Basic theory of the gamma function derived from Euler's limit definition

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Contents

1. Euler's limit, and the associated product and series expressions

Euler's integral definition of the gamma function, valid for Re $z > 0$, is $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$. In 1729, Euler developed another definition of the gamma function as the limit of a certain expression. To motivate this expression, observe that for positive integers k and n,

$$(k-1)! = \frac{(n+k)!}{k(k+1)\dots(n+k)}$$

and

$$\frac{(n+k)!}{n^k n!} = \frac{(n+1)(n+2)\dots(n+k)}{n^k} = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{k}{n}\right).$$

Denote this by $C(n,k)$. Then $C(n,k) \to 1$ as $n \to \infty$ and $(k-1)! = C(n,k)\Gamma_n(k)$, where

$$\Gamma_n(k) = \frac{n^k n!}{k(k+1)\dots(k+n)}.$$

So we have shown that $\lim_{n\to\infty} \Gamma_n(k) = (k-1)!$.

Now define, for complex z other than 0 and negative real integers,

$$\Gamma_n(z) = \frac{n^z n!}{z(z+1)\dots(z+n)},$$

where n^z means $e^{z \log n}$. We will show that $\lim_{n\to\infty} \Gamma_n(z)$ actually exists for all complex z except 0 and negative integers (a pleasant alternative proof for the real case is given in section 5). This is Euler's limit definition of the gamma function. Here we will show how to derive the basic properties of the gamma function from this definition. Some of them can be proved equally easily from the integral definition, but others cannot. In particular, Euler's limit equates to certain infinite product expressions, which are sometimes adopted as yet another equivalent definition. These expressions, or their logarithmic equivalent, imply differentiability, together with a series expression, not for $\Gamma'(z)$ itself, but for $\Gamma'(z)/\Gamma(z)$. This, in turn, leads to numerous further identities and estimations.

Ultimately, a full understanding of the gamma function requires a judicious combination of both definitions. Equivalence of the two definitions is proved in section 7, but this section does not depend on the earlier ones, and could be read first.

Most of the methods presented here are standard, e.g. see [Cop, chapter 9]. The methods described in sections 5 and 6 are less well known; they have appeared in my articles [Jam1], [Jam2].

We will actually work with the following slight variant of Euler's limit: let

$$G_n(z) = \frac{n^z (n-1)!}{z(z+1)\dots(z+n-1)}. \tag{1}$$

The numerator $n^z(n-1)!$ can equally be written as $n^{z-1}n!$. Clearly,

$$\Gamma_n(z) = \frac{n}{z+n}\, G_n(z),$$

which shows that convergence of either implies convergence of the other, with the same limit. Note that $G_n(1) = 1$, while $\Gamma_n(1) = n/(n+1)$, which suggests that in some sense G_n is more "natural" than Γ_n. We observe also that

$$G_n(z+1) = \frac{nz}{n+z}\, G_n(z),$$

so once we have established convergence, the functional equation $\Gamma(z+1) = z\Gamma(z)$ will follow immediately. (This will establish again that $\Gamma(n) = (n-1)!$.) Note also that $G_n(z+1) = z\Gamma_n(z)$.

We will work with $\log G_n(z)$. For positive, real z, this presents no problem, and the following proofs become simpler in places. In the complex case, we need to take some care over logarithms. Any w such that $e^w = z$ is called a logarithm of z. For all $z \neq 0$, we can express z as $re^{i\theta}$ with $-\pi < \theta \leq \pi$, and the *principal* logarithm of z, which we denote by $\log z$, is $\log r + i\theta$. (Note that this applies to negative real x, with $\theta = -\pi$.) Write $\mathbb{R}^- = \{x \in \mathbb{R} : x \leq 0\}$. The following properties apply:

(L1) For positive real x, $\log(z/x) = \log z - \log x$.

(L2) For $z \in \mathbb{C} \setminus \mathbb{R}^-$, $\log z = \int_{[1,z]} \frac{1}{\zeta}\, d\zeta$, where $[1, z]$ is the straight-line path from 1 to z.

(L3) For $z \in \mathbb{C} \setminus \mathbb{R}^-$, $\log z$ is differentiable, with derivative $1/z$.

(L4) $\log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^n/n$ for all z with $|z| < 1$.

(L5) by (L4), we have $|\log(1 + z) - z| \leq |z|^2$ for $|z| \leq \frac{1}{2}$.

We now prove convergence, at the same time deriving two alternative product expressions for $\Gamma(z)$, together with corresponding series expressions for a logarithm of $\Gamma(z)$. These define differentiable functions, showing that $\Gamma(z)$ is differentiable, and also deliver series expressions for $\Gamma'(z)/\Gamma(z)$. We summarise all these identities in the following (necessarily fairly lengthy) statement.

1.1 THEOREM. *For all complex z except 0 and negative integers, $G_n(z)$ tends to a non-zero limit (which we denote by $\Gamma(z)$) as $n \to \infty$, and $\Gamma(z+1) = z\Gamma(z)$. Also, $\Gamma(z)$ equates to both the infinite products*

$$\frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} e^{z/n} \left(1 + \frac{z}{n}\right)^{-1}, \tag{2}$$

$$\frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1}, \tag{3}$$

where γ is Euler's constant. A logarithm of $\Gamma(z)$ (not necessarily the principal value!) is given by $L(z) = \lim_{n\to\infty} L_n(z)$, where

$$L_n(z) = z \log n + \log[(n-1)!] - \sum_{r=0}^{n-1} \log(z+r). \tag{4}$$

Further,

$$\begin{aligned} L(z) &= -\log z - \gamma z + \sum_{n=1}^{\infty} \left[\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right], &(5)\\ &= -\log z + \sum_{n=1}^{\infty} \left[z \log\left(1 + \frac{1}{n}\right) - \log\left(1 + \frac{z}{n}\right)\right] &(6) \end{aligned}$$

where γ is Euler's constant and the logarithms are principal values.

Further, the gamma function is differentiable at all such points, and writing $\Gamma'(z)/\Gamma(z) = \psi(z)$, we have:

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{7}$$

also $\psi(z) = \lim_{n\to\infty} \psi_n(z)$, where

$$\psi_n(z) = \log n - \sum_{r=0}^{n-1} \frac{1}{r+z}, \tag{8}$$

and

$$\begin{aligned} \psi(z) &= -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right) &(9)\\ &= -\frac{1}{z} + \sum_{n=1}^{\infty} \left[\log\left(1 + \frac{1}{n}\right) - \frac{1}{n+z}\right]. &(10) \end{aligned}$$

Proof. Let z be as stated. Clearly, $L_n(z)$, as defined in (4), is a logarithm of $G_n(z)$, so convergence of $L_n(z)$ to a limit $L(z)$ implies convergence of $G_n(z)$ to $e^{L(z)}$ (which we denote by $\Gamma(z)$). As already mentioned, the identity $\Gamma(z+1) = z\Gamma(z)$ then follows at once.

