Γ**REC. Some results on** $\Gamma(1/x)$ **and** $\psi(1/x)$

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Preliminaries: the function $\psi(x)$

As usual, we use the notation $\psi(x)$ for $\Gamma'(x)/\Gamma(x)$. We will present some results for $\Gamma(1/x)$ and corresponding ones for $\psi(1/x)$. Even the results on $\Gamma(1/x)$ will require a number of properties of $\psi(x)$. We outline some of them here.

It is elementary that $\psi(x)$ is strictly increasing on $(0,\infty)$: this equates to the fact that $\log\Gamma(x)$ is convex. Also, $\psi(x_0) = 0$, where $x_0 \approx 1.46163$, so $\psi(x)$ is negative on $(0, x_0)$ and positive on (x_0, ∞). Of course, x_0 is the minimum point for $\Gamma(x)$. Further, $\psi(1) = -\gamma$ and $\psi(x+1) = \psi(x) + 1/x$ for all $x > 0$. Consequently, $\psi(x) + \frac{1}{x}$ tends to $-\gamma$ as $x \to 0$ and is greater than $-\gamma$ for $x > 0$.

The series expression for $\psi(x)$ is

$$\psi(x) = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+x}\right).$$

We will not really use this, but we will use the resulting (rather more pleasant) series expressions for $\psi'(x)$ and $\psi''(x)$:

$$\psi'(x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^2}, \qquad \psi''(x) = -\sum_{n=0}^{\infty}\frac{2}{(n+x)^3}. \tag{1}$$

Some results for $\Gamma(1/x)$

Write $\Gamma(1/x) = G(x)$. We consider this function for positive x. First, some immediate facts. Clearly, $G(1) = G(\frac{1}{2}) = 1$ and $G(x) < 1$ on $(\frac{1}{2}, 1)$, while $G(x) > 0$ on $(0, \frac{1}{2})$ and $(1, \infty)$. Also, $G(x)$ is decreasing on $(0, 1/x_0]$ and increasing on $[1/x_0, \infty)$, where $1/x_0 \approx 0.68417$.

Γ**REC1**. *For* $x \geq 1$*, we have* $x - \gamma \leq \Gamma(1/x) \leq x$*, and* $\Gamma(1/x) - x \to -\gamma$ *as* $x \to \infty$.

Proof. It is elementary that $1/y - \gamma \leq \Gamma(y) \leq 1/y$ for $0 < y < 1$, and $\Gamma(y) - 1/y \to -\gamma$ as $y \to 0$. Substitute $x = 1/y$. $\square$

Clearly, $G'(x) = -\frac{1}{x^2}\Gamma'(1/x)$. In particular, $G'(1) = -\Gamma'(1) = \gamma$ and $G'(\frac{1}{2}) = -4\Gamma'(2) = -4(1-\gamma)$.

Γ**REC2** PROPOSITION [JJ, Lemma 2]. *The function* $\Gamma(1/x)$ *is convex for all* $x > 0$.

Proof. Let $0 < x_1 < x_2$, and write $y_j = 1/x_j$ ($j = 1, 2$). Choose a, b so that $\Gamma(1 + y_j) = ay_j + b$ for $j = 1, 2$. Since Γ is convex, $\Gamma(1 + y) \leq ay + b$ for $y_2 \leq y \leq y_1$. Also, since $\Gamma(1 + y) = y\Gamma(y)$, we have $\Gamma(y) \leq a + b/y$ for $y_2 \leq y \leq y_1$, with equality at y_1 and y_2. So $\Gamma(1/x) \leq a + bx$ for $x_1 \leq x \leq x_2$, with equality at x_1 and x_2, so that $a + bx$ is the linear function agreeing with $\Gamma(1/x)$ at these points. $\square$

Note. In general, if a function f is convex and increasing, then $f(1/x)$ is convex (which implies ΓREC2 for $x > x_0$), but the example $f(x) = 1/(x + 1)$ shows that this is not true for decreasing f.

Γ**REC3** COROLLARY. *The function $x^2\Gamma'(x)$ increases with x.*

Proof. By ΓREC2, $-G'(x) = \frac{1}{x^2}\Gamma'(1/x)$ decreases with x. Apply to $y = 1/x$. $\square$

(As far as I know, a direct proof of ΓREC3 is not entirely trivial.)

