The incomplete gamma functions

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These notes incorporate the *Math. Gazette* article [Jam1], with some extra material.

Definitions and elementary properties

Recall the integral definition of the gamma function: $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt$ for $a > 0$.

By splitting this integral at a point $x \geq 0$, we obtain the two *incomplete gamma functions*:

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}\,dt, \tag{1}$$

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt. \tag{2}$$

$\Gamma(a,x)$ is sometimes called the *complementary incomplete gamma function*. These functions were first investigated by Prym in 1877, and $\Gamma(a,x)$ has also been called *Prym's function*. There are not many books that give these functions much space. Massive compilations of results about them can be seen stated without proof in [Erd, chapter 9] and [Olv, chapter 8]. Here we offer a small selection of these results, with proofs and some discussion of context.

Clearly, $\Gamma(a,0) = \Gamma(a)$ and

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a) \tag{3}$$

for all $x \geq 0$ and $a > 0$. Also, $\Gamma(1,x) = e^{-x}$ and $\gamma(1,x) = 1 - e^{-x}$.

For $x > 0$, the integral (2) converges for all real a, so we regard it as defining $\Gamma(a,x)$ for all such a. In particular, $\Gamma(0,x)$ is the "exponential integral" $\int_x^\infty t^{-1}e^{-t}\,dt$. This case has a number of features of its own: it is discussed in companion notes [Jam2].

The integral (1) only converges for $a > 0$, but in fact the definition of $\gamma(a,x)$ can be extended to negative a, as we see later.

Note on complex a and x. The definition of $\gamma(a,x)$ makes sense for complex a with Re $a > 0$, and the definition of $\Gamma(a,x)$ for all complex a. Also, with due interpretation of the integrals, one can replace x by a complex variable z. However, in these notes we confine ourselves to the case where a and x are real. Some of the results, which interested readers will be able to recognise, apply without change when a is complex.

Of course, $\gamma(a,x)$ and $\Gamma(a,x)$ can be considered both as functions of x (for fixed a) and as functions of a (for fixed x). Our emphasis will be firmly on them as functions of

x. First, some simple facts. Since the integrand is non-negative, so are $\gamma(a,x)$ and $\Gamma(a,x)$. For fixed a, $\gamma(a,x)$ is an increasing function of x, with $\lim_{x\to\infty}\gamma(a,x)=\Gamma(a)$, and $\Gamma(a,x)$ is a decreasing function of x with $\lim_{x\to\infty}\Gamma(a,x)=0$ (this applies also for $a\le 0$). By the fundamental theorem of calculus, we have

$$\frac{d}{dx}\gamma(a,x) = -\frac{d}{dx}\Gamma(a,x) = x^{a-1}e^{-x}. \tag{4}$$

If $a>1$, this is largest when $x=a-1$.

We record some inequalities that follow very easily from the integrals (1) and (2), and shed some light on their nature for small and large x. Firstly, since $e^{-x}\le e^{-t}\le 1$ for $0\le t\le x$, we have $t^{a-1}e^{-x}\le t^{a-1}e^{-t}\le t^{a-1}$ for such t. Now $\int_0^x t^{a-1}\,dt = x^a/a$, so

$$e^{-x}\,\frac{x^a}{a}\le \gamma(a,x)\le \frac{x^a}{a}. \tag{5}$$

Hence $\gamma(a,x)\sim x^a/a$ as $x\to 0^+$. (Recall that the notation $f(x)\sim g(x)$ as $x\to x_0$ means $f(x)/g(x)\to 1$ as $x\to x_0$.)

Secondly, if $a\ge 1$, then $t^{a-1}\ge x^{a-1}$ for $t\ge x$, hence

$$\Gamma(a,x)\ge x^{a-1}\int_x^\infty e^{-t}\,dt = x^{a-1}e^{-x}, \tag{6}$$

and clearly the opposite holds for $a\le 1$. For $a<0$, another inequality, comparable to the left-hand side of (5), and stronger than (6) for small x, is

$$\Gamma(a,x)\le e^{-x}\int_x^\infty t^{a-1}\,dt = e^{-x}\frac{x^a}{-a}. \tag{7}$$

We remark that (5) gives $\gamma(a,a)\ge a^{a-1}e^{-a}$, while (6) gives $\Gamma(a,a)\ge a^{a-1}e^{-a}$. Hence we have shown, with minimal effort, that $\Gamma(a)\ge 2a^{a-1}e^{-a}$ for $a\ge 1$, an elementary inequality of the same type as Stirling's formula.

Further inequalities and estimations for $\Gamma(a,x)$ will be presented later.

