Inequalities for gamma function ratios

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Write

$$R(x,y) = \frac{\Gamma(x+y)}{\Gamma(x)}.$$

Inequalities for this ratio have interesting applications, and have been considered by a number of writers over a long period. In a *Monthly* article [7], Wendel showed that

$$x(x+y)^{y-1} \le R(x,y) \le x^y \qquad \text{for } 0 \le y \le 1. \tag{1}$$

Wendel's method was an ingenious application of Hölder's inequality to the integral definition of the gamma function. Note that both inequalities are exact when y is 0 or 1, since $R(x,0) = 1$ and $R(x,1) = x$.

Without reference to Wendel, and by quite different methods, Gautschi [4] proved

$$x(x+1)^{y-1} \le R(x,y) \le x^y \qquad \text{for } 0 \le y \le 1. \tag{2}$$

The lower bound in (2) is weaker than the one in (1), and is not exact when $y = 0$. Gautschi only stated the result for integer values of x, but his method does not require this. He also established the more elaborate lower bound $x\exp[(y-1)\psi(x+1)]$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$, which implies (2) because $\psi(x) < \log x$. Refinements have appeared in many later articles, e.g. [3, 5, 6]. Most of them take the form of expressions in terms of $\psi(x)$, but one of Kershaw's bounds is $x(x + \frac{y}{2})^{y-1}$. Here we confine attention to bounds of the simple type seen in (1) and (2).

Artin's classic book [2] was published in 1931, long before either Wendel or Gautschi. In it (p. 14) we find the statement

$$(x-1)^y \le R(x,y) \le x^y \qquad \text{for } 0 \le y \le 1, \tag{3}$$

(actually only stated for integers x). Again, the lower bound is weaker than the one in (1), though this is not quite so transparent, and is not exact at 1. Artin did not state (3) as a result in its own right, but only as a step in the proof of another theorem. Perhaps for this reason, his result appears to have been overlooked by most later writers, including Wendel and Gautschi. Indeed, inequalities of this type have generally been referred to as "Gautschi-type inequalities," with scant respect to either Artin or Wendel.

As well as having appeared earlier, Artin's method is much simpler than those of the later writers - indeed, as we shall see, it makes the result seem almost trivial! The only properties of the gamma function required are the identity $\Gamma(x+1) = x\Gamma(x)$ and convexity of $\log \Gamma(x)$. The method applies to any function with these properties, and consequently also leads to a pleasant proof of the Bohr-Mollerup theorem stating that $\Gamma(x)$ is essentially the only such function, which was Artin's purpose.

Bounds for $R(x,y)$ are not only of interest for y in the interval $[0,1]$, and in fact the key to a satisfactory treatment (even for the case $0 \le y \le 1$) is to consider the whole range of values of y. Observe that (1), (2) and (3) all have x^y on the right-hand side. We now prove, by a slight modification of Artin's method, a pleasantly simple result comparing $R(x,y)$ with x^y for various values of y. We will then show that this result is enough to encapsulate all the inequalities of this kind: without further effort, a complete system of upper and lower bounds, in turn for $0 \le y \le 1$, $1 \le y \le 2$ and $y > 2$, is delivered by suitable substitutions.

Theorem 1. *Let f be any function such that $\log f(x)$ is convex and $f(x+1) = xf(x)$ for all $x > 0$ (in particular, the gamma function). Write $R(x,y) = f(x+y)/f(x)$. Then*

$$R(x,y) \le x^y \qquad \text{for } 0 \le y \le 1, \tag{4}$$

$$R(x,y) \ge x^y \qquad \text{for } y \ge 1 \text{ and for } y < 0. \tag{5}$$

Proof. For fixed x, let $F(y) = \log f(x+y) - y \log x$ for all $y > -x$. Then F is convex and

$$F(1) = \log f(x+1) - \log x = \log f(x) = F(0).$$

It follows that $F(y) \le \log f(x)$ for $0 \le y \le 1$ and $F(y) \ge \log f(x)$ for $y \ge 1$ and for $y \le 0$. This equates to (4) and (5). $\square$

Clearly, equality holds when y is 0 or 1, and if $\log f(x)$ is strictly convex, then strict inequality holds for other y.