Now if f is a convex, differentiable function on $(0, \infty)$ satisfying $f(1/x) = f(x)$, then $f'(1) = 0$, hence $f(x)$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, so the least value of $f(x)$ occurs at $x = 1$. Applied to $\Gamma(x) + \Gamma(1/x)$, this gives:

Γ**REC4** PROPOSITION. *Let $S(x) = \Gamma(x) + \Gamma(1/x)$. Then $S(x)$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, and $S(x) \geq 2$ for all $x > 0$.* $\square$

The inequality $S(x) \geq 2$ was first shown by Gautschi [Gau]. The monotonicity on $(0, 1]$ and $[1, \infty)$ was shown in [Al3, Lemma 2] by more elaborate methods.

The following inequality, which clearly implies $S(x) \geq 2$, was also obtained by Gautschi, by a different method.

Γ**REC5** PROPOSITION. *We have $\Gamma(x)\Gamma(1/x) \geq 1$ for all $x > 0$.*

Proof. Write $F(x) = \Gamma(x)\Gamma(1/x)$. From the identity $\Gamma(1 + x) = x\Gamma(x)$, we have $F(x) = \Gamma(1 + x)\Gamma(1 + \frac{1}{x})$. Since $\log\Gamma(x)$ is convex,

$$\log F(x) = \log\Gamma(1 + x) + \log\Gamma\left(1 + \frac{1}{x}\right) \geq 2\log\Gamma\left(1 + \frac{x}{2} + \frac{1}{2x}\right).$$

Now $x + \frac{1}{x} \geq 2$, so $1 + \frac{x}{2} + \frac{1}{2x} \geq 2$, hence $\log F(x) \geq 2\log\Gamma(2) = 0$. $\square$

Clearly, equality holds for $x = 1$, and strict inequality for other x. We will show later that $\Gamma(x)\Gamma(1/x)$ is convex, which of course implies ΓREC5.

The harmonic mean of a and b is

$$H(a,b) = \frac{2ab}{a+b}.$$

Let $H(x)$ denote the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$. In the same article [Gau], Gautschi showed that $H(x) \geq 1$ for all x: clearly, this implies ΓREC5. Note that it equates to the inequality $2\Gamma(x)\Gamma(1/x) \geq \Gamma(x) + \Gamma(1/x)$. Here we present a proof that is perhaps a little simpler than Gautschi's.

ΓREC6 LEMMA. *The function $x\psi(x)$ is convex for all $x > 0$.*

Proof. By the series expressions (1),

$$\begin{aligned}
\frac{d^2}{dx^2}[x\psi(x)] &= x\psi''(x) + 2\psi'(x) \\
&= -2\sum_{n=0}^{\infty} \frac{x}{(n+x)^3} + 2\sum_{n=0}^{\infty} \frac{1}{(n+x)^2} \\
&= 2\sum_{n=0}^{\infty} \frac{n}{(n+x)^3} > 0. \qquad \square
\end{aligned}$$

ΓREC7 LEMMA. *Let $h(x) = x\psi(x) + \gamma\Gamma(x)$. Then $h(x) > 0$ for $x > 1$ and $h(x) < 0$ for $\frac{2}{3} \leq x < 1$.*

Proof. By ΓREC6, $h(x)$ is convex. Since $\psi(1) = -\gamma$, we have $h(1) = 0$. The statement follows if $h(\frac{2}{3}) < 0$. Resorting to calculated values, we have $\psi(\frac{2}{3}) \approx -1.31823$, so $\frac{2}{3}\psi(\frac{2}{3}) \approx -0.87882$, and $\Gamma(\frac{2}{3}) \approx 1.35412$, so $\gamma\Gamma(\frac{2}{3}) \approx 0.78162$. $\square$

Note. The reliance on calculated values is somewhat inelegant, and suppresses unseen work. It can be avoided using elementary inequalities and no more than the values of $\pi^{1/2}$, γ and $\zeta(2)$ (or crude approximations to these values), as follows. By the convexity of the gamma function, we have

$$\Gamma(\tfrac{2}{3}) = \tfrac{3}{2}\Gamma(\tfrac{5}{3}) \leq \Gamma(\tfrac{3}{2}) + \tfrac{1}{2}\Gamma(2) = \tfrac{1}{2}\pi^{1/2} + \tfrac{1}{2} < 1.38623,$$

hence $\gamma\Gamma(\frac{2}{3}) < 0.8002$. Meanwhile, for $0 < x < 1$, we have the power series

$$\psi(1-x) = \gamma - \sum_{n=1}^{\infty} \zeta(n+1)x^n < -\gamma - \zeta(2)x - \sum_{n=2}^{\infty} x^n,$$

which gives $\psi(\frac{2}{3}) < -1.29219$, hence $\frac{2}{3}\psi(\frac{2}{3}) < -0.8614$.