We mention some equivalent forms given by simple substitutions. For $c>0$, the substitution $ct=u$ gives

$$\int_0^x t^{a-1}e^{-ct}\,dt = \int_0^{cx}\left(\frac{u}{c}\right)^{a-1}e^{-u}\frac{1}{c}\,du = \frac{1}{c^a}\gamma(a,cx), \tag{8}$$

and similarly $\int_x^\infty t^{a-1}e^{-ct}\,dt = \frac{1}{c^a}\Gamma(a,cx)$. Note the case $x=1$: $\gamma(a,c)=c^a\int_0^1 t^{a-1}e^{-ct}\,dt$.

The substitution $t=x+u$ gives

$$\Gamma(a,x) = e^{-x}\int_0^\infty (x+u)^{a-1}e^{-u}\,du. \tag{9}$$

The substitution $u = t^n$ gives

$$\int_0^x e^{-t^n}\,dt = \tfrac{1}{n}\int_0^{x^n} e^{-u}u^{\frac{1}{n}-1}\,du = \tfrac{1}{n}\gamma(\tfrac{1}{n}, x^n).$$

We include here some brief remarks about $\gamma(a,x)$ and $\Gamma(a,x)$ as functions of a (which, however, will not be used below). Now t^{a-1} decreases with a for $0 < t < 1$, and increases with a for $t > 1$. Consequently, $\gamma(a,x)$ decreases with a for fixed $x \le 1$, and $\Gamma(a,x)$ increases with a for fixed $x \ge 1$. Recall, however, that $\Gamma(a,0) = \Gamma(a)$ decreases for $0 < a < a_0$, where $a_0 \approx 1.4616$, and increases for $a > a_0$. By (5) and (6), $\gamma(a,x)$ and $\Gamma(a,x)$ tend to infinity as $a \to \infty$ for fixed $x > 1$. Also, for any $x > 0$,

$$\frac{\gamma(a,x)}{\Gamma(a)} < \frac{\gamma(a,x)}{\gamma(a,a)} \le \frac{e^a x^a}{a^a},$$

hence $\gamma(a,x)/\Gamma(a) \to 0$ and $\Gamma(a,x)/\Gamma(a) \to 1$ as $a \to \infty$. In the same way as for the gamma function, one can show that $\gamma(a,x)$ and $\Gamma(a,x)$ are convex (even log-convex) functions of a, and that they are differentiable, with (for example)

$$\frac{d}{da}\gamma(a,x) = \int_0^x t^{a-1}e^{-t}\log t\,dt.$$

Integration by parts; two basic identities; evaluation for positive integer a

The most basic property of the gamma function is the identity $\Gamma(a+1) = a\Gamma(a)$. We now show how this identity decomposes into two companion ones for the incomplete gamma functions. This is achieved by a very simple integration by parts. Clarity and simplicity are gained by stating the basic result for general integrals of the same type. Given a function $f(t)$ on $(0,\infty)$, write

$$I_f(x) = \int_0^x f(t)e^{-t}\,dt, \qquad J_f(x) = \int_x^\infty f(t)e^{-t}\,dt$$

if these integrals exist.

If $I_f(x)$ and $I_{f'}(x)$ exist and $f(0) = 0$, then for $x > 0$,

$$I_f(x) = \Big[-f(t)e^{-t}\Big]_0^x + \int_0^x f'(t)e^{-t}\,dt = -f(x)e^{-x} + I_{f'}(x). \tag{10}$$

Similarly, if $J_f(x)$ and $J_{f'}(x)$ exist and $f(x)e^{-x} \to 0$ as $x \to \infty$, then

$$J_f(x) = f(x)e^{-x} + J_{f'}(x). \tag{11}$$

If $f(t) = t^a$, then $I_f(x) = \gamma(a+1,x)$ and $I_{f'}(x) = a\gamma(a,x)$, and similarly for J_f and $J_{f'}$. Also, $f(0) = 0$ if $a > 0$, and $\lim_{x\to\infty} f(x)e^{-x} = 0$ for any a, so we conclude:

THEOREM 1. *For $x > 0$ and $a > 0$,*

$$\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x}. \tag{12}$$

For $x > 0$ and all a,

$$\Gamma(a+1, x) = a\Gamma(a, x) + x^a e^{-x}. \qquad \square \tag{13}$$

Added together, (12) and (13), with (3), reproduce the identity $\Gamma(a+1) = a\Gamma(a)$.