Corollary 1.1. *For f as in Theorem 1, we have*

$$R(x,y) \le x(x+1)^{y-1} \qquad \text{for } 1 \le y \le 2, \tag{6}$$

and the opposite holds for $0 \le y \le 1$ and $y > 2$. Also,

$$R(x,y) \ge x(x+y)^{y-1} \qquad \text{for } 0 \le y \le 1, \tag{7}$$

and the opposite holds for $y > 1$. Further,

$$R(x,y) \ge (x-1)^y \qquad \text{for } 0 \le y \le 1. \tag{8}$$

Proof. For (6), if $1 \leq y \leq 2$, then (4) gives

$$R(x+1, y-1) = \frac{f(x+y)}{f(x+1)} \leq (x+1)^{y-1},$$

so that $f(x+y) \leq x(x+1)^{y-1} f(x)$. By (5), the opposite holds for $0 \leq y \leq 1$ and $y \geq 2$.

For (7), if $0 \leq y \leq 1$, then (4) gives

$$R(x+y, 1-y) = \frac{f(x+1)}{f(x+y)} \leq (x+y)^{1-y},$$

hence $f(x+y) \geq x(x+y)^{y-1} f(x)$. By (5), the opposite holds for $y \geq 1$.

The inequality (8) can be deduced from (7), but the following argument is easier. By (5),

$$R(x-1, y+1) = \frac{f(x+y)}{f(x-1)} \geq (x-1)^{y+1},$$

so $f(x+y) \geq (x-1)^y f(x)$. □

Both (6) and (7) are exact at the end points of the stated intervals for y.

We reassemble these inequalities for the three intervals for y, in some cases choosing the better of two available options:

Theorem 2. *For f as in Theorem 1, we have*

$$x(x+y)^{y-1} \leq R(x,y) \leq x^y \qquad \textit{for } 0 \leq y \leq 1, \tag{9}$$

$$x^y \leq R(x,y) \leq x(x+1)^{y-1} \qquad \textit{for } 1 \leq y \leq 2, \tag{10}$$

$$x(x+1)^{y-1} \leq R(x,y) \leq x(x+y)^{y-1} \qquad \textit{for } y \geq 2. \tag{11}$$

Note that (9) reproduces (1), and (10) amounts to the reverse of (2).

This system of inequalities amounts to a comprehensive set of bounds for $R(x, y)$ of the type we are considering. If it seems rather complicated, reflect that it describes the true situation, and that we have arrived at it by a very simple process. It is, of course, entirely typical for inequalities involving powers to reverse at certain values of the index. At the cost of a loss of accuracy, we can in fact extract one simple pair of bounds valid for all $y \geq 0$:

$$(x-1)^y \leq R(x,y) \leq x(x+y)^{y-1}. \tag{12}$$

It is worth restating (9) specifically for the gamma function in the case where x is a positive integer n. Since $\Gamma(n) = (n-1)!$, it equates to

$$(n+y)^{y-1} n! \leq \Gamma(n+y) \leq n^{y-1} n! \qquad \text{for } 0 \leq y \leq 1, \tag{13}$$

giving a pair of bounds for the gamma function between integers.

These inequalities have numerous applications. We describe four of them.

Application 1: $\Gamma(x+y)/\Gamma(x) \sim x^y$ *as* $x \to \infty$ *with* y *fixed.* (This was the objective of Wendel's article.) To show this, first suppose that $0 \le y \le 1$. Then (9) gives

$$\left(1+\frac{y}{x}\right)^{y-1} \le \frac{R(x,y)}{x^y} \le 1,$$

hence $R(x,y) \sim x^y$. For $y > 1$, the statement now follows from the fact that $R(x, y+1) = (1+\frac{y}{x})R(x,y)$.

