The real exponential integrals

Notes by G.J.O. Jameson

The functions $E(x)$ and $E^*(x)$

Define, for suitable x,

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt, \qquad E^*(x) = \int_0^x \frac{1-e^{-t}}{t}\,dt. \tag{1}$$

$E(x)$, as well as various mutations and equivalent forms, is known as the "incomplete exponential integral". It is the special case $\Gamma(0,x)$ of the "incomplete gamma function", defined by $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt$ (e.g. see [Jam3]). Notation sometimes used in specialist literature is $E_1(x)$ for our $E(x)$ and $\mathrm{Ein}(x)$ for $E^*(x)$.

Clearly, $E(x)$ is defined for all $x > 0$, but not for $x = 0$, since $\int_0^1 e^{-t}/t\,dt$ is divergent. Since the integrand is positive, $E(x)$ is non-negative and decreasing. By the fundamental theorem of calculus, $E'(x) = -e^{-x}/x$. A simple inequality for $E(x)$ is

$$E(x) \le \frac{1}{x}\int_x^\infty e^{-t}\,dt = \frac{e^{-x}}{x}. \tag{2}$$

In particular, $E(x)$, and even $e^x E(x)$, tends to 0 as $x \to \infty$.

There is no problem about convergence of $E^*(x)$ at 0. In fact, $0 < 1 - e^{-t} \le t$ for $t > 0$, so $0 < (1-e^{-t})/t < 1$, and hence $0 < E^*(x) \le x$ for $x > 0$. In particular, $E^*(x) \to 0$ as $x \to 0^+$. A better inequality for $x > 1$ is

$$E^*(x) < E^*(1) + \int_1^x \frac{1}{t}\,dt < \log x + 1. \tag{3}$$

Also, $E^*(x)$ is increasing and $(E^*)'(x) = (1-e^{-x})/x$.

By integrating the series

$$\frac{1-e^{-t}}{t} = \sum_{n=1}^\infty (-1)^{n-1}\frac{t^{n-1}}{n!},$$

we obtain a power series expression for $E^*(x)$:

$$E^*(x) = \sum_{n=1}^\infty (-1)^{n-1}\frac{x^n}{n!n} = x - \frac{x^2}{2!2} + \frac{x^3}{3!3} - \cdots, \tag{4}$$

from which, in principle, values can be calculated (though in practice the calculation is only pleasant for fairly small x). For example, eight terms are enough to give $E^*(1) \approx 0.79660$ to five decimal places.

We mention some equivalent forms given by substitutions. For $a > 0$, the substitution $at = u$ gives

$$\int_x^\infty \frac{e^{-at}}{t}\,dt = \int_{ax}^\infty \frac{e^{-u}}{u}\,du = E(ax), \tag{5}$$

$$\int_0^x \frac{1-e^{-at}}{t}\,dt = \int_0^{ax} \frac{1-e^{-u}}{u}\,du = E^*(ax), \tag{6}$$

In particular, taking $x = 1$, we obtain

$$E(a) = \int_1^\infty \frac{e^{-at}}{t}\,dt, \qquad E^*(a) = \int_0^1 \frac{1-e^{-at}}{t}\,dt. \tag{7}$$

The substitution $t = x + u$ in (1) gives

$$E(x) = e^{-x}\int_0^\infty \frac{e^{-u}}{x+u}\,du. \tag{8}$$

This expression can be used to define $E(z)$ for any complex z not on the negative real line. As a contour integral, it equates to the integral of $e^{-\zeta}/\zeta$ on the horizontal half-line defined by $\zeta = z+u$ for $u \geq 0$. Not very helpfully, the notation $\mathrm{Ei}(z)$ is sometimes used for $-E(-z)$. We will not discuss the complex function $E(z)$ in these notes.

Substituting $e^{-u} = v$ in (8), we obtain

$$E(x) = e^{-x}\int_0^1 \frac{1}{x - \log v}\,dv. \tag{9}$$

The numbers $E(1)$ and $E^*(1)$ have particular significance. Integration by parts gives

$$E(1) = \Big[e^{-t}\log t\Big]_1^\infty + \int_1^\infty e^{-t}\log t\,dt = \int_1^\infty e^{-t}\log t\,dt. \tag{10}$$

Since $(1-e^{-t})\log t \to 0$ as $t \to 0^+$, we have similarly

$$E^*(1) = \Big[(1-e^{-t})\log t\Big]_0^1 - \int_0^1 e^{-t}\log t\,dt = -\int_0^1 e^{-t}\log t\,dt. \tag{11}$$

To relate $E^*(x)$ and $E(x)$, observe that

$$E^*(x) - E^*(1) = \int_1^x \frac{1-e^{-t}}{t}\,dt = \log x - \int_1^x \frac{e^{-t}}{t}\,dt = \log x - E(1) + E(x),$$

so

$$E(x) = E^*(x) - \log x + c, \tag{12}$$

where c is constant, in fact $c = E(1) - E^*(1)$. Furthermore, we have two ways to express c as a limit. Since $\lim_{x\to 0^+} E^*(x) = 0$, we have

$$c = \lim_{x\to 0^+}[E(x) + \log x] \tag{13}$$

and since $\lim_{x\to\infty} E(x) = 0$, we have

$$-c = \lim_{x\to\infty} [E^*(x) - \log x]. \tag{14}$$

Also, by (10) and (11), we have

$$c = \int_0^\infty e^{-t} \log t \, dt. \tag{15}$$

This is known as the (complete) exponential integral (as already noted, e^{-t}/t does not give a convergent integral on $(0, \infty)$). Readers familiar with the gamma function will recognise that it equates to $\Gamma'(1)$.