The process in (10) and (11) can be repeated by application to f' and higher derivatives. In the case of (11), the statement is:

THEOREM 2. *Suppose that $J_{f^{(r)}}(x)$ exists for $0 \le r \le k$ and that $f^{(r)}(x)e^{-x} \to 0$ as $x \to \infty$ for $0 \le r \le k-1$. Then*

$$J_f(x) = e^{-x}[f(x) + f'(x) + \cdots f^{(k-1)}(x)] + J_{f^{(k)}}(x). \qquad \square \tag{14}$$

With $f(x) = x^a$, the condition $\lim_{x\to\infty} f^{(r)}(x)e^{-x} = 0$ is, of course, satisfied for all r. The corresponding development of (10) is trickier to apply, because it requires the condition $f^{(r)}(0) = 0$ for successive r, which will fail for some r.

One can write out explicitly what (14) says for $\Gamma(a, x)$ in general: we return to this later. However, for positive integers a, it simplifies and delivers at once a closed expression for $\Gamma(a, x)$. For more pleasant notation, we will state this with $a = n + 1$: note that $\Gamma(n+1, x) = \int_x^\infty t^n e^{-t}\, dt$. Of course, once we have evaluated $\Gamma(n+1, x)$, the value of $\gamma(n+1, x)$ is given by $\gamma(n+1, x) = n! - \Gamma(n+1, x)$.

If $f(x) = x^n$, then $f^{(n+1)}(x) = 0$, so the expression in (14) terminates, and we have

$$\Gamma(n+1, x) = J_f(x) = e^{-x}[f(x) + f'(x) + \cdots f^{(n)}(x)]. \tag{15}$$

The bracketed sum simply comprises successive derivatives until they become zero. With no further effort, we can write down the first few cases:

$$\Gamma(2, x) = e^{-x}(x+1), \qquad \Gamma(3, x) = e^{-x}(x^2 + 2x + 2),$$

$$\Gamma(4, x) = e^{-x}(x^3 + 3x^2 + 6x + 6).$$

We now give an expression for the general case. For this purpose, write

$$e_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!},$$

the exponential series truncated after $n + 1$ terms.

THEOREM 3. *For integers $n \geq 1$ and $x \geq 0$,*

$$\Gamma(n+1, x) = n! e_n(x) e^{-x}. \tag{16}$$

Proof. For $f(x) = x^n$, we have

$$f^{(k)}(x) = n(n-1)\ldots(n-k+1)x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}.$$

Now applying (15) and substituting r for $n-k$, we obtain

$$\Gamma(n+1, x) = n! e^{-x} \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} = n! e^{-x} \sum_{r=0}^{n} \frac{x^r}{r!}. \qquad \square$$

This displays $\Gamma(n+1, x)$ in a rather pleasing way as a fraction of $\Gamma(n+1) = n!$, and shows that $\Gamma(n+1, x)/n! \to 1$ as $n \to \infty$ for any fixed x. It also shows that $\Gamma(n+1, x) \sim x^n e^{-x}$ as $x \to \infty$, supplementing (6).

While this derivation was, surely, attractive enough, it is instructive to see a second, equally efficient proof.

Alternative proof of (16). By (9), the binomial expansion and the fact that $\Gamma(n-r+1) = (n-r)!$, we have

$$\begin{aligned}
\Gamma(n+1, x) &= e^{-x} \int_0^\infty (x+u)^n e^{-u}\, du \\
&= e^{-x} \sum_{r=0}^{n} \binom{n}{r} x^r \int_0^\infty u^{n-r} e^{-u}\, du \\
&= e^{-x} \sum_{r=0}^{n} \binom{n}{r} x^r \Gamma(n-r+1) \\
&= e^{-x} \sum_{r=0}^{n} \frac{n!}{r!} x^r. \qquad \square
\end{aligned}$$

Note. Write I_n for $\gamma(n+1, 1) = \int_0^1 t^n e^{-t}\, dt$. It has been known for students (including my students) to be set the exercise of evaluating, say, I_3 by repeated application of the recurrence relation implied by (12), starting with I_0. After at least as much work as either of the methods just described, the student arrives (barring accidents) at the answer $I_3 = 6 - 16e^{-1}$, but this approach does little to reveal the pattern, and the formula for general n.

I am indebted to the *Math. Gazette* referee for pointing out that Theorem 3 leads to the following neat proof of the irrationality of e. By (16), $\Gamma(n+1, 1) = B_n e^{-1}$, where

$B_n = \sum_{r=0}^{n} \frac{n!}{r!}$, which is an integer. So $\gamma(n+1,1) = A_n - B_n e^{-1}$, where $A_n = n!$. Note that $\gamma(n+1,1) > 0$. Suppose now that $e = p/q$, where p and q are integers. Then $p\gamma(n+1,1) = A_n p - B_n q$: for each n, this is positive and an integer, hence at least 1. But by (5), $\gamma(n+1,1) \le \frac{1}{n+1}$, so $p\gamma(n+1,1) \to 0$ as $n \to \infty$, a contradiction!