We will rewrite $L_n(z)$ in several ways. Since $\log[(n-1)!] = \sum_{r=1}^{n-1} \log r$, we have (using (L1))

$$\begin{aligned} L_n(z) &= z \log n - \log z + \sum_{r=1}^{n-1} [\log r - \log(z+r)] \\ &= z \log n - \log z - \sum_{r=1}^{n-1} \log\left(1 + \frac{z}{r}\right). \end{aligned} \tag{11}$$

Also, write $H_n = \sum_{r=1}^{n} \frac{1}{r}$. It is well known that $H_n = \log n + \gamma + \eta_n$, where $0 \le \eta_n \le \frac{1}{n}$. Hence $\log n = H_n - \gamma - \eta_n = H_{n-1} - \gamma + \delta_n$, where $\delta_n = \frac{1}{n} - \eta_n$, so that $0 \le \delta_n \le \frac{1}{n}$. Substituting this expression for $\log n$ in (11), we obtain

$$L_n(z) = -\log z - \gamma z + \sum_{r=1}^{n-1} \left[\frac{z}{r} - \log\left(1 + \frac{z}{r}\right)\right] + \delta_n z.$$

By (L5),

$$\left|\frac{z}{r} - \log\left(1 + \frac{z}{r}\right)\right| \le \frac{|z|^2}{r^2}$$

for $r > 2|z|$. It follows that $L_n(z)$ converges to $L(z)$, as defined by (5). By the "M-test", convergence is uniform on bounded subsets of $\mathbb{C} \setminus \mathbb{R}^-$. Identity (2) follows.

Since $\log n = \sum_{r=1}^{n-1} \log(1 + \frac{1}{r})$, we can also rewrite (11) as

$$L_n(z) = -\log z + \sum_{r=1}^{n-1} \left[z \log\left(1 + \frac{1}{r}\right) - \log\left(1 + \frac{z}{r}\right)\right].$$

Taking the limit as $n \to \infty$, we obtain (6), and hence (3).

For the discussion of differentiability, restrict attention first to z in $\mathbb{C} \setminus \mathbb{R}^-$. By (L3), $L_n'(z) = \psi_n(z)$, as defined by (8). By the general theorem on uniformly convergent sequences of complex functions, it follows that $L(z)$ is differentiable on this set, and $L'(z) = \lim_{n\to\infty} \psi_n(z)$. Termwise differentiation of (5) and (6) gives (9) and (10).

Hence $\Gamma(z) = e^{L(z)}$ is differentiable, with $\Gamma'(z) = L'(z)e^{L(z)}$, so that $L'(z) = \Gamma'(z)/\Gamma(z)$ (which we denote by $\psi(z)$). Differentiation of the functional equation gives $\Gamma'(z+1) = z\Gamma'(z) + \Gamma(z)$ and hence

$$\psi(z+1) = \frac{\Gamma'(z+1)}{z\Gamma(z)} = \psi(z) + \frac{1}{z}.$$

It just remains to consider negative, non-integer real x. By repeated application of $\Gamma(z+1) = z\Gamma(z)$, we see that $\Gamma(z)$ is also differentiable at these points, and (7) still holds. It is clear that the series in (9) converges: denote its sum (temporarily) by $\phi(z)$. Clearly, $\phi(z+1) = \phi(z) + 1/z$. With (7), this shows that $\phi(x) = \psi(x)$ also for negative, non-integer x. It is easily shown that the expressions in (8) and (10) are equivalent to (9). □

Clearly, $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ and $\psi(\overline{z}) = \overline{\psi(z)}$

The product (2) is called the *Weierstrass product* for the gamma function. It can be adopted as yet another alternative definition of the function: some writers do this. Clearly, $\Gamma(1+z)$ is given by the same infinite product with the factor $1/z$ removed.

The notation $\psi(z)$ is standard within the context of the theory of the gamma function. It is sometimes called the *digamma function*. We will describe its properties alongside those of $\Gamma(z)$; in some instances, they are rather more pleasant. Clearly, $\psi(1+z)$ is given by the series (9) with the term $-1/z$ removed. Also,

$$\psi(z_1) - \psi(z_2) = \sum_{n=0}^{\infty} \left(\frac{1}{n+z_2} - \frac{1}{n+z_1} \right),$$

and we have the following simple series expression for $\psi'(z)$:

1.2. *We have*

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}. \tag{12}$$

Proof. Since $1/n - 1/(n+z) = z/[n(n+z)]$, the series in (9) is uniformly convergent on bounded discs, so termwise differentiation is valid. □

So, for example, $\psi'(1) = \zeta(2) = \pi^2/6$, and $\psi'(2) = \zeta(2) - 1$. In the notation of the "Hurwitz zeta function", $\psi'(z)$ equals $\zeta(2, z)$. Of course, series for higher derivatives are given by repeated differentiation.

Remark 1. Euler's limit defines the gamma function for all z except negative integers, whereas the integral definition only applies for $\operatorname{Re} z > 0$. Also, it has automatically delivered the fact that $\Gamma(z) \neq 0$.

Remark 2. A variant of the proof, avoiding the process of choosing complex logarithms, is to prove convergence of the products in (2) and (3) directly, manipulating $G_n(z)$ in the same fashion as we did with $L_n(z)$. One needs the "M-test" for infinite products: *If* $|a_n(z)| \leq M_n$ *on a set* E *and* $\sum_{n=1}^{\infty} M_n$ *is convergent, then* $\prod_{n=1}^{\infty}(1 + a_n(z))$ *is uniformly convergent on* E. One needs a lemma to the effect that $e^z/(1+z) = 1 + a(z)$, where $|a(z)| \leq 2|z|^2$ for $|z| \leq \frac{1}{2}$.

Remark 3. For the real case, a proof of differentiability within real analysis would apply the following result: *If (f_n) is a sequence of functions with continuous derivatives, such that $f_n \to f$ pointwise and $f_n' \to g$ uniformly on an open interval I, then f is differentiable on I, with $f' = g$.* After relating (8) to (9) by substituting for $\log n$ as before, this means that we require uniform convergence of the series in (9) on bounded intervals. This is easily seen:

$$\frac{1}{n} - \frac{1}{n+x} = \frac{1}{n(n+x)} < \frac{1}{n^2}.$$

With minor modifications, this approach applies equally to the complex case.

Remark 4. Since $L_n(1+z) - L_n(z) = \log n - \log(n+z) + \log z$, we have $L(1+z) = L(z) + \log z$. Hence expressions for $L(1+z)$ are obtained by leaving out the term $-\log z$ in (5) and (6).

2. Elementary applications

Applications of the identities in Theorem 1 are very numerous: it is quite a challenge to arrange them in a coherent order! We start with some very simple ones.

2.1. *We have $z\Gamma(z) \to 1$ as $z \to 0$. So $\Gamma(z)$ has a simple pole at 0, with residue 1.*

Proof. By continuity, $z\Gamma(z) = \Gamma(z+1) \to \Gamma(1) = 1$ as $z \to 0$. □

The next result describes a very basic property of the real gamma function.