ΓREC8 THEOREM. *Let $H(x)$ be the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$. Then $H(x)$ is increasing for $x \geq 1$ and decreasing for $0 < x \leq 1$, so $H(x) \geq 1$ for all $x > 0$. Equivalently,*

$$\Gamma(x) + \Gamma\left(\frac{1}{x}\right) \leq 2\Gamma(x)\Gamma\left(\frac{1}{x}\right). \tag{2}$$

Proof. Write $u(x) = 1/\Gamma(x)$. Then $2/H(x) = u(x) + u(1/x)$: denote this by $v(x)$. We show that $v(x)$ is decreasing, so that $H(x)$ is increasing, for $x \geq 1$. This is trivial for $x \geq \frac{3}{2}$, since both $u(x)$ and $u(1/x)$ are then decreasing. Now

$$u'(x) = -\frac{\Gamma'(x)}{\Gamma(x)^2} = -\frac{\psi(x)}{\Gamma(x)},$$

so

$$v'(x) = u'(x) - \frac{1}{x^2}u'\left(\frac{1}{x}\right) = -\frac{\psi(x)}{\Gamma(x)} + \frac{\psi(1/x)}{x^2\Gamma(1/x)}.$$

Writing $y = 1/x$, we can restate this as follows:

$$xv'(x) = \frac{y\psi(y)}{\Gamma(y)} - \frac{x\psi(x)}{\Gamma(x)}.$$

If $1 < x \leq \frac{3}{2}$, then $\frac{2}{3} \leq y < 1$, and, by ΓREC7,

$$\frac{x\psi(x)}{\Gamma(x)} > -\gamma > \frac{y\psi(y)}{\Gamma(y)},$$

hence $v'(x) < 0$, as required. □

Alzer [Al1] has generalized Gautschi's theorem by showing that the power mean $M_r[\Gamma(x), \Gamma(1/x)] \geq 1$ for all $r \geq r_0$, where $r_0 \approx -3.20464$.

Some estimations for $\psi'(x)$ and $\psi''(x)$

Our further results will need some bounds for $\psi'(x)$ and $\psi''(x)$, which we derive from the series expressions (1). Recall the process of estimating the sum of a series by the corresponding integral. Let f be a decreasing, non-negative function such that $\int_0^\infty f(t)\,dt$ converges (to I). Simple integral estimation then gives $\sum_{n=1}^\infty f(n) \leq I \leq \sum_{n=0}^\infty f(n)$. If f is convex, the trapezium formula overestimates the integral on each interval of length 1, so the upper bound can be reduced to $\frac{1}{2}f(0) + \sum_{n=1}^\infty f(n)$. So in this case, we have

$$I + \tfrac{1}{2}f(0) \leq \sum_{n=0}^{\infty} f(n) \leq I + f(0).$$

Applying this to the series for $\psi'(x)$ and $\psi''(x)$, we deduce:

Γ**REC9** PROPOSITION. *For all $x > 0$,*

$$\frac{1}{x} + \frac{1}{2x^2} \leq \psi'(x) \leq \frac{1}{x} + \frac{1}{x^2}, \tag{3}$$

$$-\frac{1}{x^2} - \frac{2}{x^3} \leq \psi''(x) \leq -\frac{1}{x^2} - \frac{1}{x^3}. \quad \square \tag{4}$$

Note. Further degrees of accuracy are provided by Euler-Maclaurin summation. For example,

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - r_2(x),$$

where $0 \le r_2(x) \le 1/(30x^5)$.