It is clear from (5) and (6) that functions like $x^{-a}\gamma(a,x)$ and $e^x\Gamma(a,x)$ are particularly relevant. Using (12) and (13), we can derive very satisfactory expressions for the derivatives of these functions. By (4) and (12), we have

$$\begin{aligned} \frac{d}{dx}\frac{\gamma(a,x)}{x^a} &= -\frac{a\gamma(a,x)}{x^{a+1}} + \frac{x^{a-1}e^{-x}}{x^a} \\ &= \frac{1}{x^{a+1}}[-a\gamma(a,x) + x^a e^{-x}] \\ &= -\frac{\gamma(a+1,x)}{x^{a+1}}. \end{aligned} \tag{17}$$

and similarly for $\Gamma(a,x)$. So if $f(a,x) = x^{-a}\gamma(a,x)$, then $\frac{d}{dx}f(a,x) = -f(a+1,x)$. We can deduce at once that the nth derivative is $(-1)^n f(a+n,x)$.

By (4) and (13), we have

$$\frac{d}{dx}[e^x\Gamma(a,x)] = e^x[\Gamma(a,x) - x^{a-1}e^{-x}] = e^x(a-1)\Gamma(a-1,x), \tag{18}$$

and similarly for $\gamma(a,x)$. In particular, $e^x\Gamma(a,x)$ increases with x when $a \ge 1$, and decreases when $a < 1$.

Series expressions for $\gamma(a,x)$ and extension to $a < 0$

We now give an explicit power-type series expression for $\gamma(a,x)$.

THEOREM 4. *For $a > 0$ and $x > 0$,*

$$\gamma(a,x) = x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(a+n)} = x^a\left(\frac{1}{a} - \frac{x}{a+1} + \frac{x^2}{2!(a+2)} - \cdots\right). \tag{19}$$

Proof. This expression is obtained at once by termwise integration on $[0,x]$ of the series

$$t^{a-1}e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{a+n-1}.$$

Termwise integration is justified by uniform convergence of the power series for e^{-t} on bounded intervals (after separating out the first term if $a < 1$). $\square$

In principle, (19) enables us to calculate $\gamma(a,x)$, although in practice the calculation is only pleasant for fairly small x.

For a fixed $x > 0$, the series (19) converges for all a except 0 and negative integers, so we take it as the definition of $\gamma(a, x)$ for such a. We will show that the extended function still satisfies the basic identity (12).

For the gamma function itself, the usual procedure is to extend the definition by the identity $\Gamma(a + 1) = a\Gamma(a)$: given $\Gamma(a + 1)$, this defines $\Gamma(a)$. Meanwhile, $\Gamma(a, x)$ is already defined for all a. We show that the two extensions are compatible, in the sense that the identity (3) still holds.

PROPOSITION 5. *For all a except 0 and negative integers, and all $x > 0$,*

$$\gamma(a + 1, x) = a\gamma(a, x) - x^a e^{-x}. \tag{20}$$

If the definition of $\Gamma(a)$ is extended in the way just stated, then $\gamma(a, x) + \Gamma(a, x) = \Gamma(a)$ for all such a.

Proof. We have

$$\begin{aligned} a\gamma(a, x) &= x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \frac{a}{a+n} \\ &= x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \left(1 - \frac{n}{a+n}\right) \\ &= x^a e^{-x} - x^a \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n-1)!(a+n)} \\ &= x^a e^{-x} + x^a \sum_{m=0}^{\infty} \frac{(-1)^m x^{m+1}}{m!(a+m+1)} \\ &= x^a e^{-x} + \gamma(a + 1, x). \end{aligned}$$

Now write $\gamma(a, x) + \Gamma(a, x) = F(a, x)$. We know that $F(a, x) = \Gamma(a)$ for $a > 0$. By (13) and (20), $F(a + 1, x) = aF(a, x)$. Hence if we know that $F(a + 1, x) = \Gamma(a + 1)$, it follows that $F(a, x) = \Gamma(a)$. By repeated backwards steps of length 1, it now follows that $F(a, x) = \Gamma(a)$ for all a except 0 and negative integers. □

A satisfying application of Proposition 5 is that it gives an explicit formula for the extended function $\Gamma(a)$. For this purpose, we only need to take $x = 1$, obtaining:

$$\Gamma(a) = \gamma(a, 1) + \Gamma(a, 1) = \int_1^{\infty} t^{a-1} e^{-t}\, dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(a+n)}.$$

In fact, as an alternative to the procedure described above, one can adopt this as the definition of $\Gamma(a)$ for such a. If this approach is chosen, then the conclusion from (20) is that the gamma function (extended in this way) still satisfies $\Gamma(a + 1) = a\Gamma(a)$.

