f(t) f(t-1) \geq (t - \frac{1}{12})^6$ where we previously used $(t - \frac{1}{2})^6$. This leads to $\delta_3(x) < 2/[45(x - \frac{1}{12})^5]$ and $r_3(n, x) \leq 1/[180(n+x + \frac{5}{12})^4]$.

So $\frac{1}{2} \log f(n) - \frac{1}{180n^4}$ approximates H_n with error $O(n^{-5})$. With n chosen to be only 10, the resulting lower and upper estimates for γ are 0.57721559 and 0.57721577 to eight d.p. (compare the actual value 0.57721566.)

Again, the case $x = -\frac{1}{2}$ gives an estimation for sums of odd reciprocals.

COROLLARY 3.1. *Let $U_n = \sum_{r=1}^n \frac{1}{2r-1}$. Then*

$$U_n - \frac{1}{2}\gamma - \log 2 = \frac{1}{4} \log(n^2 + \frac{1}{12}) - \rho_3(n), \quad (29)$$

where

$$\frac{1}{360(n + \frac{1}{2})^4} \leq \rho_3(n) \leq \frac{1}{360(n - \frac{1}{2})^4}.$$

Proof. Just note that $f(n - \frac{1}{2}) = n^2 + \frac{1}{12}$. □

A Stirling-type formula derived from Theorem 3 would involve the rather unpleasant antiderivative of $\log f(x)$.

The estimate for $H_n - \gamma$ in Theorem 3 has a rather surprising application to a residual term that arises in the Dirichlet divisor problem. See [Jam1].

Further note on the selection of c . The choice $c = \frac{1}{3}$ in Proposition 3 emerged from the proof. We now show directly how this choice in Theorem 3 can be determined ahead of the actual proof. Using (1), we show that if $f(n) = n^2 + n + c$ and $H_n - \gamma = \frac{1}{2} \log f(n) + O(1/n^3)$, then $c = \frac{1}{3}$. By (1),

$$2H(n) - 2\gamma = 2 \log n + \frac{1}{n} - \frac{1}{6n^2} + O\left(\frac{1}{n^4}\right),$$

so

$$\log f(n) - 2 \log n = \frac{1}{n} - \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right).$$

But by the logarithmic series,

$$\begin{aligned} \log f(n) - 2 \log n &= \log \left(1 + \frac{1}{n} + \frac{c}{n^2}\right) \\ &= \frac{1}{n} + \frac{c}{n^2} - \frac{1}{2} \left(\frac{1}{n} + \frac{c}{n^2}\right)^2 + O\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n} + \frac{c - \frac{1}{2}}{n^2} + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Hence $c - \frac{1}{2} = -\frac{1}{6}$, so $c = \frac{1}{3}$.

Comparison of Theorems 2 and 3. The estimates in Theorems 2 and 3 are clearly of similar degrees of accuracy. We compare them, using the logarithmic series. For this

purpose, write $n + x = y$ and

$$\log\left(y + \frac{1}{2}\right) + \frac{1}{24\left(y + \frac{1}{2}\right)^2} = L(y).$$

We restate Theorem 2 in the form

$$L(y) - \frac{7}{960y^4} \leq H_n(x) - C(x) \leq L(y) - \frac{7}{960(y+1)^4} \quad (30)$$

and Theorem 3 in the form

$$\frac{1}{2} \log f(y) - \frac{1}{180y^4} \leq H_n(x) - C(x) \leq \frac{1}{2} \log f(y) - \frac{1}{180(y+1)^4}. \quad (31)$$

Now $f(y) = \left(y + \frac{1}{2}\right)^2 + \frac{1}{12}$, so

$$\log f(y) = 2 \log\left(y + \frac{1}{2}\right) + \log\left(1 + \frac{1}{12\left(y + \frac{1}{2}\right)^2}\right).$$

