

The Frullani integrals

Notes by G.J.O. Jameson

We consider integrals of the form

$$I_f(a, b) = \int_0^\infty \frac{f(ax) - f(bx)}{x} dx,$$

where f is a continuous function (real or complex) on $(0, \infty)$ and $a, b > 0$. If $f(x)$ tends to a non-zero limit at 0, then the separate integrals of $f(ax)/x$ and $f(bx)/x$ diverge at 0, and a similar comment applies at infinity. The point is that under suitable conditions, the integral of the *difference* converges.

The basic theorem is as follows. I do not know many books or articles where proofs are given: two references (for which I am grateful to Nick Lord) are [Fer, p. 134–135] and [Tr].

FRU1 THEOREM. *Consider the following conditions:*

(C1) $f(x) \rightarrow c_0$ as $x \rightarrow 0^+$;

(C2) $f(x) \rightarrow c_\infty$ as $x \rightarrow \infty$;

(C3) *there exists K such that $|F(x)| \leq K$ for all $x > 0$, where $F(x) = \int_0^x f(t) dt$.*

Under conditions (C1) and (C2), we have

$$I_f(a, b) = (c_0 - c_\infty)(\log b - \log a). \quad (1)$$

Under conditions (C1) and (C3), we have

$$I_f(a, b) = c_0(\log b - \log a). \quad (2)$$

Proof. We may assume that $b > a$: interchanging a and b then gives the case $b < a$. Let $0 < \delta < X$. The substitution $ax = y$ gives

$$\int_\delta^X \frac{f(ax)}{x} dx = \int_{a\delta}^{aX} \frac{f(y)}{y} dy.$$

So

$$\begin{aligned} \int_\delta^X \frac{f(ax) - f(bx)}{x} dx &= \int_{a\delta}^{aX} \frac{f(y)}{y} dy - \int_{b\delta}^{bX} \frac{f(y)}{y} dy \\ &= \int_{a\delta}^{b\delta} \frac{f(y)}{y} dy - \int_{aX}^{bX} \frac{f(y)}{y} dy \\ &= I(\delta) - I(X), \end{aligned}$$

where we write

$$I(r) = \int_{ar}^{br} \frac{f(y)}{y} dy$$

for any $r > 0$. Let $\varepsilon > 0$ be given. Under condition (C1), there exists $\delta_0 > 0$ such that if $0 < y \leq b\delta_0$, then $|f(y) - c_0| \leq \varepsilon$. Then for $\delta \leq \delta_0$, we have $I(\delta) = c_0(\log b - \log a) + r_1(\delta)$, where

$$r_1(\delta) = \int_{a\delta}^{b\delta} \frac{f(y) - c_0}{y} dy,$$

hence

$$|r_1(\delta)| \leq \int_{a\delta}^{b\delta} \frac{\varepsilon}{y} dy = \varepsilon(\log b - \log a).$$

So $I(\delta) \rightarrow c_0(\log b - \log a)$ as $\delta \rightarrow 0^+$.

In exactly the same way, condition (C2) implies that $I(X) \rightarrow c_\infty(\log b - \log a)$ as $X \rightarrow \infty$.

Under condition (C3), integration by parts gives

$$I(X) = \left[\frac{F(y)}{y} \right]_{aX}^{bX} + \int_{aX}^{bX} \frac{F(y)}{y^2} dy,$$

hence

$$|I(X)| \leq \frac{2K}{aX} + K \int_{aX}^{bX} \frac{1}{y^2} dy < \frac{3K}{aX},$$

so $I(X) \rightarrow 0$ as $X \rightarrow \infty$. (Of course, this also shows that $\int_1^\infty \frac{f(x)}{x} dx$ converges; this statement could be taken as the hypothesis instead of (C3)). \square

In particular, $I_f(1, b)$ equals $(c_0 - c_\infty) \log b$ in case (1), and $c_0 \log b$ in case (2).

We record a number of particular examples which are transparently cases of (1) or (2), without repeatedly writing out the integral expressions:

$f(x)$	$I_f(a, b)$
e^{-x}	$\log b - \log a$
$1/(1+x^2)$	$\log b - \log a$
$\cos x$	$\log b - \log a$
$e^{-x} \cos x$	$\log b - \log a$
$\sin x/x$	$\log b - \log a$
$\tan^{-1} x$	$\frac{\pi}{2}(\log a - \log b)$
$\tanh x$	$\log a - \log b$

Among these examples, $\cos x$ is the only one satisfying (C3), but not (C2). A simple alternative proof for this case is given in [Jam, p. 280].

Many further examples of Frullani integrals are given in [AABM].

We now state a simple extension of the Theorem.

FRU2. Let $G(x) = \sum_{j=1}^n m_j f(a_j x)$, where $a_j > 0$ for $1 \leq j \leq n$ and $\sum_{j=1}^n m_j = 0$. If f satisfies (C1) and (C2), then

$$\int_0^\infty \frac{G(x)}{x} dx = (c_\infty - c_0) \sum_{j=1}^n m_j \log a_j. \quad (3)$$

If f satisfies (C1) and (C3), the same applies with c_∞ replaced by 0.

Proof. Write $M_j = m_1 + \dots + m_j$. By Abel summation, since $M_n = 0$,

$$G(x) = \sum_{j=1}^{n-1} M_j [f(a_j x) - f(a_{j+1} x)].$$

By (1),

$$\int_0^\infty \frac{G(x)}{x} dx = (c_\infty - c_0) \sum_{j=1}^{n-1} M_j (\log a_j - \log a_{j+1}) = (c_\infty - c_0) \sum_{j=1}^n m_j \log a_j. \quad \square$$

Double integral method for (1).