Since the identity $\gamma(a,x)+\Gamma(a,x)=\Gamma(a)$ still applies, so does the identity $\frac{d}{dx}\gamma(a,x) = -\frac{d}{dx}\Gamma(a,x) = x^{a-1}e^{-x}$. In particular, $\gamma(a,x)$ is an increasing function of x. However, it may well be negative: indeed, (20) shows that it is certainly negative for $-1 < a < 0$.

Further note on Theorem 4. Write $s_n(t) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} t^r$. It is easily proved by repeated integration that $s_{2n+1}(t) \le e^{-t} \le s_{2n}(t)$ for all $t > 0$. Hence $|e^{-t} - s_n(t)| \le t^{n+1}/(n+1)!$, from which one can justify the termwise integration directly. Also, if we write

$$s_n(a,x) = \int_0^x t^{a-1} s_n(t)\, dt = x^a \sum_{r=0}^{n} \frac{(-1)^r x^r}{r!(a+r)},$$

it follows that $s_{2n+1}(a,x) \le \gamma(a,x) \le s_{2n}(a,x)$. In particular,

$$\gamma(a,x) \ge x^a \left(\frac{1}{a} - \frac{x}{a+1} \right).$$

This, together with (5), shows that $\gamma(a,x) \sim 1/a$ as $a \to 0^+$ for any fixed $x > 0$.

We now give a second series expression for $\gamma(a,x)$. We will use the following well-known beta integral: for integers $n \ge 0$,

$$B(a, n+1) =: \int_0^1 u^{a-1}(1-u)^n\, du = \frac{n!}{a(a+1)\dots(a+n)}.$$

PROPOSITION 6. *For $a > 0$ and $x > 0$,*

$$\gamma(a,x) = x^a e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{a(a+1)\dots(a+n)}. \tag{21}$$

Proof. Again applying termwise integration of an exponential series, we have

$$\begin{aligned} e^x \gamma(a,x) &= \int_0^x t^{a-1} e^{x-t}\, dt \\ &= \sum_{n=0}^{\infty} \int_0^x \frac{t^{a-1}(x-t)^n}{n!}\, dt \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (xu)^{a-1}[x(1-u)]^n x\, du \qquad \text{(substitute } t = xu) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^{a+n} B(a, n+1) \\ &= \sum_{n=0}^{\infty} \frac{x^{a+n}}{a(a+1)\dots(a+n)}. \qquad \square \end{aligned}$$

Again, the series (21) converges for all a except 0 and negative integers. Denote, temporarily, the function so defined by $G(a,x)$. It is straightforward to show that

$G(a+1,x) = aG(a,x) - x^a e^{-x}$ (we leave the details to the reader). By this and (20), it follows that $G(a,x) = \gamma(a,x)$ for all such a.

Some integrals

We now evaluate some integrals of expressions involving the incomplete gamma functions. Since these are already defined as integrals, the integrals considered will appear as double integrals, and evaluation will be achieved by reversing them (which is valid, because the integrands are positive). The answers will be in terms of the gamma function itself.

First, consider $\int_0^\infty x^{-p-1}\gamma(a,x)\,dx$. Convergence at infinity clearly requires $p > 0$. Since, by (5), $\gamma(a,x) \sim x^a/a$ as $x \to 0^+$, convergence at 0 requires $p < a$. The need for these conditions also shows up clearly in the following proof.

PROPOSITION 7. *For $0 < p < a$,*

$$\int_0^\infty \frac{1}{x^{p+1}}\gamma(a,x)\,dx = \frac{1}{p}\Gamma(a-p). \tag{22}$$

Proof. Reversing the double integral, we have

$$\begin{aligned}\int_0^\infty \frac{1}{x^{p+1}}\gamma(a,x)\,dx &= \int_0^\infty \frac{1}{x^{p+1}}\int_0^x t^{a-1}e^{-t}\,dt\,dx \\ &= \int_0^\infty t^{a-1}e^{-t}\left(\int_t^\infty \frac{1}{x^{p+1}}\,dx\right)\,dt \\ &= \frac{1}{p}\int_0^\infty t^{a-p-1}e^{-t}\,dt \\ &= \frac{1}{p}\Gamma(a-p). \qquad \square\end{aligned}$$

An alternative proof is by integrating by parts, using (4) for $\gamma'(a,x)$. For this method, (5) is needed to identify the limit at 0.