2.2 THEOREM. *For real $x > 0$, $\psi(x)$ is strictly increasing and strictly concave, hence $\log \Gamma(x)$ and $\Gamma(x)$ are strictly convex.*

Proof. It is clear from (9) that $\psi(x)$ is strictly increasing, hence $\log \Gamma(x)$ is strictly convex. (Convexity, though not strict convexity, already follows from (4)). It is elementary that if $f(x)$ is strictly convex, then so is $e^{f(x)}$: hence $\Gamma(x)$ is strictly convex. □

Hence, of course, $\Gamma'(x)$ is (strictly) increasing and $\psi'(x)$ is decreasing (this also follows from (12)).

By strict convexity and the values $\Gamma(1) = \Gamma(2) = 1$, we have at once:

2.3. *(i) For $1 \le x \le 2$, $\Gamma(x) < 1$.*

(ii) For $0 < x \le 1$, $\Gamma(x)$ is decreasing and $\Gamma(x) > 1$.

(iii) For $x \ge 2$, $\Gamma(x)$ is increasing and $\Gamma(x) > 1$. □

Further inequalities follow easily, by the functional equation:

2.4. *(i) For* $0 < x < 1$, $\Gamma(x) < 1/x$.

(ii) For $1 < x < 2$, $\Gamma(x) > \max(1/x, x-1)$.

Proof. (i) Then $x\Gamma(x) = \Gamma(1+x) < 1$.

(ii) Then $1 + x > 2$, so $x\Gamma(x) = \Gamma(1+x) > 1$. Also, $0 < x - 1 < 1$, so $\Gamma(x-1) > 1$, hence $\Gamma(x) = (x-1)\Gamma(x-1) > x - 1$. □

Clearly, $\Gamma(x)$ attains its least value at a point x_0 in $(1, 2)$: this is the point where $\psi(x_0) = 0$. By (ii), the least value is not less than the common value of $1/x$ and $x - 1$ where these intersect, i.e. $\frac{1}{2}(\sqrt{5}+1) \approx 0.618$. In fact, it is known that $x_0 \approx 1.4616$, with $\Gamma(x_0) \approx 0.8856$.

Example. A typical application of convexity of $\log\Gamma(x)$ is: $\log\Gamma(a+x)+\log\Gamma(a-x) \geq 2\log\Gamma(a)$ for $0 < x < a$, hence $\Gamma(a+x)\Gamma(a-x) \geq \Gamma(a)^2$. In particular, $\Gamma(1+x)\Gamma(1-x) \geq 1$ and $\Gamma(2+x)\Gamma(2-x) \geq 1$ for $0 \leq x \leq 1$. (We shall see later that there is an explicit expression for these products.)

The next result describes the behaviour of $\Gamma(z)$ on vertical lines.

2.5. *For* $z = x + iy$, *we have* $|\Gamma(z)| \leq |\Gamma(x)|$. *Also,* $|\Gamma(x+iy)|$ *decreases with* y *for* $y > 0$, *and* $|\Gamma(x+iy)| \leq |\Gamma(x+1)|/y$, *so* $\Gamma(x+iy) \to 0$ *as* $y \to \infty$.

Proof. We have $|n^z| = n^x$ and $|z + r| \geq |x + r|$, so $|G_n(z)| \leq |G_n(x)|$. Also, $|z + r| = |x + r + iy|$ increases with y, so $|G_n(z)|$ decreases with y. Further,

$$|\Gamma(x+iy)| = \frac{|\Gamma(x+1+iy)|}{|x+iy|} \leq \frac{|\Gamma(x+1)|}{y}.$$

(By considering $x + k + iy$, one can show that $|\Gamma(x + iy| = O(y^{-k})$.) □

We now identify the values of ψ and Γ' at positive integers.

2.6 PROPOSITION. *We have* $\psi(1) = \Gamma'(1) = -\gamma$.

Proof. In (8), we have $\psi_n(1) = \log n - H_{n-1} \to \gamma$ as $n \to \infty$. □

Recall the notation $H_n = \sum_{r=1}^{n} \frac{1}{r}$. By (7), we now deduce further:

2.7. *We have* $\psi(2) = \Gamma'(2) = 1 - \gamma$, *and for integers* $n \geq 2$, $\psi(n) = H_{n-1} - \gamma$. □

Example. $\psi(3) = \frac{3}{2} - \gamma$, hence $\Gamma'(3) = 3 - 2\gamma$. Also, $\psi(4) = \frac{11}{6} - \gamma$, so $\Gamma'(4) = 11 - 6\gamma$.

We can now strengthen 2.1:

2.8. *We have* $\Gamma(z) - 1/z \to -\gamma$ *as* $z \to 0$.

Proof. By 2.6, we have

$$\Gamma(z) - \frac{1}{z} = \frac{1}{z}\Big(\Gamma(1+z) - 1\Big) \to \Gamma'(1) = -\gamma \quad \text{as } z \to 0. \qquad \square$$

We can deduce the following further inequalities, using the fact that convex functions lie above their tangents, together with the values at 1 and 2:

2.9. *For* $0 < x < 1$, *we have:*

(i) $\log\Gamma(1+x) \geq -\gamma x$, *hence* $\Gamma(1+x) \geq e^{-\gamma x}$;

(ii) $\Gamma(1+x) \geq 1 - \gamma x$, *so* $1/x - \gamma \leq \Gamma(x) < 1/x$;

(iii) $\Gamma(1+x) \geq \gamma + (1-\gamma)x$. □

The two lower bounds for $\Gamma(1+x)$ in 2.9 intersect at $x = 1-\gamma$, improving our previous estimate for the minimum value $\Gamma(x_0)$ to $1 - \gamma + \gamma^2 \approx 0.7559$.

We now state some corresponding results for $\psi(z)$:

2.10. *When* $z \to 0$, *we have* $z\psi(z) \to -1$, $\psi(z) + 1/z \to -\gamma$ *and* $z^2\Gamma'(z) \to -1$. *For real* x *with* $0 \leq x \leq 1$, *we have* $\psi(1+x) = \psi(x) + 1/x \geq x - \gamma$.

Proof. By continuity, $\psi(z)+1/z = \psi(1+z) \to \psi(1) = -\gamma$ as $z \to 0$. Hence $z\psi(z) \to -1$. With 2.1, this gives $z^2\Gamma'(z) = z\Gamma(z).\, z\psi(z) \to -1$. Since ψ is concave, $\psi(1) = -\gamma$ and $\psi(2) = 1-\gamma$, we have $\psi(1+x) \geq x - \gamma$ for real x in $[0,1]$. □

Assuming, for the moment, equivalence with the integral definition of the gamma function, we can deduce the "exponential integral", direct evaluation of which is not entirely trivial (e.g. see [Jam4]):

2.11 PROPOSITION. *We have*

$$\int_0^\infty e^{-t}\log t\, dt = -\gamma.$$

Proof. By the integral definition, $\Gamma'(1)$ equates to this integral. □

Example. Recall from 1.2 that $\psi'(1) = \zeta(2)$. Now clearly

$$\psi'(z) = \frac{\Gamma''(z)}{\Gamma(z)} - \frac{\Gamma'(z)^2}{\Gamma(z)^2}.$$

Hence $\Gamma''(1) = \psi'(1) + \Gamma'(1)^2 = \zeta(2) + \gamma^2$.