Application 2: Binomial coefficients. Let

$$K_n(y) = (-1)^n \binom{-y}{n} = \binom{n+y-1}{n} = \frac{y(y+1)\dots(y+n-1)}{n!}.$$

This is the coefficient of t^n in the series for $(1-t)^{-y}$. Clearly

$$K_n(y) = \frac{\Gamma(n+y)}{\Gamma(y)n!} = \frac{R(n,y)}{n\Gamma(y)},$$

so we have by (9) and (10)

$$\frac{(n+y)^{y-1}}{\Gamma(y)} \le K_n(y) \le \frac{n^{y-1}}{\Gamma(y)} \qquad \text{for } 0 \le y \le 1, \tag{14}$$

$$\frac{n^{y-1}}{\Gamma(y)} \le K_n(y) \le \frac{(n+1)^{y-1}}{\Gamma(y)} \qquad \text{for } 1 \le y \le 2, \tag{15}$$

and (11) gives a similar estimation for $y \ge 2$. These can be regarded as bounds for $K_n(y)$, with $\Gamma(y)$ regarded as "known". Given the value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the case $y = \frac{1}{2}$ can be written as follows:

$$\frac{1}{[(n+\frac{1}{2})\pi]^{1/2}} \le \frac{1.3.\dots(2n-1)}{2.4.\dots(2n)} \le \frac{1}{(n\pi)^{1/2}}, \tag{16}$$

which amounts to the Wallis product stated as a pair of inequalities.

Application 3: Bounds for $\Gamma(y-n)$. If $0 < y < 1$, then $\Gamma(y-n)$ alternates sign in such a way that $(-1)^n\Gamma(y-n)$ is positive. Using Euler's reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, we can give bounds for this quantity that form a natural companion to (13). Indeed, by Euler's formula,

$$\Gamma(y-n)\Gamma(n+1-y) = \frac{\pi}{\sin\pi(y-n)} = (-1)^n\frac{\pi}{\sin\pi y}$$

and by (13), applied to $1-y$, we have $(n+1)^{-y}n! < \Gamma(n+1-y) < n^{-y}n!$. Hence, for $0 < y < 1$ and integers $n \ge 1$,

$$\frac{\pi}{\sin\pi y}\frac{n^y}{n!} < (-1)^n\Gamma(y-n) < \frac{\pi}{\sin\pi y}\frac{(n+1)^y}{n!}. \tag{17}$$

Application 4: The Bohr-Mollerup theorem: *Suppose that* $f(x)$ *is defined for* $x > 0$ *and: (i)* $\log f(x)$ *is convex, (ii)* $f(x+1) = xf(x)$, *(iii)* $f(1) = 1$. *Then* $f(x) = \Gamma(x)$.

We outline Artin's method. It uses Euler's limit expression for the gamma function, which can be stated as follows: $\Gamma(x) = \lim_{n\to\infty} G_n(x)$, where

$$G_n(x) = \frac{n^{x-1}n!}{x(x+1)\dots(x+n-1)}.$$

(According to taste, this can either be taken as the definition of the gamma function, or derived from the integral definition; note that, either way, convexity of $\log\Gamma(x)$ is an immediate consequence.) Equally, $\Gamma(x) = \lim_{n\to\infty} H_n(x)$, where

$$H_n(x) = \frac{(n+1)^{x-1}n!}{x(x+1)\dots(x+n-1)}.$$

It is clearly enough to prove the theorem for $1 < x < 2$ (for consistency with our earlier notation, we now write y for x). By (10), for integers n we then have $n^y \le R(n,y) \le n(n+1)^{y-1}$. Clearly, $f(n) = (n-1)!$, so

$$n^{y-1}n! \le f(n+y) \le (n+1)^{y-1}n!.$$

Since $f(n+y) = y(y+1)\dots(y+n-1)f(y)$, this equates to $G_n(y) \le f(y) \le H_n(y)$, so $f(y) = \lim_{n\to\infty} G_n(y) = \Gamma(y)$.

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