Half-integer estimates for harmonic sums and the digamma function: de Temple's method

Notes by G.J.O. Jameson

Write H_n for the harmonic sum $\sum_{r=1}^n \frac{1}{r}$. The following well-known estimation can be established by Euler-Maclaurin summation, or by the logarithmic and binomial series:

$$H_n - \gamma = \log n + \frac{1}{2n} - \frac{1}{12n^2} + r(n), \tag{1}$$

where γ is Euler's constant and

$$0 \le r(n) \le \frac{1}{120n^4}.$$

Since $\log(n+\frac{1}{2}) = \log n + \frac{1}{2n} + O(\frac{1}{n^2})$, it is a natural idea to absorb the term $\frac{1}{2n}$ into the logarithmic term and compare $H_n - \gamma$ directly with $\log(n+\frac{1}{2})$. This was done by De Temple [DT]. His result (reproduced in [BM, p. 181–2]) states that

$$H_n - \gamma = \log(n + \tfrac{1}{2}) + r_1(n), \tag{2}$$

where

$$\frac{1}{24(n+1)^2} \le r_1(n) \le \frac{1}{24n^2}.$$

De Temple also stated without proof the following more accurate estimation:

$$H_n - \gamma = \log(n + \tfrac{1}{2}) + \frac{1}{24(n+\frac{1}{2})^2} - r_2(n), \tag{3}$$

where

$$\frac{7}{960(n+1)^4} \le r_2(n) \le \frac{7}{960n^4}.$$

We will prove both these estimations. Without any more effort, the same method applies to the more general sum

$$H_n(x) = \sum_{r=1}^n \frac{1}{r+x}.$$

With this notation, $H_n = H_n(0)$. We write $C(x) = \lim_{n\to\infty}[H_n(x) - \log n]$. The existence of the limit $C(x)$ follows from elementary integral estimation of the sum $H_n(x)$, but it will also be established in the proof to follow. Clearly, $C(0) = \gamma$. We will establish a variant of De Temple's estimations applying to $H_n(x) - C(x)$.

In fact, $C(x)$ can be expressed in terms of the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. From Euler's limit formula for the gamma function, we have $\psi(x+1) = \lim_{n\to\infty} \psi_n(x+1)$, where

$$\psi_n(x+1) = \log n - \sum_{r=1}^n \frac{1}{x+r} = \log n - H_n(x).$$

So in fact $C(x) = -\psi(x+1)$. The resulting estimations will equate to a variant of Stirling's formula.

Note that

$$\log(x+\tfrac{1}{2}) - \log(x-\tfrac{1}{2}) = \int_{x-1/2}^{x+1/2} \frac{1}{t}\,dt.$$

The mid-point approximation to this integral is $1/x$. The proofs are based on estimates (at various levels of accuracy) comparing $\log(x+\frac{1}{2}) - \log(x-\frac{1}{2})$ with $1/x$.

PROPOSITION 1. *For $x > \frac{1}{2}$,*

$$\log(x+\tfrac{1}{2}) - \log(x-\tfrac{1}{2}) = \frac{1}{x} + \delta_1(x), \tag{4}$$

where

$$\frac{1}{12x^3} \le \delta_1(x) \le \frac{1}{12(x-\frac{1}{2})^3}.$$

Proof. Let $\delta_1(x)$ be defined by (4). Then $\delta_1(x) \to 0$ as $x \to \infty$, so $\delta_1(x) = -\int_x^\infty \delta_1'(t)\,dt$. Now

$$\begin{aligned}
-\delta_1'(t) &= \frac{1}{t-\frac{1}{2}} - \frac{1}{t+\frac{1}{2}} - \frac{1}{t^2} \\
&= \frac{1}{t^2-\frac{1}{4}} - \frac{1}{t^2} \\
&= \frac{1}{4t^2(t^2-\frac{1}{4})} \\
&> \frac{1}{4t^4}.
\end{aligned}$$