ΓREC10. *The function $x^2\psi'(x)$ is increasing for all $x > 0$.*

Proof. We have

$$x^2\psi'(x) = \sum_{n=0}^{\infty} \frac{x^2}{(n+x)^2}.$$

For each n,

$$\frac{x}{n+x} = 1 - \frac{1}{n+x},$$

which is increasing with x. □

ΓREC11. *The function $x\psi'(x)$ is decreasing for all $x > 0$.*

Proof. By (3) and (4), we have

$$\frac{d}{dx}[x\psi'(x)] = \psi'(x) + x\psi''(x) \le \frac{1}{x} + \frac{1}{x^2} - \left(\frac{1}{x} + \frac{1}{x^2}\right) = 0. \qquad \square.$$

Note. The last two results are special cases of [Al2, Lemma 1], which states: for integers $k \ge 1$, $x^{k+1}|\psi^{(k)}(x)|$ is increasing, while $x^k|\psi^{(k)}(x)|$ is decreasing. The proof in [Al2] uses the integral representation of $\psi(x)$. The first statement can be proved easily as above. The second statement is also a special case of the following result from [Jam1]: *Given a function f, let $S(x) = \sum_{n=0}^{\infty} f(n+x)$ and $I(x) = \int_0^{\infty} f(t+x)\,dt$. If $f'(x)/f(x)$ is increasing, then $S(x)/I(x)$ is decreasing.*

Some results for $\psi(1/x)$

Recall that $\psi(y) + \frac{1}{y}$ tends to $-\gamma$ as $y \to 0$ and is greater than $-\gamma$ for $y > 0$. Hence $\psi(1/x) + x$ tends to $-\gamma$ as $x \to \infty$ and is greater than $-\gamma$ for all $x > 0$. A slightly stronger inequality, which we will need later, is:

ΓREC12. *For $x > 1$,*

$$\psi(1/x) \ge \frac{1}{x} - x - \gamma. \tag{5}$$

Proof. Since $\psi(1) = -\gamma$, $\psi(2) = 1 - \gamma$ and $\psi(x)$ is concave, we have $\psi(x) + \frac{1}{x} = \psi(1+x) \ge x - \gamma$ for $0 < x \le 1$. Substitute $1/x$ for x. □

Now using ΓREC10 and ΓREC11, we deduce two results for $\psi(1/x)$:

Γ**REC13** PROPOSITION. *The function $\psi(1/x)$ is convex.*

Proof. We have

$$\frac{d}{dx}\psi\left(\frac{1}{x}\right) = -\frac{1}{x^2}\psi'\left(\frac{1}{x}\right) = -y^2\psi'(y),$$

where $y = 1/x$. By ΓREC10, this is decreasing with y, hence increasing with x. $\square$

Γ**REC14** PROPOSITION [Jam2]. *For all $x > 0$, we have*

$$\psi(x) + \psi(1/x) \le -2\gamma. \tag{6}$$

Proof. Write $\psi(x) + \psi(1/x) = P(x)$. Then

$$P'(x) = \psi'(x) - \frac{1}{x^2}\psi'\left(\frac{1}{x}\right) = \frac{1}{x}\left[x\psi'(x) - \frac{1}{x}\psi'\left(\frac{1}{x}\right)\right].$$

Let $x > 1$ and write $y = 1/x$. Then $x > y$, so by ΓREC11, $x\psi'(x) \le y\psi'(y)$, hence $P'(x) \le 0$. So $P(x)$ is decreasing on $[1,\infty)$, hence increasing on $(0,1]$, and the maximum value is $P(1) = -2\gamma$. $\square$

Of course, equality holds when $x = 1$, and it is clear from the preceding proofs that strict inequality holds for other x. Below, we will prove the stronger result that $P(x)$ is concave.

We now present some results on products and the harmonic mean, given in [AJ].

Γ**REC15** PROPOSITION. *For $0 \le y < 1$, we have*

$$\psi(1+y)\psi(1-y) \le \gamma^2. \tag{7}$$

Proof. If $x_0 - 1 < y \le 1$, then $\psi(1-y) < 0 < \psi(1+y)$, so (7) certainly holds. So assume that $y \le x_0 - 1$ ($< \frac{1}{2}$). By the power series $-\psi(1+y) = \gamma - \zeta(2)y + \zeta(3)y^2 - \ldots$,

$$-\psi(1+y) \le \gamma - \zeta(2)y + \zeta(3)y^2$$

since the series has alternate signs. Note that the right-hand side is positive for $0 \le y \le \frac{1}{2}$. Also,

$$\begin{aligned} -\psi(1-y) &\le \gamma + \zeta(2)y + \zeta(3)\sum_{n=2}^{\infty} y^n \\ &= \gamma + \zeta(2)y + \zeta(3)\frac{y^2}{1-y} \\ &\le \gamma + \zeta(2)y + 2\zeta(3)y^2. \end{aligned}$$