One can deduce further inequalities from the concavity of ψ, for example $\psi(n+x) \geq H_{n-1} - \gamma + \frac{x}{n}$ for $0 \leq x \leq 1$ and (from the tangent at 1) $\psi(1+x) \leq \zeta(2)x - \gamma$ for all $x > 0$. However, a more pleasant estimation is obtained, in elegant style, as follows:

2.12 PROPOSITION. *For all* $x > 0$,

$$\log x - \frac{1}{x} \leq \psi(x) \leq \log x. \tag{13}$$

Proof. We have $\log\Gamma(x+1) - \log\Gamma(x) = \log x$. But by the mean-value theorem, $\log\Gamma(x+1) - \log\Gamma(x) = \psi(\xi)$ for some ξ in $(x, x+1)$. Since ψ is increasing, $\psi(x) \leq \log x < \psi(x+1)$. Since $\psi(x+1) = \psi(x) + \frac{1}{x}$, this equates to (13). (Also, $\psi(x) \geq \log(x-1)$, but this is weaker than (13).) □

By integration, we obtain corresponding inequalities for $\Gamma(x)$.

2.13 COROLLARY. *For all* $x > 0$,

$$x^{x-1}e^{1-x} \leq \Gamma(x) \leq x^x e^{1-x}. \tag{14}$$

Proof. Note that $\int_1^x \psi(t)\,dt = \log\Gamma(x)$. So, integration of (13) on $[1, x]$ gives

$$(x-1)\log x - x + 1 \leq \log\Gamma(x) \leq x\log x - x + 1,$$

which equates to (14). □

The estimations in (13) and (14) can be greatly improved: the outcome is Stirling's formula. A simple exposition is given in [Jam3].

A similar proof shows that $\psi'(x)$ is like $1/x$ for large x:

2.14. *For all* $x > 0$,

$$\frac{1}{x} \leq \psi'(x) \leq \frac{1}{x} + \frac{1}{x^2},$$

hence $\psi(x) - \log x$ *is increasing and* $\psi(x) - \log x + 1/x$ *is decreasing.*

Proof. By the mean-value theorem, $\frac{1}{x} = \psi(x+1) - \psi(x) = \psi'(\xi)$ for some ξ in $(x, x+1)$. Since ψ' is decreasing, $\psi'(x) \geq \frac{1}{x} \geq \psi'(x+1)$. Also, $\psi'(x+1) = \psi'(x) - 1/x^2$. □

Alternatively, 2.14 can be proved by integral estimation of the series (12) for $\psi'(x)$.

By the functional equation, we have

$$\Gamma(n+z+1) = z(z+1)\ldots(z+n)\Gamma(z).$$

We describe several consequences of this. First, $\Gamma(x)$ has alternate signs on the intervals between negative integers. In fact, if $x \in (-n-1, -n)$, then $x+n < 0 < x+n+1$, so $(-1)^{n+1}\Gamma(x) > 0$. Also, we can deduce the nature of $\Gamma(z)$ at integers $-n$:

2.15. *For positive integers n, we have $\lim_{z\to -n}(z+n)\Gamma(z) = (-1)^n/n!$, so $\Gamma(z)$ has a simple pole at $-n$, with residue $(-1)^n/n!$.*

Proof. We have

$$(z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n-1)} \to \frac{\Gamma(1)}{(-n)\dots(-2)(-1)} = \frac{(-1)^n}{n!} \quad \text{as } z \to -n. \qquad \square$$

2.16. *For fixed z, we have $\Gamma(n+z) \sim n^{z-1}n!$ as $n \to \infty$.*

Proof. By the definition (1) of $G_n(z)$,

$$G_n(z) = n^{z-1}n!\frac{\Gamma(z)}{\Gamma(n+z)},$$

so $\Gamma(n+z) = n^{z-1}n!\Gamma(z)/G_n(z)$. The statement follows. $\square$

Similarly, we have for $\psi(z)$:

2.17. *For positive integers n, $\psi(z)$ has a simple pole at $-n$, and $\psi(-n+z) + 1/z \to H_n - \gamma$ as $z \to 0$.*

Proof. This follows from the fact that

$$\psi(-n+z) = \psi(z) + \frac{1}{1-z} + \frac{1}{2-z} + \cdots + \frac{1}{n-z},$$

together with 2.10. $\square$

It follows that $\psi(x)$ strictly increases from $-\infty$ to ∞ on each interval $(-n, -n+1)$.

Next, we establish the values of $\Gamma(\frac{1}{2})$ and $\psi(\frac{1}{2})$. We assume the *Wallis product*, which can be stated as follows: let

$$W_n = \frac{2.4.\dots(2n)}{1.3.5.\dots(2n-1)(2n+1)^{1/2}}.$$

Then $W_n \to \sqrt{(\pi/2)}$ as $n \to \infty$.

2.18 PROPOSITION. *We have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.*

Proof. We have

$$G_n(\tfrac{1}{2}) = \frac{n!}{\frac{1}{2}(1+\frac{1}{2})\dots(n-\frac{1}{2})n^{1/2}} = \frac{2.4.\dots(2n-2)(2n)}{1.3.5.\dots(2n-1)n^{1/2}} = W_n\frac{(2n+1)^{1/2}}{n^{1/2}},$$

which tends to $\sqrt{(\pi/2)}\sqrt{2} = \sqrt{\pi}$ as $n \to \infty$. $\square$

Note that, from the integral definition, we have $\Gamma(\frac{1}{2}) = \int_{-\infty}^{\infty} e^{-x^2}\,dx$, so this is one way to evaluate this integral.

2.19. *We have*

$$\psi(\tfrac{1}{2}) = -\gamma - 2\log 2, \qquad \psi(\tfrac{3}{2}) = 2 - \gamma - 2\log 2 \approx 0.0365,$$

hence $x_0 < \frac{3}{2}$ *and* $\Gamma(x)$ *is strictly increasing for* $x \geq \frac{3}{2}$.

Proof. By (9),

$$\begin{aligned}\psi(\tfrac{1}{2}) &= -2 - \gamma + 2\sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n+1}\right)\\ &= -2 - \gamma + 2(1 - \log 2)\\ &= -\gamma - 2\log 2. \qquad \square\end{aligned}$$

3. Three identities

3.1 THEOREM ("Euler's reflection formula"). *For all non-integer* z,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{15}$$

Proof. We assume the well-known identity $\sin \pi z = \pi z \prod_{n=1}^{\infty}(1 - z^2/n^2)$. Recall (2):

$$\Gamma(z) = \frac{1}{z}\, e^{-\gamma z} \prod_{n=1}^{\infty} e^{z/n}\left(1 + \frac{z}{n}\right)^{-1}.$$

Hence also

$$\Gamma(1-z) = -z\Gamma(-z) = e^{\gamma z} \prod_{n=1}^{\infty} e^{-z/n}\left(1 - \frac{z}{n}\right)^{-1},$$

so

$$\Gamma(z)\Gamma(1-z) = \frac{1}{z}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)^{-1} = \frac{\pi}{\sin \pi z}. \qquad \square$$

This result can be derived equally from the expression for $G_n(z)$. Note that it can also be written $\Gamma(1+z)\Gamma(1-z) = \pi z/(\sin \pi z)$ (compare the example following 2.4). The value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ is a special case. Also, for example, $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$. Another application is:

3.2 COROLLARY. *For real* $y \neq 0$,

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}, \qquad |\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh \pi y}.$$

Proof. By (15), $\Gamma(iy)\Gamma(1-iy) = \pi/(\sin \pi iy) = \pi/(i \sinh \pi y)$. But this also equals $-iy\Gamma(iy)\Gamma(-iy) = -iy|\Gamma(iy)|^2$. The second identity follows.