Hence

$$\delta_1(x) > \int_x^\infty \frac{1}{4t^4}\,dt = \frac{1}{12x^3}.$$

Also, $t^2(t^2-\frac{1}{4}) > (t-\frac{1}{2})^4$, so

$$\delta_1(x) < \int_x^\infty \frac{1}{4(t-\frac{1}{2})^4}\,dt = \frac{1}{12(x-\frac{1}{2})^3}. \qquad \square$$

THEOREM 1. *Let $n \ge 1$ and $x > -1$. Let $C(x) = \lim_{n\to\infty}(H_n(x) - \log n)$. Then*

$$H_n(x) - C(x) = \log(n+x+\tfrac{1}{2}) + r_1(n,x), \tag{5}$$

where

$$\frac{1}{24(n+x+1)^2} \le r_1(n,x) \le \frac{1}{24(n+x)^2}.$$

In particular,

$$H_n - \gamma = \log(n+\tfrac{1}{2}) + r_1(n), \tag{6}$$

where

$$\frac{1}{24(n+1)^2} \le r_1(n) \le \frac{1}{24n^2}$$

Also, (5) holds with $n = 0$ *for* $x > 0$ *(with* $H_0(x) = 0$*), so that*

$$\psi(x+1) = \log(x + \tfrac{1}{2}) + r_1(x), \tag{7}$$

where

$$\frac{1}{24(x+1)^2} \le r_1(x) \le \frac{1}{24x^2}.$$

Proof. Apply (4) to $r + x$ for $1 \le r \le n$ and add: we obtain

$$\log(n + x + \tfrac{1}{2}) - \log(x + \tfrac{1}{2}) = H_n(x) + \sum_{r=1}^{n} \delta_1(r + x),$$

equivalently

$$H_n(x) - \log(n + x + \tfrac{1}{2}) = -\log(x + \tfrac{1}{2}) - \sum_{r=1}^{n} \delta_1(r + x). \tag{8}$$

Take the limit as $n \to \infty$. Clearly, $\log(n + x + \frac{1}{2}) - \log n \to 0$, so we obtain

$$C(x) = -\log(x + \tfrac{1}{2}) - \sum_{r=1}^{\infty} \delta_1(r + x). \tag{9}$$

Taking the difference between (8) and (9), we obtain

$$H_n(x) - \log(n + x + \tfrac{1}{2}) - C(x) = \sum_{r=n+1}^{\infty} \delta_1(r + x). \tag{10}$$

Denote this by $r_1(n, x)$. The condition $x > -1$ ensures that the inequality (4) applies to $\delta_1(r + x)$ for $r \ge 2$. By integral estimation, we now have for $n \ge 1$,

$$r_1(n, x) \ge \sum_{r=n+1}^{\infty} \frac{1}{12(r+x)^3} > \int_{n+1}^{\infty} \frac{1}{12(t+x)^3}\, dt = \frac{1}{24(n+x+1)^2}.$$

At the same time,

$$r_1(n, x) \le \sum_{r=n+1}^{\infty} \frac{1}{12(r + x - \frac{1}{2})^3}.$$

The function $1/(t+x)^3$ is convex, and convex functions $h(t)$ satisfy $h(y - \frac{1}{2}) \le \int_{y-1}^{y} h(t)\, dt$, hence

$$r_1(n, x) \le \int_{n}^{\infty} \frac{1}{12(t+x)^3}\, dt = \frac{1}{24(n+x)^2}.$$

For $x > 0$, the case $n = 0$ follows similarly from (9) (note that (4) now applies also to $\delta_1(1 + x)$). Since $C(x) = -\psi(x + 1)$, this equates to (7). □

Note. In (6) and (7), we have used the notation $r_1(n)$ for $r_1(n,0)$ and $r_1(x)$ for $r_1(0,x)$.

The case $x = -\frac{1}{2}$ gives the following estimation for sums of odd reciprocals.