In particular, $\int_0^\infty \frac{1}{x^a}\gamma(a,x)\,dx = 1/(a-1)$ for $a > 1$.

Now consider $\int_0^\infty x^{p-1}\Gamma(a,x)\,dx$, possibly with $a < 0$. If $a > 0$, then convergence at 0 requires $p > 0$. For $a < 0$, we will see later (refining (7)) that $\Gamma(a,x) \sim -x^a/a$ as $x \to 0^+$, so convergence at 0 requires $a + p > 0$.

PROPOSITION 8. *Suppose that either (i) $a > 0$ and $p > 0$ or (ii) $a \le 0$ and $p > -a$. Then*

$$\int_0^\infty x^{p-1}\Gamma(a,x)\,dx = \frac{1}{p}\Gamma(a+p). \tag{23}$$

In particular, for $a > -1$,

$$\int_0^\infty \Gamma(a,x)\,dx = \Gamma(a+1). \tag{24}$$

Proof. Under either condition, $p > 0$. Reversing the double integral, we have

$$\begin{aligned}\int_0^\infty x^{p-1}\Gamma(a,x)\,dx &= \int_0^\infty x^{p-1}\int_x^\infty t^{a-1}e^{-t}\,dt\,dx \\ &= \int_0^\infty t^{a-1}e^{-t}\left(\int_0^t x^{p-1}\,dx\right)\,dt \\ &= \frac{1}{p}\int_0^\infty t^{a+p-1}e^{-t}\,dt \\ &= \frac{1}{p}\Gamma(a+p). \qquad \square\end{aligned}$$

PROPOSITION 9. *For $a > 0$ and $c > 0$,*

$$\int_0^\infty e^{-cx}\gamma(a,x)\,dx = \frac{\Gamma(a)}{c(c+1)^a}, \tag{25}$$

$$\int_0^\infty e^{-cx}\Gamma(a,x)\,dx = \frac{\Gamma(a)}{c}\left(1 - \frac{1}{(c+1)^a}\right). \tag{26}$$

Proof. Note that the substitution $bt = u$ gives $\int_0^\infty t^{a-1}e^{-bt}\,dt = \Gamma(a)/b^a$. We have

$$\begin{aligned}\int_0^\infty e^{-cx}\gamma(a,x)\,dx &= \int_0^\infty e^{-cx}\int_0^x t^{a-1}e^{-t}\,dt\,dx \\ &= \int_0^\infty t^{a-1}e^{-t}\left(\int_t^\infty e^{-cx}\,dx\right)\,dt \\ &= \int_0^\infty t^{a-1}e^{-t}\frac{e^{-ct}}{c}\,dt \\ &= \frac{1}{c}\int_0^\infty t^{a-1}e^{-(c+1)t}\,dt \\ &= \frac{\Gamma(a)}{c(c+1)^a}.\end{aligned}$$

We deduce (26) using (3) and $\int_0^\infty e^{-cx}\,dx = 1/c$. $\square$

In particular, $\int_0^\infty e^{-x}\gamma(a,x)\,dx = \frac{1}{2^a}\Gamma(a)$.

By integrating (13), one can show that (26) also holds for $-1 < a < 0$ (we leave the details to the reader). For the case $a = 0$, one can show (see [Jam2]) that

$$\int_0^\infty e^{-cx}\Gamma(0,x)\,dx = \frac{1}{c}\log(1+c).$$

Further results for $\Gamma(a, x)$

Recall from (6) that $\Gamma(a, x) \geq x^{a-1}e^{-x}$ for $a \geq 1$, together with the opposite inequality for $a < 1$. We now elaborate on this comparison, giving a restricted reverse inequality when $a > 1$ and showing that $\Gamma(a, x) \sim x^{a-1}e^{-x}$ as $x \to \infty$.

PROPOSITION 10. *If* $a \geq 1$ *and* $e^x > 2^a$, *then*

$$\Gamma(a, x) \leq 2^a x^{a-1} e^{-x}. \tag{27}$$

For all a, *we have* $\Gamma(a, x) \sim x^{a-1}e^{-x}$ *as* $x \to \infty$.

Proof. First, let $a \geq 1$. By (9),

$$e^x \Gamma(a, x) = \int_0^\infty (x + u)^{a-1} e^{-u}\, du.$$

Since $x + u \leq 2x$ when $0 \leq u \leq x$ and $x + u \leq 2u$ when $u \geq x$,

$$\begin{aligned} e^x \Gamma(a, x) &\leq \int_0^x (2x)^{a-1} e^{-u}\, du + \int_x^\infty (2u)^{a-1} e^{-u}\, du \\ &< 2^{a-1}[x^{a-1} + \Gamma(a, x)], \end{aligned}$$

so

$$(e^x - 2^{a-1})\Gamma(a, x) \leq 2^{a-1} x^{a-1}$$

Clearly, (27) follows.