So

$$\psi(1+y)\psi(1-y) \le \gamma^2 - c_2 y^2 - c_3 y^3 + c_4 y^4,$$

where

$$c_2 = \zeta(2)^2 - 3\gamma\zeta(3) \approx 0.6244, \qquad c_3 = \zeta(2)\zeta(3), \qquad c_4 = 2\zeta(3)^2.$$

Clearly, $c_3 > \frac{1}{2}c_4$, so $c_3 y^3 > c_4 y^4$ for $0 < y \le \frac{1}{2}$. Since $c_2 > 0$, (10) follows. □

Note. Equality holds only for $y = 0$. In the opposite direction we have, clearly, $\psi(1+y)\psi(1-y) \ge \gamma^2 - \zeta(2)^2 y^2$ for y sufficiently close to 0.

Γ**REC16** PROPOSITION. *For positive $x \ne 1$, we have*

$$\psi(x)\psi(1/x) < \gamma^2. \tag{8}$$

Proof. It is sufficient to prove (8) for $x > 1$. If $x \ge x_0$, then $\psi(x)\psi(1/x) \le 0$. Assume that $x < x_0$ and write $x = 1+y$. Then $1/x > 1-y$, so $\psi(1-y) < \psi(1/x)$. Since $\psi(1+y) < 0$, we have from (7)

$$\psi(x)\psi(1/x) = \psi(1+y)\psi(1/x) < \psi(1+y)\psi(1-y) \le \gamma^2. \qquad \square$$

Γ**REC17** THEOREM. *For all $x > 0$, the harmonic mean of $\psi(x)$ and $\psi(1/x)$ is not less than $-\gamma$. In other words,*

$$\frac{2\psi(x)\psi(1/x)}{\psi(x) + \psi(1/x)} \ge -\gamma. \tag{9}$$

Proof. By (6) and (8), since $\psi(x) + \psi(1/x) < 0$,

$$2\psi(x)\psi(1/x)\,\frac{1}{\psi(x)+\psi(1/x)} > 2\gamma^2\,\frac{1}{\psi(x)+\psi(1/x)} > 2\gamma^2\,\frac{1}{-2\gamma} = -\gamma. \qquad \square$$

Convexity of $\log[\Gamma(x)\Gamma(1/x)]$ *and concavity of* $\psi(x) + \psi(1/x)$

Write $F(x) = \Gamma(x)\Gamma(1/x)$. We will show that $\log F(x)$, hence also $F(x)$, is convex; this result is given in [Jam2]. Of course, it implies ΓREC5. Two preliminary remarks will help to set the context for this statement. Firstly, in general, a product of two convex functions is convex when both are increasing or both decreasing, but not otherwise. Secondly, while $\log\Gamma(x)$ is convex, $\log\Gamma(1/x)$ is not.

We will use a number of estimations for $\psi(x)$, $\psi'(x)$ and $\psi''(x)$, including (3), (4) and (5). Trivially, since $\psi(x) \ge -\gamma$ for $x \ge 1$, we have

$$\psi(1/x) \ge -\gamma \quad \text{for } 0 < x \le 1. \tag{10}$$

Next, note that rewritten for $1/x$, (3) and (4) become:

$$x + \tfrac{1}{2}x^2 \le \psi'(1/x) \le x + x^2, \tag{11}$$

$$-x^2 - 2x^3 \le \psi''(1/x) \le -x^2 - x^3. \tag{12}$$

For $0 < x < 1$, better bounds are found by simply taking the first term of the series:

$$\psi'(x) \ge \frac{1}{x^2}, \qquad \psi''(x) \le -\frac{2}{x^3}, \tag{13}$$

so also $\psi'(1/x) \ge x^2$ and $\psi''(1/x) \le -2x^3$ for $x \ge 1$.

ΓREC18 THEOREM. *The function* $L(x) = \log\Gamma(x) + \log\Gamma(1/x)$, *is convex. Hence* $\Gamma(x)\Gamma(1/x)$ *is log-convex, so convex.*

Proof. We have

$$L'(x) = \psi(x) - \frac{1}{x^2}\psi(1/x),$$
$$L''(x) = \psi'(x) + \frac{2}{x^3}\psi(1/x) + \frac{1}{x^4}\psi'(1/x).$$

Note first that for $0 < x < 1/x_0 \approx 0.68417$, we have $\psi(1/x) > 0$, hence $L''(x) > 0$.