COROLLARY 1.1. *Let* $U_n = \sum_{r=1}^{n} \frac{1}{2r-1}$. *Then*

$$U_n - \tfrac{1}{2}\gamma - \log 2 = \tfrac{1}{2}\log n + \rho_1(n), \tag{11}$$

where

$$\frac{1}{48(n+\frac{1}{2})^2} \le \rho_1(n) \le \frac{1}{48(n-\frac{1}{2})^2}.$$

Proof. Note that $2U_n = H_n(-\frac{1}{2})$. Also, $2U_n = 2H_{2n} - H_n$; it follows easily that $2U_n - \log n \to \gamma + 2\log 2$ as $n \to \infty$. The statement is the case $x = -\frac{1}{2}$ in Theorem 1. $\square$

Integration of (7) delivers the following variant of Stirling's formula:

COROLLARY 1.2. *For a certain constant c, we have*

$$\log\Gamma(x) = (x - \tfrac{1}{2})\log(x - \tfrac{1}{2}) - x + c - P_1(x), \tag{12}$$

where

$$\frac{1}{24x} \le P_1(x) \le \frac{1}{24(x-1)}.$$

Proof. Write (7) in the form $\psi(t) = \log(t - \frac{1}{2}) + r_1(t-1)$ and integrate on $[\frac{3}{2}, x]$ to obtain

$$\log\Gamma(x) - \log\Gamma(\tfrac{3}{2}) = (x - \tfrac{1}{2})\log(x - \tfrac{1}{2}) - x + \tfrac{3}{2} + \int_{3/2}^{x} r_1(t-1)\,dt.$$

Let $I = \int_{3/2}^{\infty} r_1(t-1)\,dt$. Then (12) holds with $c = \log\Gamma(\frac{3}{2}) + \frac{3}{2} + I$ and

$$P_1(x) = \int_x^{\infty} r_1(t-1)\,dt \le \int_x^{\infty} \frac{1}{24(t-1)^2}\,dt = \frac{1}{24(x-1)},$$

and similarly $P_1(x) \ge 1/24x$. $\square$

As in most proofs of Stirling's formula, one now needs to invoke the Wallis product (or some equivalent) to evaluate c.

We now improve upon the estimate in Proposition 1, and derive the generalised verson of (3).

PROPOSITION 2. *For* $x > \frac{1}{2}$,

$$\log(x + \tfrac{1}{2}) - \log(x - \tfrac{1}{2}) = \frac{1}{x} + \frac{1}{24}\left(\frac{1}{(x-\frac{1}{2})^2} - \frac{1}{(x+\frac{1}{2})^2}\right) - \delta_2(x), \tag{13}$$

where

$$\frac{7}{240x^5} \le \delta_2(x) \le \frac{7}{240(x-\frac{1}{2})^5}.$$

Proof. Use (13) as the definition of $\delta_2(x)$. Using the expression for $\delta_1'(t)$ from Proposition 1, we obtain

$$\begin{aligned} -\delta_2'(t) &= -\frac{1}{4t^2(t^2-\frac{1}{4})} + \frac{1}{12(t-\frac{1}{2})^3} - \frac{1}{12(t+\frac{1}{2})^3} \\ &= \frac{G(t)}{12t^2(t^2-\frac{1}{4})^3}, \end{aligned}$$

where

$$\begin{aligned} G(t) &= -3(t^2-\tfrac{1}{4})^2 + t^2\Big((t^2+\tfrac{1}{2})^3 - (t^2-\tfrac{1}{2})^3\Big) \\ &= -3(t^4 - \tfrac{1}{2}t^2 + \tfrac{1}{16}) + t^2(3t^2+\tfrac{1}{4}) \\ &= \tfrac{7}{4}t^2 - \tfrac{3}{16} \\ &= \tfrac{7}{4}(t^2 - \tfrac{3}{28}). \end{aligned}$$

Hence

$$-\delta_2'(t) > \frac{7}{48t^2(t^2-\frac{1}{4})^2} > \frac{7}{48t^6}$$

and

$$\delta_2(x) = -\int_x^\infty \delta_2'(t)\,dt > \int_x^\infty \frac{7}{48t^6}\,dt = \frac{7}{240x^5}.$$