Now take any a. By (13), $\Gamma(a, x) = x^{a-1}e^{-x} + (a - 1)\Gamma(a - 1, x)$. If $a \geq 2$, then by (27), $\Gamma(a - 1, x) \leq 2^{a-1}x^{a-2}e^{-x}$ for large enough x. By (6), if $a < 2$, this holds with 2^{a-1} replaced by 1. Hence in both cases $\Gamma(a, x) = x^{a-1}e^{-x}(1 + O(1/x))$ as $x \to \infty$. $\square$

We now consider the behaviour of $\Gamma(a, x)$ for x close to 0. Of course, for $a > 0$, $\Gamma(a, x)$ simply tends to the finite limit $\Gamma(a)$ as $x \to 0^+$. The case $a = 0$ (the exponential integral) is rather special: we deal with it next.

PROPOSITION 11. *There is a constant* c *such that* $\Gamma(0, x) + \log x \to c$ *as* $x \to 0^+$.

Proof. Define the companion function

$$E^*(x) = \int_0^x \frac{1 - e^{-t}}{t}\, dt.$$

Since $0 < 1 - e^{-t} \leq t$ for $t > 0$, this integral is well-defined and $0 \leq E^*(x) \leq x$ for $x > 0$. Now

$$E^*(1) - E^*(x) = \int_x^1 \frac{1 - e^{-t}}{t}\, dt = -\log x - \Gamma(0, x) + \Gamma(0, 1),$$

so $\Gamma(0,x) = E^*(x) - \log x + c$, where $c = \Gamma(0,1) - E^*(1)$. The statement follows. $\square$

Note. One can show that $c = -\gamma$: e.g. see [Jam].

Recall from (7) that $\Gamma(a,x) \le -x^a e^{-x}/a$ for $a < 0$. We now develop this into an asymptotic estimate for x close to 0.

PROPOSITION 12. *For $a < 0$, $\Gamma(a,x) \sim -x^a/a$ as $x \to 0^+$.*

Proof. By (13),

$$ax^{-a}\Gamma(a,x) = x^{-a}\Gamma(a+1,x) - e^{-x}.$$

The stated result follows if we can show that $x^{-a}\Gamma(a+1,x) \to 0$ as $x \to 0^+$. If $-1 < a < 0$, this follows from the fact that $\Gamma(a+1,x)$ tends to $\Gamma(a+1)$. If $a = -1$, it follows from Proposition 11, since $x \log x \to 0$ as $x \to 0^+$. If $a < -1$, then (7) gives $\Gamma(a+1,x) \le -x^{a+1}/(a+1)$: again, the required limit follows. $\square$

We now formulate what (14) says when applied to $\Gamma(a,x)$ where a is not a positive integer. The outcome is a system of approximations to $\Gamma(a,x)$ that are effective for large x. Before stating the result for general a, we describe the case $a = 0$.

Example. Consider $\Gamma(0,x)$. We have $f(x) = 1/x$, so $f^{(k)}(x) = (-1)^k/x^{k+1}$. Let

$$S_k(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \cdots + (-1)^{k-1}\frac{(k-1)!}{x^k}.$$

If k is even, then $f^{(k)}(x)$, hence also $J_{f^{(k)}}(x)$, is positive, so by (14), $\Gamma(0,x) \ge e^{-x}S_k(x)$. The opposite holds if k is odd.

Now consider general a. To simplify the statement, we write

$$P_k(a) = (a-1)(a-2)\ldots(a-k)$$

for $k \ge 1$ (also $P_0(a) = 1$). (Here I am departing from the more-or-less standard notation, which would be $(-1)^k(1-a)_k$.) Note that if $f(x) = x^{a-1}$, then $f^{(r)}(x) = P_r(a)x^{a-r-1}$. Also, once k is larger than a, the numbers $P_k(a)$ are alternately positive and negative.