We now consider the cases $\frac{1}{4} \le x \le 1$ and $x > 1$ separately. We will estimate $\psi'(y)$ (where y is x or $1/x$) by (13) if $y \le 1$ and by (3) or (11) if $y \ge 1$.

Case $\frac{1}{4} \le x \le 1$. We estimate $\psi'(x)$ by (13), $\psi(1/x)$ by (10) and $\psi'(1/x)$ by (11), to obtain

$$\begin{aligned} x^4 L''(x) &\ge x^2 - 2\gamma x + (x + \tfrac{1}{2}x^2) \\ &= x\left(\tfrac{3}{2}x - (2\gamma - 1)\right) \\ &> 0 \qquad \text{for } x \ge \tfrac{1}{4}. \end{aligned}$$

Case $x > 1$. We now estimate $\psi'(x)$ by (3), $\psi(1/x)$ by (5) and $\psi'(1/x)$ by (13), to obtain

$$\begin{aligned} x^4 L''(x) &\ge x^4\left(\frac{1}{x} + \frac{1}{2x^2}\right) + 2x\left(\frac{1}{x} - x - \gamma\right) + x^2 \\ &= x^3 - \tfrac{1}{2}x^2 - 2\gamma x + 2. \end{aligned}$$

Denote this by $p(x)$. Then $p(1) = \frac{5}{2} - 2\gamma > 0$ and $p'(x) = 3x^2 - x - 2\gamma > 0$, hence $p(x) > 0$ and $L''(x) > 0$, for $x \ge 1$. □

Note. This reasoning shows that $\log\Gamma(1/x)$ itself is convex on $(0, 1]$. However, for $x > 1/(x_0 - 1)$, we have $\psi(1/x) < -x$ (since then $\psi(y) + \frac{1}{y} = \psi(y+1) < 0$, where $y = 1/x$). With (11), this shows that $\log\Gamma(1/x)$ is concave for such x.

We now prove, by similar steps, that $\psi(x)+\psi(1/x)$ is concave. Of course, this implies (6). We need an estimate corresponding to (5) for $\psi'(x)$.

Γ**REC19** LEMMA. *For $x>1$, we have*

$$\psi'(1/x) \le x^2 + \zeta(2) - \frac{1}{x}. \tag{14}$$

Proof. Since $\psi'(1)=\zeta(2)$, $\psi'(2)=\zeta(2)-1$ and $\psi'(x)$ is convex, we have $\psi'(1+x) \le \zeta(2)-x$ for $0 \le x \le 1$. Also, $\psi'(1+x)=\psi'(x)-1/x^2$, so $\psi'(x) \le 1/x^2+\zeta(2)-x$ for $0<x\le 1$. Substitute $1/x$ for x to obtain (14). □

Γ**REC20** THEOREM. *Let $P(x)=\psi(x)+\psi(1/x)$. Then $P(x)$ is concave on* $(0,\infty)$.

Proof. We have

$$P''(x) = \psi''(x) + \frac{2}{x^3}\psi'(1/x) + \frac{1}{x^4}\psi''(1/x).$$

Case $0<x\le 1$. Estimate $\psi''(x)$ by (13), $\psi'(1/x)$ by (11) and $\psi''(1/x)$ by (12) to obtain

$$\begin{aligned}
x^4 P''(x) &\le -2x + 2x(x+x^2) - (x^2+x^3) \\
&= x(x^2+x-2) \\
&\le 0 \quad \text{for } 0<x\le 1.
\end{aligned}$$

Case $x>1$. Estimate $\psi''(x)$ by (4), $\psi'(1/x)$ by (14) and $\psi''(1/x)$ by (13) to obtain

$$\begin{aligned}
x^4 P''(x) &\le -x^4\left(\frac{1}{x^2}+\frac{1}{x^3}\right) + 2x\left(x^2+\zeta(2)-\frac{1}{x}\right) - 2x^3 \\
&= -x^2 + Cx - 2,
\end{aligned}$$

where $C=2\zeta(2)-1$. Now $C<\frac{5}{2}$, from which we see that $x^2-Cx+2>0$ for all x. □.

Results relating to $\Gamma(x+\frac{1}{x})$

The function $x+\frac{1}{x}$ is convex, with least value 2. Since Γ is convex and increasing on $[2,\infty)$, it follows that $\Gamma(x+\frac{1}{x})$ is a convex function of x. Its least value is $\Gamma(2)=1$.

The following result, greatly strengthening ΓREC5, was first proved in [GL].