Also

$$-\delta_2'(t) < \frac{7}{48(t^2-\frac{1}{4})^3} < \frac{7}{48(t-\frac{1}{2})^6},$$

so that

$$\delta_2(x) < \frac{7}{240(x-\frac{1}{2})^5}. \qquad \square$$

THEOREM 2. *With the notation of Theorem 1, we have for $n \ge 1$ and $x > -1$,*

$$H_n(x) - C(x) = \log(n+x+\tfrac{1}{2}) + \frac{1}{24(n+x+\frac{1}{2})^2} - r_2(n,x), \tag{14}$$

where

$$\frac{7}{960(n+x+1)^4} \le r_2(n,x) \le \frac{7}{960(n+x)^4}.$$

In particular,

$$H_n - \gamma = \log(n+\tfrac{1}{2}) + \frac{1}{24(n+\frac{1}{2})^2} - r_2(n), \tag{15}$$

where

$$\frac{7}{960(n+1)^4} \le r_2(n) \le \frac{7}{960n^4}.$$

Also, for $x > 0$,

$$\psi(x+1) = \log(x+\tfrac{1}{2}) + \frac{1}{24(x+\frac{1}{2})^2} - r_2(x), \tag{16}$$

where

$$\frac{7}{960(x+1)^4} \le r_2(x) \le \frac{7}{960x^4}.$$

Proof. Modify the proof of Theorem 1 as follows. Applying (13) to $r+x$ for $1 \le r \le n$ and adding, we obtain

$$H_n(x) - \log(n+x+\tfrac{1}{2}) = -\log(x+\tfrac{1}{2}) - \frac{1}{24(x+\frac{1}{2})^2} + \frac{1}{24(n+x+\frac{1}{2})^2} + \sum_{r=1}^{n} \delta_2(r+x).$$

Taking the limit as $n \to \infty$, we obtain

$$C(x) = -\log(x+\tfrac{1}{2}) - \frac{1}{24(x+\frac{1}{2})^2} + \sum_{r=1}^{\infty} \delta_2(r+x).$$

and taking the difference, we have

$$H_n(x) - \log(n+x+\tfrac{1}{2}) - C(x) = \frac{1}{24(n+x+\frac{1}{2})^2} - r_2(n,x),$$

where

$$r_2(n,x) = \sum_{r=n+1}^{\infty} \delta_2(r+x).$$

By (13) and integral estimation as before, we obtain the stated bounds for $r_2(n,x)$. □

COROLLARY 2.1. *Let $U_n = \sum_{r=1}^{n} \frac{1}{2r-1}$. Then*

$$U_n - \tfrac{1}{2}\gamma - \log 2 = \tfrac{1}{2}\log n + \frac{1}{48n^2} - \rho_2(n), \tag{17}$$

where

$$\frac{7}{1920(n+\frac{1}{2})^4} \le \rho_2(n) \le \frac{7}{1920(n-\frac{1}{2})^4}. \qquad \square$$

Note that (17) is considerably sharper than the estimate of the same type found by applying (1) to H_{2n} and H_n.

Reasoning as in Corollary 1.2, with (7) replaced by (16), we obtain:

COROLLARY 2.2. *With c as in Corollary 1.2, we have*

$$\log\Gamma(x) = (x-\tfrac{1}{2})\log(x-\tfrac{1}{2}) - x + c - \frac{1}{24(x-\frac{1}{2})} + P_2(x), \tag{18}$$

where

$$\frac{7}{2880x^3} \le P_2(x) \le \frac{7}{2880(x-1)^3}. \qquad \square$$

We derive a lower bound of a rather different type, which implies one given in [Ch]. For simplicity, we only state it for H_n instead of $H_n(x)$.