As usual, there is a companion result for $\psi(z)$. For positive real numbers, it follows by logarithmic differentiation of Euler's formula, but this is more delicate in the complex case, and it is just as easy to prove it directly from the relevant series expressions.

3.3 PROPOSITION. *For all non-integer z,*

$$\psi(1-z) - \psi(z) = \pi \cot \pi z.$$

Proof. We use the series

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

By (8), we have

$$\begin{aligned} \psi_n(1-z) - \psi_n(z) &= \sum_{r=0}^{n-1} \frac{1}{r+z} - \sum_{r=0}^{n-1} \frac{1}{r+1-z} \\ &= \sum_{r=0}^{n-1} \frac{1}{r+z} - \sum_{r=1}^{n} \frac{1}{r-z} \\ &= \frac{1}{z} + \sum_{r=1}^{n-1} \left(\frac{1}{z+r} + \frac{1}{z-r} \right) + \frac{1}{n-z}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain the statement. □

Examples. $\psi(\frac{1}{2} - n) = \psi(\frac{1}{2} + n)$ for integers n; $\psi(\frac{3}{4}) - \psi(\frac{1}{4}) = \pi$.

3.4 PROPOSITION ("Legendre's duplication formula"). *For $2z$ not equal to 0 or a negative integer,*

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{16}$$

Note. For positive integers z, this statement follows easily from the functional equation and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. Recall that

$$G_n(z) = \frac{n^{z-1} n!}{z(z+1)\dots(z+n-1)}.$$

After doubling the terms to remove fractions, the expressions for $G_n(z)$ and $G_n(z+\frac{1}{2})$ interlace to give

$$G_n(z)G_n(z+\tfrac{1}{2}) = \frac{n^{2z-\frac{3}{2}} 2^{2n} (n!)^2}{(2z)(2z+1)\dots(2z+2n-1)}.$$

The trick is now to consider $G_{2n}(2z)$, not $G_n(2z)$. We have

$$G_{2n}(2z) = \frac{(2n)^{2z-1}(2n)!}{(2z)(2z+1)\dots(2z+2n-1)}.$$

So

$$\frac{G_n(z)G_n(z+\frac{1}{2})}{G_{2n}(2z)} = \frac{2^{2n}(n!)^2}{2^{2z-1}n^{1/2}(2n)!}.$$

By the Wallis product,

$$\frac{2^{2n}(n!)^2}{n^{1/2}(2n)!} \to \sqrt{\pi} \quad \text{as } n \to \infty,$$

which gives (16). □

The identity $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$ follows again.

4. The power series for $\psi(z)$ and $L(z)$

We start with $\psi(z)$. We actually state the power series for $\psi(1+z)$. Since $\psi(1+z) = \psi(z) + \frac{1}{z}$, the series for $\psi(z)$ itself is simply obtained by subtracting $1/z$.

4.1 THEOREM. *For* $|z| < 1$,

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)z^n = -\gamma + \zeta(2)z - \zeta(3)z^2 + \cdots. \tag{17}$$

Proof. By (9) and (7),

$$\psi(1+z) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right).$$

By the geometric series, for $|z| < k$,

$$\begin{aligned}\frac{1}{k} - \frac{1}{k+z} = \frac{z}{k(k+z)} &= \frac{z}{k^2(1+\frac{z}{k})} \\ &= \frac{z}{k^2}\sum_{m=0}^{\infty}(-1)^m\frac{z^m}{k^m} \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{k^{n+1}}.\end{aligned}$$

Assuming that reversal of the order of summation is valid, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{z}{k(k+z)} &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{k^{n+1}} \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}z^n\sum_{k=1}^{\infty}\frac{1}{k^{n+1}} \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)z^n.\end{aligned}$$

The series $\sum_{n=1}^\infty \zeta(n+1)|z|^n$ converges for all z with $|z|<1$, since $\zeta(n+1)<2$ for all n. Hence the double series is absolutely convergent, and the reversal is indeed valid. □

Alternatively, (17) follows from the values of $\psi^{(n)}(1)$ implied by repeated differentiation as in 1.2.

Note. It is easily shown by integral estimation that $\zeta(m)-1 \le 1/2^{m-1}$ for $m \ge 3$. It follows that the series $\sum_{n=1}^\infty (-1)^{n+1}[\zeta(n+1)-1]z^n$ converges for $|z|<2$. In the proof of the theorem, if we first separate out the term $k=1$, which contributes $z/(1+z)$, the term $\zeta(n+1)$ is replaced by $\zeta(n+1)-1$ in the ensuing calculations, so we obtain

$$\psi(1+z) = -\gamma + \frac{z}{1+z} + \sum_{n=1}^\infty (-1)^{n+1}[\zeta(n+1)-1]z^n$$

for $|z|<2$ (at $z=-1$, this holds in the sense of a limiting value, given the elementary fact that $\sum_{n=2}^\infty [\zeta(n)-1]=1$).

Now consider $L(z)$, as defined in section 1. Recall (Remark 4) that $L(1+z)=L(z)+\log z$.

4.2 THEOREM. *With $L(z)$ defined as in Theorem 1.1, we have for $|z|<1$*

$$L(1+z) = -\gamma z + \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)}{n} z^n. \tag{18}$$

Proof. By (5) and Remark 4, we have

$$L(1+z) = -\gamma z + \sum_{k=1}^\infty \left[\frac{z}{k} - \log\left(1+\frac{z}{k}\right)\right],$$

Now $z-\log(1+z) = \sum_{n=2}^\infty (-1)^n z^n/n$, so

$$L(1+z) = -\gamma z + \sum_{k=1}^\infty \sum_{n=2}^\infty (-1)^n \frac{z_n}{nk^n}.$$

Reversing the order of summation as before, we obtain (18). □

Of course, differentiation of the series in (18) gives (17), so either of these series can be used to prove the other.

Again, by separating out the term $k=1$, we can deduce the following variant, valid for $|z|<2$ except for real $z\le -1$,

$$L(1+z) = (1-\gamma)z - \log(1+z) + \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)-1}{n} z^n.$$

This implies that (18) is also valid for $z = 1$. Since $L(2) = 0$, it equates to the identity

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} = \gamma,$$

which can, of course, be proved more directly.