The evaluation of c will be our main theorem, and we turn to it shortly. But there are some facts that can be deduced without knowing the value of c. Firstly, $E(x) \sim -\log x$ as $x \to 0^+$, hence $xE(x) \to 0$ as $x \to 0^+$. Another deduction is the following integral:

PROPOSITION 1. *For a, $b > 0$,*

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt = \log b - \log a. \tag{16}$$

Proof. By (6) and (12),

$$\begin{aligned} \int_0^x \frac{e^{-at} - e^{-bt}}{t} \, dt &= E^*(bx) - E^*(ax) \\ &= E(bx) - E(ax) + \log bx - \log ax \\ &= E(bx) - E(ax) + \log b - \log a \\ &\to \log b - \log a \qquad \text{as } x \to \infty. \qquad \square \end{aligned}$$

A well-known alternative proof of (16) is by expressing the integrand as $\int_a^b e^{-ty} \, dy$ and reversing the double integral. It is also a special case of the "Frullani integral" (see [Fer, p. 133–135]).

Determination of c; the complete exponential integral

We will use (14) to evaluate c.

LEMMA 1. *For $0 \le t \le n$,*

$$\left(1 - \frac{t^2}{n}\right) e^{-t} \le \left(1 - \frac{t}{n}\right)^n \le e^{-t}. \tag{17}$$

Proof. From the series for e^x and $1/(1-x)$, we have $1 + x \le e^x \le 1/(1-x)$, and hence $(1 - x^2)e^{-x} \le 1 - x \le e^{-x}$ for $0 < x < 1$. Substitute $x = t/n$ and take the nth power to obtain

$$\left(1-\frac{t^2}{n^2}\right)^n e^{-t} \le \left(1-\frac{t}{n}\right)^n \le e^{-t}.$$

for $0 \le t \le n$. Now $(1-a)^n \ge 1-na$ for $0 \le a \le 1$, so (17) follows. $\square$

THEOREM 2. *We have $c = -\gamma$, where γ is Euler's constant. Hence*

$$\int_0^\infty e^{-t} \log t \, dt = -\gamma, \tag{18}$$

$$E^*(x) = E(x) + \log x + \gamma. \tag{19}$$

Proof. By (14), it is enough to show that $E^*(n) - \log n \to \gamma$ as $n \to \infty$. Since $e^{-t} = \lim_{n\to\infty}(1-\frac{t}{n})^n$, it seems at least plausible that $E^*(n)$ is approximated by K_n, where

$$\begin{aligned} K_n &= \int_0^n \frac{1}{t}\left[1-\left(1-\frac{t}{n}\right)^n\right] dt \\ &= \int_0^1 \frac{1}{u}[1-(1-u)^n] \, du \\ &= \int_0^1 \frac{1-v^n}{1-v} \, dv \\ &= \int_0^1 (1+v+\cdots+v^{n-1}) \, dv \\ &= 1+\frac{1}{2}+\cdots+\frac{1}{n}, \end{aligned}$$

so that $K_n - \log n \to \gamma$ as $n \to \infty$.

To make this precise, we have by (17)

$$1-e^{-t} \le 1-\left(1-\frac{t}{n}\right)^n \le 1-e^{-t}+\frac{1}{n}t^2 e^{-t}.$$

It follows that $E^*(n) \le K_n \le E^*(n) + \Delta_n$, where

$$\Delta_n = \frac{1}{n}\int_0^n te^{-t} \, dt < \frac{1}{n}. \qquad \square$$

COROLLARY 2.1. *For all $x > 0$,*

$$-\log x - \gamma < E(x) < -\log x - \gamma + x, \tag{20}$$

$$\log x + \gamma < E^*(x) < \log x + \gamma + \frac{e^{-x}}{x}. \qquad \square \tag{21}$$

Of course, (20) is effective for small x, and (21) for large x.

Also, we can deduce particular values of $E(x)$, for example $E(1) = E^*(1) - \gamma \approx 0.21938$.

The integral (18), together with most of the equivalent versions described below, was discovered by Euler. The proof given here follows [WW, p. 242]. A slight variant is given in [Lo]. We now record some integrals derived from it, or equivalent to it.

First, the substitution $at = u$ (where $a > 0$) gives

$$\int_0^\infty e^{-at}\log t\,dt = \int_0^\infty e^{-u}(\log u - \log a)\frac{1}{a}\,du = -\frac{1}{a}(\gamma + \log a), \tag{22}$$

which we can rewrite rather pleasantly as $\int_0^\infty e^{-at}(\log t + \gamma)\,dt = -\frac{1}{a}\log a$.