COROLLARY 2.3. *For all $n \ge 1$,*

$$H_n - \gamma \ge \log(n + \tfrac{1}{2}) + \frac{1}{24(n + \frac{1}{2} + \frac{1}{4n})^2}. \tag{19}$$

Proof. By elementary algebra, for a, $b > 0$,

$$\frac{1}{a^2} - \frac{1}{(a+b)^2} > \frac{2b}{(a+b)^3}.$$

Hence for $n \ge 2$ (with $c \le 1$ to be chosen),

$$\frac{1}{(n+\frac{1}{2})^2} - \frac{1}{(n + \frac{1}{2} + \frac{c}{n})^2} > \frac{2c}{n(n+1)^3}.$$

To derive (19) from (15), we require this to be not less than $7/(40n^4)$, which equates to $[(n+1)/n]^3 \le \frac{80}{7}c$. With $c = \frac{1}{4}$, this holds for $n \ge 3$, and one can check that (19) also holds for $n = 1$ and $n = 2$. $\square$

Chen actually gives the lower bound $1/[24(n+a)^2]$, where $a \approx 0.55106$, and upper bound $1/[24(n + \frac{1}{2})^2]$.

Negoi [Neg] demonstrated that the n^{-2} term in (15) can be absorbed into the log term by considering $\log h(n)$, where $h(n) = n + \frac{1}{2} + \frac{1}{24n}$. His proof was by another variation of De Temple's method, entailing a rather laborious calculation. Here we show how to deduce his result (in fact, a slightly stronger one) from Theorem 2.

COROLLARY 2.4. *Let $h(n) = n + \frac{1}{2} + \frac{1}{24n}$. Then*

$$H_n - \gamma = \log h(n) - \rho(n), \tag{20}$$

where

$$\frac{1}{48(n+1)^3} \le \rho(n) \le \frac{1}{48(n+\frac{1}{2})^3}.$$

(Negoi's statement has $\rho(n) \le 1/(48n^3)$.)

Proof. Write $n + \frac{1}{2} = m$. By Theorem 2,

$$H_n - \gamma \le \log m + \frac{1}{24m^2}$$

(we do not need the term $r_2(n)$). Now $m = h(n) - \frac{1}{24n}$, so by the inequality $\log(1-x) < -x$, we have

$$\log m \le \log h(n) - \frac{1}{24nh(n)}.$$

The stated upper bound in (20) will follow if we can show that

$$\frac{1}{nh(n)} - \frac{1}{m^2} \ge \frac{1}{2m^3},$$

which equates to $2m[m^2 - nh(n)] \ge nh(n)$. One checks that $m^2 - nh(n) = \frac{1}{2}n + \frac{5}{24}$, so that

$$2m[m^2 - nh(n)] = (n + \tfrac{1}{2})(n + \tfrac{5}{12}) > nh(n).$$

The lower bound is proved in a similar way, using

$$\log h(n) \le \log m + \frac{1}{24mn}.$$

Of course, the term $r_2(n)$ must be taken into account. We omit the details. $\square$

Comparison with $\frac{1}{2}\log(n^2 + n + \frac{1}{3})$

In a development of Negoi's result, Lu [Lu] has demonstrated that approximation to the accuracy $O(n^{-4})$ is obtained by comparing instead with $\frac{1}{2}\log f(n)$, where $f(n) = n^2+n+\frac{1}{3}$. He showed that

$$H_n - \gamma = \tfrac{1}{2}\log f(n) - r_3(n), \tag{21}$$

where $n^4 r_3(n) \to \frac{1}{180}$ as $n \to \infty$.

The relation of (21) to (15) is reflected by the fact that $\log(n + \frac{1}{2}) = \frac{1}{2}\log(n^2 + n + \frac{1}{4})$. Also, (21) explains the choice of the term $\frac{1}{24n}$ in Negoi's result (20), because this ensures that $h(n)^2 = n^2 + n + \frac{1}{3} + O(\frac{1}{n})$.

By a further application of De Temple's method, we will present a generalisation of Lu's result, applying to $H_n(x) - C(x)$ instead of $H_n - \gamma$, and incorporating explicit upper and lower bounds.