The power series for $\Gamma(1+z)$ is considerably less pleasant. The coefficients are best expressed in terms of integrals. This series is discussed in separate notes ΓPS.

5. Alternative proof of convergence in the real case

We now present an attractive alternative proof of convergence of $G_n(x)$ in the real case, based on the principle that bounded, monotonic sequences converge. The method has appeared in [Jam2]. It delivers some useful inequalities without further work. (However, readers can omit or defer this section).

Define a companion sequence $H_n(x)$ by

$$H_n(x) = \frac{(n+1)^{x-1} n!}{x(x+1)\dots(x+n-1)}.$$

Then $H_n(1) = H_n(2) = 1$ and

$$H_n(x) = \left(1 + \frac{1}{n}\right)^{x-1} G_n(x).$$

Hence $G_n(x)$ and $H_n(x)$ have the same limit (if either converges), and $G_n(x) < H_n(x)$ for $x > 1$, while $G_n(x) > H_n(x)$ for $0 < x < 1$

We use the following well-known inequality:

5.1 LEMMA. *For all* $t > 0$, $(1+t)^p > 1+pt$ *if* $p > 1$, *and* $(1+t)^p < 1+pt$ *if* $0 < p < 1$.

Proof. Let $f(t) = (1+t)^p$. By the mean-value theorem,

$$(1+t)^p - 1 = f(t) - f(0) = pt(1+\xi)^{p-1}$$

for some $\xi \in (0, t)$. Clearly, $(1+\xi)^{p-1} > 1$ if $p > 1$, and the reverse holds if $0 < p < 1$: the statement follows. □

5.2. *(i)* $G_n(x)$ *increases (strictly) with* n *for* $x > 1$ *and decreases for* $0 < x < 1$.

(ii) $H_n(x)$ *increases with* n *for* $x > 2$ *and for* $0 < x < 1$, *and decreases for* $1 < x < 2$.

Proof. Let $\delta_n(x) = G_{n+1}(x)/G_n(x)$. Then

$$\delta_n(x) = \left(\frac{n+1}{n}\right)^x \frac{n}{n+x} = \left(1 + \frac{1}{n}\right)^x \left(1 + \frac{x}{n}\right)^{-1}.$$

By the Lemma, $\delta_n(x) > 1$ for $x > 1$ and $\delta_n(x) < 1$ for $0 < x < 1$.

Now let $\mu_{n-1}(x) = H_n(x)/H_{n-1}(x)$. Then

$$\mu_{n-1}(x) = \left(\frac{n+1}{n}\right)^{x-1} \frac{n}{n+x-1} = \left(1+\frac{1}{n}\right)^{x-1} \left(1+\frac{x-1}{n}\right)^{-1}.$$

By the Lemma, $\mu_{n-1}(x) > 1$ for $x > 2$ and $0 < x < 1$, while $\mu_{n-1}(x) < 1$ for $1 < x < 2$. □

We have already seen that $G_n(x)$ is constant when $x = 1$, and $H_n(x)$ is constant when $x = 1$ and $x = 2$.

5.3 THEOREM. *For all real x except* 0 *and negative integers, $G_n(x)$ and $H_n(x)$ tend to a common limit (which we denote by $\Gamma(x)$) as $n \to \infty$. Furthermore, we have:*

$$H_n(x) \le \Gamma(x) \le G_n(x) \qquad \text{for } 0 \le x \le 1, \tag{19}$$

$$G_n(x) \le \Gamma(x) \le H_n(x) \qquad \text{for } 1 \le x \le 2, \tag{20}$$

$$G_n(x) \le H_n(x) \le \Gamma(x) \qquad \text{for } x \ge 2. \tag{21}$$

Proof. First, let $1 \le x \le 2$. Then $G_n(x)$ is increasing, $H_n(x)$ is decreasing and $G_n(x) \le H_n(x)$ for all n. Hence $G_n(x) \le H_1(x)$ for all n, so $G_n(x)$ is bounded above, hence convergent. As already noted, it follows that $H_n(x)$ converges to the same limit. Furthermore, it is clear that (20) holds. When $0 \le x \le 1$, similar reasoning establishes convergence and (19).

To deduce the statement for all other x, just note that convergence of $G_n(x)$ to L implies convergence of $G_n(x+1)$ to xL, and conversely. Repeated steps of length 1 (both forwards and backwards) now establish convergence for other x. Also, (21) holds. □

Since $x\Gamma_n(x) = G_n(x+1)$, it follows that for all $x > 0$, $\Gamma_n(x)$ increases with n and $\Gamma_n(x) \le \Gamma(x)$.

The inequality (19) can be rewritten as a pair of bounds for $\Gamma(x)$ between integers. Since $x(x+1)\dots(x+n-1) = \Gamma(n+x)/\Gamma(x)$, we have

$$G_n(x) = n^{x-1}n!\frac{\Gamma(x)}{\Gamma(n+x)}, \qquad H_n(x) = (n+1)^{x-1}n!\frac{\Gamma(x)}{\Gamma(n+x)}.$$

Inserted into (19), this gives:

5.4 COROLLARY. *For $0 \le x \le 1$ and integers $n \ge 1$,*

$$(n+1)^{x-1}n! \le \Gamma(n+x) \le n^{x-1}n!. \tag{22}$$

For $1 \le x \le 2$, these inequalities are reversed. □

We return to inequalities of this sort in section 6, where we will show that there is no need for the restriction of n to integers.

Example. Given the value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the case $x = \frac{1}{2}$ in (19) equates to the following specific form of the Wallis product:

$$(n\pi)^{1/2} \le \frac{2.4.\ldots(2n)}{1.3\ldots(2n-1)} \le [(n+1)\pi]^{1/2}.$$

All the identities in Theorem 1.1 can now be derived as before (with, of course, no complication in the choice of $\log\Gamma(x)$). Differentiability is proved as in Remark 3.

6. Inequalities for gamma function ratios; the Bohr-Mollerup theorem

This section follows the article [Jam1], with some additional results for $\psi(x)$. The objective is to find upper and lower bounds for the ratio $\Gamma(x+y)/\Gamma(x)$, which we denote by $R(x,y)$. A simple estimation follows easily from the fact (2.12) that $\log(x-1) \le \psi(x) \le \log x$:

6.1. *For all $x > 1$ and $y > 0$,*

$$(x-1)^y \le R(x,y) \le (x+y)^y. \tag{23}$$

Hence $R(x+y) \sim x^y$ as $x \to \infty$ with y fixed.

Proof. By the mean-value theorem, $\log\Gamma(x+y) - \log\Gamma(x) = y\psi(\xi)$ for some ξ in $(x, x+y)$. By 2.12, we have $\log(x-1) \le \psi(\xi) \le \log(x+y)$. Inequality (23) follows, and it clearly implies the last statement. □

With very little extra effort, we now derive more accurate estimations for $R(x,y)$, distinguishing the cases $0 \le y \le 1$, $1 \le y \le 2$ and $y \ge 2$. Furthermore, we will use nothing more than log-convexity and the functional equation. So we consider any function $f(x)$, defined for $x > 0$, that satisfies

(A) $\quad \log f(x)$ is convex and $f(x+1) = xf(x)$ for all $x > 0$.