Γ**REC21** THEOREM. *For all $x>0$,*

$$\Gamma(x)\Gamma\left(\frac{1}{x}\right) \ge \tfrac{1}{2}\Gamma\left(1+x+\frac{1}{x}\right), \tag{15}$$

hence

$$\Gamma(x)\Gamma\left(\frac{1}{x}\right) \geq \Gamma\left(x + \frac{1}{x}\right). \tag{16}$$

Proof. Note first that (16) follows from (15), because

$$\tfrac{1}{2}\Gamma\left(1 + x + \frac{1}{x}\right) = \frac{1}{2}\left(x + \frac{1}{x}\right)\Gamma\left(x + \frac{1}{x}\right) \geq \Gamma\left(x + \frac{1}{x}\right).$$

We use the product formula

$$\Gamma(1+x) = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)^{x}\left(1 + \frac{x}{n}\right)^{-1}. \tag{17}$$

(The Weierstrass product formula, or Euler's limit expression for $\Gamma(x)$, would do equally well.) Recall that $\Gamma(x)\Gamma(\frac{1}{x}) = \Gamma(1+x)\Gamma(1+\frac{1}{x})$. Write $x + \frac{1}{x} = y$. By (17),

$$\frac{\Gamma(1+x)\Gamma(1+\frac{1}{x})}{\Gamma(1+y)} = \prod_{n=1}^{\infty} u_n(x),$$

where

$$u_n(x) = \frac{1 + \frac{y}{n}}{(1 + \frac{x}{n})(1 + \frac{1}{nx})} = \frac{1 + \frac{y}{n}}{1 + \frac{y}{n} + \frac{1}{n^2}}.$$

Now

$$\prod_{n=1}^{\infty} u_n(1) = \prod_{n=1}^{\infty}\frac{1 + \frac{2}{n}}{(1 + \frac{1}{n})^2} = \prod_{n=1}^{\infty}\frac{n(n+2)}{(n+1)^2} = \lim_{n\to\infty}\frac{n+2}{2(n+1)} = \frac{1}{2},$$

so the statement follows if we can show that $u_n(x) \geq u_n(1)$ for all $x > 0$. Now

$$\frac{u_n(x)}{u_n(1)} = \frac{(1 + \frac{y}{n})(1 + \frac{1}{n})^2}{(1 + \frac{y}{n} + \frac{1}{n^2})(1 + \frac{2}{n})}$$

and

$$\begin{aligned}
\left(1 + \frac{y}{n}\right)\left(1 + \frac{1}{n}\right)^2 - \left(1 + \frac{y}{n} + \frac{1}{n^2}\right)\left(1 + \frac{2}{n}\right) &= \frac{1}{n^2}\left(1 + \frac{y}{n}\right) - \frac{1}{n^2}\left(1 + \frac{2}{n}\right) \\
&= \frac{y-2}{n^3} \\
&\geq 0,
\end{aligned}$$

since $y \geq 2$ (and inequality is strict unless $x = 1$). □

The reverse inequality to (15), without the factor $\frac{1}{2}$, is elementary. Let $x > 1$. Since $\Gamma(1/x) \leq x$, we have

$$\Gamma(x)\Gamma\left(\frac{1}{x}\right) \leq x\Gamma(x) = \Gamma(1+x) \leq \Gamma\left(1 + x + \frac{1}{x}\right).$$

The following inequality, comparing the lower bounds in (2) and (15), was conjectured by Donald Kershaw and proved in [JJ]:

$$\Gamma(x) + \Gamma\left(\frac{1}{x}\right) \le \Gamma\left(1 + x + \frac{1}{x}\right). \tag{18}$$

With this, (15) implies (2). A second method for (18) was given in [LN], by deriving it from the inequalities

$$\frac{\Gamma(\frac{1}{x})}{\Gamma(x+\frac{1}{x})} - \frac{1}{x} \le (1+\gamma)\log x \le x - \frac{\Gamma(x)}{\Gamma(x+\frac{1}{x})}.$$

Alzer [Al3] established the following related inequality:

$$\Gamma(x) + \Gamma\left(\frac{1}{x}\right) \le C\Gamma\left(x + \frac{1}{x}\right),$$

where $C \approx 2.098$; one only needs to take $x = 2$ to see that this does not hold with $C = 2$.

Numerous further results concerning $\Gamma(1/x)$ and $\Gamma(x + 1/x)$ can be seen in the articles listed and further references listed there.

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