PROPOSITION 3. *Let $f(x) = x^2 + x + \frac{1}{3}$. Then for all $x \ge 1$,*

$$\log f(x) - \log f(x-1) = \frac{2}{x} - \delta_3(x), \tag{22}$$

where

$$\frac{2}{45x^5} < \delta_3(x) < \frac{2}{45(x-\frac{1}{2})^5}.$$

Proof. We start with $f(x) = x^2 + x + c$ and allow the choice of c to emerge from the reasoning. Let $\delta_3(x)$ be defined by (22). Now $f(x)/f(x-1) \to 1$, and hence $\delta_3(x) \to 0$, as $x \to \infty$. So $\delta_3(x) = -\int_x^\infty \delta_3'(t)\, dt$ for all $x > 0$. Now

$$\begin{aligned} -\delta_3'(t) &= \frac{2t+1}{f(t)} - \frac{2t-1}{f(t-1)} + \frac{2}{t^2} \\ &= \frac{G(t)}{t^2 f(t) f(t-1)}, \end{aligned}$$

where

$$\begin{aligned} G(t) &= t^2(2t+1)(t^2-t+c) - t^2(2t-1)(t^2+t+c) + 2(t^2+t+c)(t^2-t+c) \\ &= -t^2(2t^2-2c) + 2[t^4 + (2c-1)t^2 + c^2] \\ &= 2(3c-1)t^2 + 2c^2. \end{aligned}$$

To eliminate the t^2 term, we now choose $c = \frac{1}{3}$, so that $G(t) = \frac{2}{9}$ and

$$-\delta_3'(t) = \frac{2}{9t^2 f(t) f(t-1)}.$$

Now $f(t)f(t-1) = t^4 - \frac{1}{3}t^2 + \frac{1}{9} < t^4$ for $t \geq 1$, so

$$\delta_3(x) > \int_x^\infty \frac{2}{9t^6}\, dt = \frac{2}{45x^5}$$

for $x \geq 1$. On the other hand, $f(t) > f(t-1) > (t - \frac{1}{2})^2$, hence

$$\delta_3(x) < \int_x^\infty \frac{2}{9(t-\frac{1}{2})^6}\, dt = \frac{2}{45(x-\frac{1}{2})^5}. \qquad \square$$

THEOREM 3. *Define $H_n(x)$ and $C(x)$ as in Theorem 1. Let $f(t) = t^2 + t + \frac{1}{3}$. Then for $n \geq 1$ and $x > -1$,*

$$H_n(x) - C(x) = \tfrac{1}{2}\log f(n+x) - r_3(n,x), \tag{23}$$

where

$$\frac{1}{180(n+x+1)^4} \leq r_3(n,x) \leq \frac{1}{180(n+x)^4}.$$

In particular,

$$H_n - \gamma = \tfrac{1}{2}\log f(n) - r_3(n), \tag{24}$$

where

$$\frac{1}{180(n+1)^4} \leq r_3(n) \leq \frac{1}{180n^4}.$$

Also, (23) holds with $n = 0$ for $x > 0$, so that

$$\psi(x+1) = \tfrac{1}{2}\log f(x) - r_3(x), \tag{25}$$

where

$$\frac{1}{180(x+1)^4} \le r_3(x) \le \frac{1}{180x^4}.$$

Proof. Apply the identity (22) to $r+x$ for $1 \le r \le n$ and add: we find

$$\log f(n+x) - \log f(x) = 2H_n(x) - \sum_{r=1}^{n} \delta_3(r+x),$$

equivalently

$$2H_n(x) - \log f(n+x) = -\log f(x) + \sum_{r=1}^{n} \delta_3(r+x). \tag{26}$$

Now $f(n+x)/n^2 \to 1$, so $\log f(n+x) - 2\log n \to 0$, as $n \to \infty$. Taking the limit in (26), we see that