By proceeding in this way, we will be able to use the results to establish that $\Gamma(x)$ is in fact the only function that satisfies (A) and $f(1) = 1$.

6.2. *Suppose that f satisfies (A). Write $R(x,y) = f(x+y)/f(x)$. Then*

$$R(x,y) \le x^y \quad \text{for } 0 \le y \le 1, \tag{24}$$

$$R(x,y) \geq x^y \quad \textit{for } y \geq 1 \textit{ and for } y < 0. \tag{25}$$

Proof. For fixed x, let $F(y) = \log f(x+y) - y \log x$ for all $y > -x$. Then F is convex and

$$F(1) = \log f(x+1) - \log x = \log f(x) = F(0).$$

It follows that $F(y) \leq \log f(x)$ for $0 \leq y \leq 1$ and $F(y) \geq \log f(x)$ for $y \geq 1$ and for $y \leq 0$. This equates to (24) and (25). □

Note. A variant of this proof is as follows. Let $L(x) = \log f(x)$ and $m_L(x,y) = [L(y) - L(x)]/(y-x)$. Clearly, $m_L(x, x+1) = \log x$. If $0 < y < 1$, then convexity of L implies that $m_L(x, x+y) \leq m_L(x, x+1)$, hence $\frac{1}{y} \log R(x,y) \leq \log x$, so $R(x,y) \leq x^y$. The opposite holds when $y > 1$. □

Clearly, equality holds when y is 0 or 1, and if $\log f(x)$ is strictly convex, then strict inequality holds for other y.

By suitable substitutions, we can deduce further bounds applying for various intervals for y:

6.3. *For f satisfying (A), we have*

$$R(x,y) \leq x(x+1)^{y-1} \qquad \textit{for } 1 \leq y \leq 2, \tag{26}$$

and the opposite holds for $y \leq 1$ and $y > 2$. Also,

$$R(x,y) \geq x(x+y)^{y-1} \qquad \textit{for } 0 \leq y \leq 1, \tag{27}$$

and the opposite holds for $y > 1$.

Proof. For (26): if $1 \leq y \leq 2$, then (24) gives

$$R(x+1, y-1) = \frac{f(x+y)}{f(x+1)} \leq (x+1)^{y-1},$$

so that $f(x+y) \leq x(x+1)^{y-1} f(x)$. By (25), the opposite holds for $y \leq 1$ and $y \geq 2$.

For (27): if $0 \leq y \leq 1$, then (24) gives

$$R(x+y, 1-y) = \frac{f(x+1)}{f(x+y)} \leq (x+y)^{1-y},$$

hence $f(x+y) \geq x(x+y)^{y-1} f(x)$. By (25), the opposite holds for $y \geq 1$. □

Note that both (26) and (27) are exact at the end points of the stated intervals for y.

We reassemble these inequalities for the three intervals for y, obtaining a comprehensive system of bounds for $R(x,y)$. In some cases, we choose the better of two options on offer:

6.4 THEOREM. *For f satisfying (A), we have*

$$x(x+y)^{y-1} \le R(x,y) \le x^y \qquad \text{for } 0 \le y \le 1, \tag{28}$$

$$x^y \le R(x,y) \le x(x+1)^{y-1} \qquad \text{for } 1 \le y \le 2, \tag{29}$$

$$x(x+1)^{y-1} \le R(x,y) \le x(x+y)^{y-1} \qquad \text{for } y \ge 2. \qquad \square \tag{30}$$

If this seems complicated, reflect that we have arrived at it by a very simple process! It is quite normal for inequalities involving powers to reverse at certain values of the index.

Note. To recapture the (weaker) lower bound $(x-1)^y$ in (23), observe that by (25),

$$R(x-1, y+1) = \frac{f(x+y)}{f(x-1)} \ge (x-1)^{y+1},$$

so $f(x+y) \ge (x-1)^y f(x)$.

These inequalities have a curious history. In Artin's book [Art], published in 1931, the inequality $(x-1)^y \le R(x,y) \le x^y$ appears on p. 14 (actually stated for integer x), but only as a step in the proof of the Bohr-Mollerup theorem. The method is essentially the one given here. With no reference to Artin, and by a considerably longer method, Wendel proved (28) in 1948 [Wen]. In 1959, with no reference to Artin or Wendel, and by a different method again, Gautschi [Gau] obtained (28) with the weaker lower bound $x(x+1)^{y-1}$ (along with another estimate in terms of $\psi(x)$). Later writers have generally referred to results of this type as "Gautschi-type inequalities", with scant respect to either Artin or Wendel.

These inequalities have numerous applications. First, we restate (28) specifically for the gamma function in the case where x is a positive integer n, thereby obtaining bounds for the gamma function between integers. Since $\Gamma(n) = (n-1)!$, the statement becomes:

6.5. *For positive integers n and $0 \le y \le 1$,*

$$(n+y)^{y-1} n! \le \Gamma(n+y) \le n^{y-1} n! \qquad \text{for } 0 \le y \le 1. \qquad \square \tag{31}$$

In 5.4, the same upper bound was found by a different method, together with the slightly weaker lower bound $(n+1)^{y-1} n!$.

Another application is to binomial coefficients. Let

$$K_n(y) = (-1)^n \binom{-y}{n} = \binom{n+y-1}{n} = \frac{y(y+1)\dots(y+n-1)}{n!}.$$

This is the coefficient of t^n in the series for $(1-t)^{-y}$. Clearly

$$K_n(y) = \frac{\Gamma(n+y)}{\Gamma(y)n!} = \frac{R(n,y)}{n\Gamma(y)},$$

so by (28) and (29), we have:

6.6. *We have*

$$\frac{(n+y)^{y-1}}{\Gamma(y)} \le K_n(y) \le \frac{n^{y-1}}{\Gamma(y)} \qquad \text{for } 0 \le y \le 1,$$

$$\frac{n^{y-1}}{\Gamma(y)} \le K_n(y) \le \frac{(n+1)^{y-1}}{\Gamma(y)} \qquad \text{for } 1 \le y \le 2.$$

A similar estimate for $y > 2$ can be derived from (30). These can be regarded as bounds for $K_n(y)$, with $\Gamma(y)$ regarded as known.

Using Euler's reflection formula (3.1), we can deduce bounds for $\Gamma(y-n)$ that form a natural companion to (31).

6.7. *For $0 < y < 1$ and integers $n \ge 1$,*

$$\frac{\pi}{\sin \pi y} \frac{n^y}{n!} < (-1)^n \Gamma(y-n) < \frac{\pi}{\sin \pi y} \frac{(n+1)^y}{n!}. \tag{32}$$

Proof. By Euler's formula,

$$\Gamma(y-n)\Gamma(n+1-y) = \frac{\pi}{\sin \pi(y-n)} = (-1)^n \frac{\pi}{\sin \pi y}$$

and by (31), applied to $1-y$, we have $(n+1)^{-y} n! < \Gamma(n+1-y) < n^{-y} n!$. □

Theorem 6.4, together with Euler's limit, delivers very easily the following characterisation of the real gamma function, known as the Bohr-Mollerup theorem. We use $G_n(x)$, together with the companion expression $H_n(x)$ defined in section 5.