PROPOSITION 13. *Suppose that a is not a positive integer. Define $P_k(a)$ as above, and let*

$$\begin{aligned} S_k(a,x) &= x^{a-1} + P_1(a)x^{a-2} + \cdots + P_{k-1}(a)x^{a-k} && (28) \\ &= x^{a-1}\left(1 + \frac{P_1(a)}{x} + \cdots + \frac{P_{k-1}(a)}{x^{k-1}}\right). && (29) \end{aligned}$$

Then

$$\Gamma(a,x) = e^{-x}S_k(a,x) + R_k(a,x), \tag{30}$$

where $R_k(a,x) = P_k(a)\Gamma(a-k,x)$. *So* $\Gamma(a,x) \geq S_k(a,x)$ *if* $P_k(a) > 0$, *and* $\Gamma(a,x) \leq S_k(a,x)$ *if* $P_k(a) < 0$. *In particular, if* $P_k(a) > 0$ *and* $P_{k+1}(a) < 0$, *then*

$$e^{-x} S_k(a,x) \leq \Gamma(a,x) \leq e^{-x} S_{k+1}(a,x). \tag{31}$$

Equivalently,

$$\Gamma(a,x) = e^{-x}[S_k(a,x) + r_k(a,x)], \tag{32}$$

where $0 \leq r_k(a,x) \leq P_k(a) x^{a-k-1}$.

Proof. Apply (14) with $f(x) = x^{a-1}$. We just need to note that $J_{f^{(k)}}(x) = P_k(a)\Gamma(a-k,x)$. The stated inequalities follow from the fact that $\Gamma(a-k,x) > 0$. □

It is easy to be more specific about the parity of $P_k(a)$. If a is positive (and not an integer), let k_0 be the integer such that $a-1 < k_0 < a$. If $a \leq 0$, put $k_0 = 0$. In both cases, $P_{k_0}(a) > 0$. Now let $k > k_0$. The negative factors in $P_k(a)$ are $a - j$ for $k_0 + 1 \leq j \leq k$, so $P_k(a)$ is positive when $k - k_0$ is even, negative when $k - k_0$ is odd. So if $k - k_0$ even, then (31) applies.

Example. Consider $\Gamma(\frac{1}{2}, x)$. The statement $S_4(\frac{1}{2}, x) \leq e^x \Gamma(\frac{1}{2}, x) \leq S_3(\frac{1}{2}, x)$ equates to

$$\Gamma(\tfrac{1}{2}, x) = e^{-x} x^{-1/2} \left(1 - \frac{1}{2x} + \frac{3}{4x^2} - r_3(x)\right),$$

where $0 \leq r_3(x) \leq 15/(8x^3)$. For $x = 4$, this gives bounds 0.00815 and 0.00845.

It is important to realise that (28) is not the beginning of a convergent series $\sum_{n=0}^\infty P_n(a) x^{a-n-1}$. Indeed, $P_n(a)$ grows like $n!$, so for any x, $P_n(a)x^{-n-1}$ tends to infinity as $n \to \infty$. This is an *asymptotic series*, not a convergent one. A judicious choice of k may make the error term $r_k(a,x)$ small, but ultimately it grows large again.

Now recall Euler's identity: for $0 < a < 1$, $\Gamma(1-a)\Gamma(a) = \pi/(\sin \pi a)$. Our last result gives an integral expression for $\Gamma(1-a)\Gamma(a,x)$ which implies Euler's identity in the case $x = 0$.

PROPOSITION 14. *For* $0 < a < 1$ *and* $x \geq 0$,

$$\Gamma(1-a)\Gamma(a,x) = e^{-x} \int_0^\infty \frac{1}{u^a(1+u)} e^{-ux}\, du. \tag{33}$$

Proof. The substitution $tu = v$ gives

$$\int_0^\infty u^{-a} e^{-tu}\, du = t^{a-1}\Gamma(1-a).$$

Substituting this integral expression for $t^{a-1}\Gamma(1-a)$, we obtain

$$\begin{aligned}
\Gamma(1-a)\Gamma(a,x) &= \Gamma(1-a)\int_x^\infty t^{a-1}e^{-t}\,dt \\
&= \int_x^\infty e^{-t}\int_0^\infty u^{-a}e^{-tu}\,du\,dt \\
&= \int_0^\infty u^{-a}\int_x^\infty e^{-t(1+u)}\,dt\,du \\
&= \int_0^\infty u^{-a}\frac{e^{-(1+u)x}}{1+u}\,du. \qquad \square
\end{aligned}$$

The case $x=0$ gives

$$\Gamma(1-a)\Gamma(a) = \int_0^\infty \frac{1}{u^a(1+u)}\,du = \frac{\pi}{\sin \pi a}.$$

For $x>0$, the substitution $ux=t$ in (33) gives the alternative expression

$$\Gamma(1-a)\Gamma(a,x) = x^a e^{-x}\int_0^\infty \frac{e^{-t}}{t^a(x+t)}\,dt. \tag{34}$$

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