For monotonic f , the following is an alternative method for (1) (but not (2)). The method has been in circulation for a long time, at least for the special case $f(x) = e^{-x}$. For example, it can be seen in [Cou, p. 240].

Note that for $x, y > 0$,

$$\frac{1}{y} \frac{d}{dx} f(xy) = \frac{1}{x} \frac{d}{dy} f(xy) = f'(xy).$$

Hence

$$\int_a^b f'(xy) dy = \left[\frac{1}{x} f(xy) \right]_{y=a}^b = \frac{f(bx) - f(ax)}{x}.$$

Since $f'(x)$ is of constant sign, reversal of the following double integral is justified:

$$-I_f(a, b) = \int_0^\infty \int_a^b f'(xy) dy dx = \int_a^b \int_0^\infty f'(xy) dx dy.$$

But

$$\int_0^\infty f'(xy) dx = \left[\frac{1}{y} f(xy) \right]_{x=0}^\infty = \frac{1}{y} (c_\infty - c_0).$$

So

$$-I_f(a, b) = (c_\infty - c_0) \int_a^b \frac{1}{y} dy = (c_\infty - c_0) (\log b - \log a). \quad \square$$

Some applications of the case $f(x) = e^{-x}$.

This is undoubtedly the best known Frullani integral. Written out explicitly, the statement is

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log b - \log a. \quad (4)$$

The substitution $x = -\log y$ transforms (4) to

$$\int_0^1 \frac{y^{b-1} - y^{a-1}}{\log y} dy = \log b - \log a.$$

So, for example,

$$\int_0^1 \frac{y-1}{\log y} dy = \log 2$$

(cf. [BM, p. 97]).

Now consider the integral

$$I_1 = \int_0^\infty \frac{1}{x^2} (1 - e^{-ax})(1 - e^{-bx}) dx.$$

Integration by parts gives $I_1 = A_1 + J_1$, where

$$A_1 = \left[-\frac{1}{x} (1 - e^{-ax})(1 - e^{-bx}) \right]_0^\infty,$$

$$J_1 = \int_0^\infty \frac{1}{x} (ae^{-ax} + be^{-bx} - (a+b)e^{-(a+b)x}).$$

Since $0 < 1 - e^{-x} < x$ for $x > 0$, we have $\frac{1}{x}(1 - e^{-ax})(1 - e^{-bx}) < abx$, hence $A_1 = 0$. So by (3),

$$I_1 = J_1 = (a+b) \log(a+b) - a \log a - b \log b.$$

Next, consider the function

$$E(x) = \int_x^\infty \frac{e^{-t}}{t} dt.$$

By reversal of the implied double integral, one sees easily that $\int_0^\infty E(x) dx = 1$. Now let

$$I_2 = \int_0^\infty e^{-ax} E(x) dx,$$

where $a > 0$. Reversing the double integral and applying (4), we find

$$\begin{aligned} I_2 &= \int_0^\infty e^{-ax} \int_x^\infty \frac{e^{-t}}{t} dt dx \\ &= \int_0^\infty \frac{e^{-t}}{t} \int_0^t e^{-ax} dx dt \\ &= \int_0^\infty \frac{e^{-t}(1 - e^{-at})}{at} dt \\ &= \frac{1}{a} \log(1 + a). \end{aligned}$$

Applications of the case $f(x) = \cos x$

We describe two applications of this case. First, since $2 \sin ax \sin bx = \cos(a - b)x - \cos(a + b)x$, we have, for $a > b > 0$,

$$\int_0^\infty \frac{\sin ax \sin bx}{x} dx = \frac{1}{2} \log \frac{a + b}{a - b}.$$

Second, consider the integral

$$I_3 = \int_0^\infty \frac{\sin^5 x}{x^2} dx.$$

Since $16 \sin^5 x = 10 \sin x - 5 \sin 3x + \sin 5x$, we have by (3)

$$\begin{aligned} 16I_3 &= \left[-\frac{\sin^5 x}{x} \right]_0^\infty + \int_0^\infty \frac{1}{x} (10 \cos x - 15 \cos 3x + 5 \cos 5x) dx \\ &= -(10 \log 1 - 15 \log 3 + 5 \log 5) \\ &= 15 \log 3 - 5 \log 5. \end{aligned}$$

Integrals of this type are discussed more generally in [Tr].

References

- [AABM] Matthew Albano, Tewodros Amdeberhan, Erin Beyerstedt and Victor H. Moll, The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals, *Scientica, Series A: Mathematical Sciences* **19** (2010), 113–119.
- [BM] George Boros and Victor H. Moll, *Irresistible Integrals*, Cambridge Univ. Press (2004).
- [Cou] R. Courant, *Differential and Integral Calculus*, vol. II, Blackie & Son (1936).
- [Fer] W. L. Ferrar, *Integral Calculus*, Oxford Univ. Press (1958).
- [Jam] G. J. O. Jameson, Sine, cosine and exponential integrals, *Math. Gazette* **99** (2015), 276–289.
- [Tr] J. Trainin, Integrating expressions of the form $\frac{\sin^n x}{x^m}$ and others, *Math. Gazette* **94** (2010), 216–223.

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