6.8 THEOREM. *Suppose that $f(x)$ is defined for $x > 0$ and: (i) $\log f(x)$ is convex, (ii) $f(x+1) = xf(x)$, (iii) $f(1) = 1$. Then $f(x) = \Gamma(x)$.*

Proof. It is clearly enough to prove the theorem for $1 < x < 2$ (for consistency with the current notation, we now write y for x). By (29), for integers n we then have $n^y \le R(n,y) \le n(n+1)^{y-1}$. Clearly, $f(n) = (n-1)!$, so

$$n^{y-1} n! \le f(n+y) \le (n+1)^{y-1} n!.$$

Since $f(n+y) = y(y+1)\dots(y+n-1)f(y)$, this equates to $G_n(y) \le f(y) \le H_n(y)$, so $f(y) = \lim_{n\to\infty} G_n(y) = \Gamma(y)$. □

We now describe, in less detail, analogous results for $\psi(x)$. The corresponding quantity to consider is $\psi(x+y) - \psi(x)$. As an easy consequence of 2.14, we have:

6.9. *For $x > 1$ and $y > 0$,*

$$\frac{y}{x+y} \le \psi(x+y) - \psi(x) \le \frac{y}{x-1}.$$

Proof. By the mean-value theorem, $\psi(x+y) - \psi(x) = y\psi'(\xi)$ for some $\xi \in (x, x+y)$. Since ψ' is decreasing, 2.14 gives $\psi'(\xi) \le \psi'(x) \le 1/(x-1)$, also $\psi'(\xi) \ge \psi'(x+y) \ge 1/(x+y)$. □

Now consider any function g, defined on the positive real line, that shares the following properties of ψ:

(B) g is concave and $g(x+1) = g(x) + \frac{1}{x}$ for all $x > 0$.

6.10. *Suppose that g satisfies (B), and write $S(x,y) = g(x+y) - g(x)$. Then*

$$S(x,y) \ge \frac{y}{x} \quad \text{for } 0 \le y \le 1, \tag{33}$$

and the opposite holds for $y \ge 1$ and for $y \le 0$. Further, for $0 < y < 1$,

$$S(x,y) \le \frac{y(x+1)}{x(x+y)}. \tag{34}$$

Hence $S(x,y) \to 0$ as $x \to \infty$ with y fixed.

Proof. For fixed x, let $G(y) = g(x+y) - y/x$. Then G is concave and $G(1) = g(x+1) - \frac{1}{x} = g(x) = G(0)$. Hence $G(y) \ge g(x)$ for $0 \le y \le 1$ and $G(y) \le g(x)$ for other y. This equates to (33). For $0 < y < 1$, we deduce further

$$S(x+y, 1-y) = g(x+1) - g(x+y) \ge \frac{1-y}{x+y}$$

so that

$$g(x+y) - g(x) \le \frac{1}{x} - \frac{1-y}{x+y} = \frac{y(x+1)}{x(x+y)}.$$

□

This also implies the simpler upper bound $y/(x-1)$. Suitable substitutions give further inequalities for $1 < y < 2$ and for $y > 2$: we leave this to the reader. Instead, we show that the analogue of the Bohr-Mollerup theorem follows very easily:

6.11 PROPOSITION. *If g satisfies (B), then $g(x) = \psi(x) + c$ for some constant c.*

Proof. Fix $y > 0$, and let $g(n+y) - g(n) = \delta_n$. By 6.10, $\delta_n \to 0$ as $n \to \infty$. By (B),

$$g(n) = g(1) + \sum_{r=1}^{n-1} \frac{1}{r},$$

$$g(n+y) = g(y) + \frac{1}{y} + \sum_{r=1}^{n-1} \frac{1}{r+y},$$

hence

$$g(y) = g(1) - \frac{1}{y} + \sum_{r=1}^{n-1} \left(\frac{1}{r} - \frac{1}{r+y} \right) + \delta_n.$$

By (9), the right-hand side tends to $\psi(y) + g(1) + \gamma$ as $n \to \infty$. □

7. Equivalence of Euler's limit and the integral definition

We now establish the equivalence of Euler's limit with the integral definition $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$. It is elementary that this integral converges for all z with Re $z > 0$. We shall present the proof in terms of $\Gamma_n(z)$ rather than $G_n(z)$.

7.1 LEMMA. *For* $0 \le t < n$,

$$\left(1 - \frac{t^2}{n}\right) e^{-t} \le \left(1 - \frac{t}{n}\right)^n \le e^{-t}.$$

Proof. From the series for e^y and $1/(1-y)$, it is clear that $1 + y \le e^y \le 1/(1-y)$ for $0 \le y < 1$. Multiplying through by $(1-y)e^{-y}$, we obtain $(1-y^2)e^{-y} \le 1 - y \le e^{-y}$. Now take the nth power:

$$(1-y^2)^n e^{-ny} \le (1-y)^n \le e^{-ny}$$

for $0 \le y < 1$. Now $(1-a)^n \ge 1 - na$ for $0 \le a < 1$, so we can replace $(1-y^2)^n$ by $1 - ny^2$. Substitute $y = t/n$ to obtain the statement. □

Recall that the *beta integral* $B(a,b)$ is defined by $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$ for a, b with Re $a > 0$ and Re $b > 0$. We assume the following well-known evaluation of $B(a,n)$ for positive integers n:

$$B(a,n) = \frac{(n-1)!}{a(a+1)\dots(a+n-1)}.$$

Hence $\Gamma_n(z) = n^z B(z, n+1)$.

7.2 THEOREM. *For* Re $z > 0$, *we have*

$$\lim_{n\to\infty} \Gamma_n(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Proof. Denote the integral by $I(z)$, and let

$$I_n = \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n \, dt.$$

The substitution $t = nu$ gives

$$I_n = \int_0^1 n^{z-1} u^{z-1} (1-u)^n n \, du = n^z B(z, n+1) = \Gamma_n(z).$$

The result follows if we can show that $I_n \to I(z)$ as $n \to \infty$. The integrand in I_n converges pointwise to $t^{z-1}e^{-t}$ and, by the Lemma, has modulus not greater than $t^{x-1}e^{-t}$, where $z = x + iy$, so this statement (for those who know) is a case of the dominated convergence theorem. However, it is easily proved directly, as follows. Let $J_n = \int_0^n t^{z-1}e^{-t} \, dt$. Then $J_n \to I(z)$ as $n \to \infty$. By the Lemma,

$$0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{1}{n} t^2 e^{-t},$$

so

$$|J_n - I_n| \le \frac{1}{n} \int_0^n t^{x+1} e^{-t} \, dt < \frac{I(x+2)}{n},$$

hence $J_n - I_n \to 0$ as $n \to \infty$. □

Of course, this theorem can serve as a proof of the convergence of Euler's limit. However, we would then still need the method of Theorem 1.1 for the derivation of the product and series expressions.

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