Since $t - \frac{1}{2}t^2 \leq \log(1+t) \leq t$ for $0 < t < 1$, we have

$$L(y) - \frac{1}{576\left(y + \frac{1}{2}\right)^4} \leq \frac{1}{2} \log f(y) \leq L(y). \quad (32)$$

Given (31), apply (32) with $y + \frac{1}{2}$ replaced by y . Noting that $\frac{1}{180} + \frac{1}{576} = \frac{7}{960}$, we obtain

$$L(y) - \frac{7}{960y^4} \leq H_n(x) - C(x) \leq L(y) - \frac{1}{180(y+1)^4},$$

in which the lower bound agrees with (30), while the upper bound is slightly weaker. In fact, the upper bound stated in (30) can be derived from (31) using the inequality $\log(1+t) \leq t - \frac{1}{2}t^2 + \frac{1}{3}t^3$, after some manipulation.

Conversely, (30) implies

$$\frac{1}{2} \log f(y) - \frac{7}{960y^4} \leq H_n(x) - C(x) \leq \frac{1}{2} \log f(y) + \frac{1}{576\left(y + \frac{1}{2}\right)^4} - \frac{7}{960(y+1)^4}.$$

Both bounds are weaker than those in (31), though the upper bound differs by no more than $1/(288y^5)$.

Another method: mid-point Euler-Maclaurin summation

In [DTW], a mid-point variant of Euler-Maclaurin summation is used to derive an asymptotic expansion for $H_n - \gamma$ in terms of $(n + \frac{1}{2})^{-2k}$. The coefficients are in terms of the Bernoulli numbers $B_n(\frac{1}{2})$. The method is described in [DTW] for the special case $f(t) = 1/t$, and for general functions in my website notes [Jam2]. Taken as far as the third derivative, and expressed as a pair of inequalities, it can be stated as follows. Suppose that

f is *completely monotonic* on $(0, \infty)$, that is, $f^{(2k)}(t) \geq 0$ and $f^{(2k+1)}(t) \leq 0$ for all $k \geq 0$ and $t > 0$. Then

$$\sum_{r=m}^n f(r) = \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} f(t) dt - \frac{1}{24} \left(f'(n + \frac{1}{2}) - f'(m - \frac{1}{2}) \right) + T_2(m, n),$$

where

$$0 \leq T_2(m, n) \leq \frac{7}{5760} \left(f^{(3)}(n + \frac{1}{2}) - f^{(3)}(m - \frac{1}{2}) \right).$$

Applied with $f(t) = 1/(t+x)$, this leads to the following stronger upper bound for $r_2(n, x)$ in Theorem 2:

$$r_2(n, x) \leq \frac{7}{960(n+x+\frac{1}{2})^4}.$$

Further terms of the expansion give stronger bounds (both upper and lower), at the cost of greater complication.

References

- [BM] George Boros and Victor Moll, *Irresistible Integrals*, Cambridge Univ. Press (2004).
- [Ch] Chao-Ping Chen, Inequalities for the Euler-Mascheroni constant, *Appl. Math. Lett.* **23** (2010), 161–164.
- [DT] D. W. De Temple, A quicker convergence to Euler’s constant, *American Math. Monthly* **100** (1993), 468–470.
- [DTW] D. W. De Temple and S. H. Wang, Half integer approximations for the partial sums of the harmonic series, *J. Math. Anal. Appl.* **160** (1991), 187–190.
- [Lu] Dawei Lu, Some quicker classes of sequences convergent to Euler’s constant, *Appl. Math. Comp.* **232** (2014), 172–177.
- [Jam1] A sharp estimate for harmonic sums and the digamma function, with an application to the Dirichlet divisor problem, preprint.
- [Jam2] Euler-Maclaurin summation, at www.maths.lancs.ac.uk/~jameson.
- [Neg] Tanase Negoi, A faster convergence to Euler’s constant, *Math. Gazette* **83** (1999), 487–489.

updated 29 January 2016