$$2C(x) = -\log f(x) + \sum_{r=1}^{\infty} \delta_3(r+x). \tag{27}$$

Now taking the difference, we have

$$2H_n(x) - \log f(n+x) - 2C(x) = -2r_3(n,x), \tag{28}$$

where

$$2r_3(n,x) = \sum_{r=n+1}^{\infty} \delta_3(r+x).$$

If $x > -1$, then the inequality (22) applies to $\delta_3(r+x)$ for $r \ge 2$. By integral estimation, we now have for $n \ge 1$,

$$r_3(n,x) \ge \sum_{r=n+1}^{\infty} \frac{1}{45(r+x)^5} > \int_{n+1}^{\infty} \frac{1}{45(t+x)^5}\, dt = \frac{1}{180(n+x+1)^4}.$$

At the same time,

$$r_3(n,x) \le \sum_{r=n+1}^{\infty} \frac{1}{45(r-\frac{1}{2}+x)^5}.$$

By convexity of $1/(t+x)^5$, we deduce

$$r_3(n,x) \le \int_{n}^{\infty} \frac{1}{45(t+x)^5}\, dt = \frac{1}{180(n+x)^4}.$$

For $x > 0$, the case $n = 0$ follows similarly from (27): this equates to (25). $\square$

Note 1. The upper bounds for $\delta_3(x)$ in Proposition 3 and $r_3(n,x)$ in Theorem 1 can be slightly improved. One can verify that $t^2 f(t) f(t-1) \ge (t - \frac{1}{12})^6$ where we previously used $(t-\frac{1}{2})^6$. This leads to $\delta_3(x) < 2/[45(x-\frac{1}{12})^5]$ and $r_3(n,x) \le 1/[180(n+x+\frac{5}{12})^4]$.

So $\frac{1}{2}\log f(n) - \frac{1}{180n^4}$ approximates H_n with error $O(n^{-5})$. With n chosen to be only 10, the resulting lower and upper estimates for γ are 0.57721559 and 0.57721577 to eight d.p. (compare the actual value 0.57721566.)

Again, the case $x = -\frac{1}{2}$ gives an estimation for sums of odd reciprocals.

COROLLARY 3.1. *Let* $U_n = \sum_{r=1}^{n} \frac{1}{2r-1}$. *Then*

$$U_n - \tfrac{1}{2}\gamma - \log 2 = \tfrac{1}{4}\log(n^2 + \tfrac{1}{12}) - \rho_3(n), \tag{29}$$

where

$$\frac{1}{360(n+\frac{1}{2})^4} \le \rho_3(n) \le \frac{1}{360(n-\frac{1}{2})^4}.$$

Proof. Just note that $f(n - \frac{1}{2}) = n^2 + \frac{1}{12}$. □

A Stirling-type formula derived from Theorem 3 would involve the rather unpleasant antiderivative of $\log f(x)$.

The estimate for $H_n - \gamma$ in Theorem 3 has a rather surprising application to a residual term that arises in the Dirichlet divisor problem. See [Jam1].

Further note on the selection of c. The choice $c = \frac{1}{3}$ in Proposition 3 emerged from the proof. We now show directly how this choice in Theorem 3 can be determined ahead of the actual proof. Using (1), we show that if $f(n) = n^2+n+c$ and $H_n - \gamma = \frac{1}{2}\log f(n) + O(1/n^3)$, then $c = \frac{1}{3}$. By (1),

$$2H(n) - 2\gamma = 2\log n + \frac{1}{n} - \frac{1}{6n^2} + O\left(\frac{1}{n^4}\right),$$

so

$$\log f(n) - 2\log n = \frac{1}{n} - \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right).$$

But by the logarithmic series,

$$\begin{aligned} \log f(n) - 2\log n &= \log\left(1 + \frac{1}{n} + \frac{c}{n^2}\right) \\ &= \frac{1}{n} + \frac{c}{n^2} - \frac{1}{2}\left(\frac{1}{n} + \frac{c}{n^2}\right)^2 + O\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n} + \frac{c - \frac{1}{2}}{n^2} + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Hence $c - \frac{1}{2} = -\frac{1}{6}$, so $c = \frac{1}{3}$.

Comparison of Theorems 2 and 3. The estimates in Theorems 2 and 3 are clearly of similar degrees of accuracy. We compare them, using the logarithmic series. For this purpose, write $n+x=y$ and

$$\log(y+\tfrac{1}{2})+\frac{1}{24(y+\frac{1}{2})^2}=L(y).$$

We restate Theorem 2 in the form

$$L(y)-\frac{7}{960y^4}\le H_n(x)-C(x)\le L(y)-\frac{7}{960(y+1)^4} \tag{30}$$

and Theorem 3 in the form

$$\tfrac{1}{2}\log f(y)-\frac{1}{180y^4}\le H_n(x)-C(x)\le \tfrac{1}{2}\log f(y)-\frac{1}{180(y+1)^4}. \tag{31}$$

Now $f(y)=(y+\frac{1}{2})^2+\frac{1}{12}$, so

$$\log f(y)=2\log(y+\tfrac{1}{2})+\log\left(1+\frac{1}{12(y+\frac{1}{2})^2}\right).$$

Since $t-\frac{1}{2}t^2\le\log(1+t)\le t$ for $0<t<1$, we have

$$L(y)-\frac{1}{576(y+\frac{1}{2})^4}\le\tfrac{1}{2}\log f(y)\le L(y). \tag{32}$$

Given (31), apply (32) with $y+\frac{1}{2}$ replaced by y. Noting that $\frac{1}{180}+\frac{1}{576}=\frac{7}{960}$, we obtain

$$L(y)-\frac{7}{960y^4}\le H_n(x)-C(x)\le L(y)-\frac{1}{180(y+1)^4},$$

in which the lower bound agrees with (30), while the upper bound is slightly weaker. In fact, the upper bound stated in (30) can be derived from (31) using the inequality $\log(1+t)\le t-\frac{1}{2}t^2+\frac{1}{3}t^3$, after some manipulation.

Conversely, (30) implies

$$\tfrac{1}{2}\log f(y)-\frac{7}{960y^4}\le H_n(x)-C(x)\le\tfrac{1}{2}\log f(y)+\frac{1}{576(y+\frac{1}{2})^4}-\frac{7}{960(y+1)^4}.$$

Both bounds are weaker than those in (31), though the upper bound differs by no more than $1/(288y^5)$.

Another method: mid-point Euler-Maclaurin summation

In [DTW], a mid-point variant of Euler-Maclaurin summation is used to derive an asymptotic expansion for $H_n-\gamma$ in terms of $(n+\frac{1}{2})^{-2k}$. The coefficients are in terms of the Bernoulli numbers $B_n(\frac{1}{2})$. The method is described in [DTW] for the special case $f(t)=1/t$, and for general functions in my website notes [Jam2]. Taken as far as the third derivative, and expressed as a pair of inequalities, it can be stated as follows. Suppose that

f is *completely monotonic* on $(0, \infty)$, that is, $f^{(2k)}(t) \geq 0$ and $f^{(2k+1)}(t) \leq 0$ for all $k \geq 0$ and $t > 0$. Then

$$\sum_{r=m}^{n} f(r) = \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} f(t)\, dt - \frac{1}{24}\Big(f'(n+\tfrac{1}{2}) - f'(m-\tfrac{1}{2})\Big) + T_2(m,n),$$

where

$$0 \leq T_2(m,n) \leq \frac{7}{5760}\Big(f^{(3)}(n+\tfrac{1}{2}) - f^{(3)}(m-\tfrac{1}{2})\Big).$$

Applied with $f(t) = 1/(t+x)$, this leads to the following stronger upper bound for $r_2(n,x)$ in Theorem 2:

$$r_2(n,x) \leq \frac{7}{960(n+x+\frac{1}{2})^4}.$$

Further terms of the expansion give stronger bounds (both upper and lower), at the cost of greater complication.

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