Next, substituting in turn $e^{-t} = u$ and $e^t = u$, we obtain

$$\int_0^1 \log(-\log u)\,du = \int_1^\infty \frac{\log\log u}{u^2}\,du = -\gamma. \tag{23}$$

PROPOSITION 3. *We have*

$$\int_0^\infty te^{-t}\log t\,dt = 1 - \gamma \tag{24}$$

(Remark: this integral equates to $\Gamma'(2)$.)

Proof. Denote the integral by I. Expressing the integrand as $e^{-t}(t\log t)$ and integrating by parts, we obtain

$$I = \Big[-e^{-t}(t\log t)\Big]_0^\infty + \int_0^\infty e^{-t}(\log t + 1)\,dt = 0 - \gamma + 1. \qquad \square$$

In a similar way, one can prove by induction that

$$\int_0^\infty t^n e^{-t}\log t\,dt = n!(H_n - \gamma),$$

where $H_n = \sum_{r=1}^n \frac{1}{r}$. (Hence $\psi(n+1) = H_n - \gamma$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$.)

We now derive some further integral expressions for γ. Now that we know that $c = -\gamma$, we can apply (14) again to obtain at once:

PROPOSITION 4. *We have*

$$\gamma = \int_0^\infty \frac{1}{t}\left(\frac{1}{t+1} - e^{-t}\right)\,dt. \tag{25}$$

Proof. By (14),

$$\begin{aligned}\gamma = \lim_{x\to\infty}[E^*(x) - \log(x+1)] &= \lim_{x\to\infty}\int_0^x \left(\frac{1-e^{-t}}{t} - \frac{1}{t+1}\right)\,dt \\ &= \int_0^\infty \left(\frac{1}{t(t+1)} - \frac{e^{-t}}{t}\right)\,dt. \qquad \square\end{aligned}$$

Alternatively, (25) can be derived from (18) by using (16) to substitute for $\log x$ and reversing the resulting double integral.

PROPOSITION 5. *We have*

$$\gamma = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{e^{-t}}{t} \right) dt. \tag{26}$$

Proof. This will follow from (25) if we prove that

$$\int_0^\infty \left(\frac{1}{t(t+1)} - \frac{1}{e^t - 1} \right) dt = 0.$$

For $\delta > 0$, denote by I_δ this integral reduced to the interval $[\delta, \infty)$. Noting that $1/(e^t - 1) = e^{-t}/(1 - e^{-t})$, we have

$$I_\delta = \log(1+\delta) - \log\delta + \log(1 - e^{-\delta}).$$

Now $\frac{1}{\delta}(1 - e^{-\delta}) \to 1$ as $\delta \to 0^+$, hence $I = \lim_{\delta\to 0^+} I_\delta = 0$. □

Alternatively, one can prove (26) directly and derive (18) [BM, Proposition 9.3.3]. The substitution $e^{-t} = x$ in (26) gives:

COROLLARY 5.1. *We have*

$$\gamma = \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\log x} \right) dx. \qquad \square \tag{27}$$

(25) and (26) are the cases $x = 1$ in the two standard integral expressions for $\psi(x) = \Gamma'(x)/\Gamma(x)$ [AAR, p. 26].

PROPOSITION 6. [Sco] *We have*

$$\gamma = 2\int_0^\infty \frac{1}{t}(e^{-t^2} - e^{-t})\, dt. \tag{28}$$

Proof. Recall that $\gamma = E^*(1) - E(1)$. Substituting $t = u^2$, we have

$$E(1) = \int_1^\infty \frac{e^{-t}}{t}\, dt = 2\int_1^\infty \frac{e^{-u^2}}{u}\, du,$$

hence

$$E(1) = 2E(1) - E(1) = 2\int_1^\infty \frac{1}{t}(e^{-t} - e^{-t^2})\, dt.$$

Similarly,

$$E^*(1) = \int_0^1 \frac{1 - e^{-t}}{t}\, dt = 2\int_0^1 \frac{1 - e^{-u^2}}{u}\, du,$$

so

$$E^*(1) = 2E^*(1) - E^*(1) = 2\int_0^1 \frac{1}{t}(e^{-t^2} - e^{-t})\,dt.$$

Statement (28) follows. $\square$

Calculation of γ. The identity $\gamma = E^*(x) - E(x) - \log x$ has been used for the calculation of γ to great degrees of accuracy. For a suitably chosen x, one can calculate $E^*(x)$ and $\log x$, and estimate $E(x)$ by its asymptotic expansion (which we discuss next), or simply choose x so that $E(x)$ is suitably small. For a survey of this topic, see [GS], [Lag] or [BM, chap. 9].

Asymptotic expression for $E(x)$

The inequality (2) is enough for many purposes, but there is an asymptotic expression giving a better estimation for large x, if wanted. It is helpful to describe the process more generally, as follows. Suppose that $f(t)$ and all its derivatives tend to 0 as $t \to \infty$, and write $I_f(x) = \int_x^\infty e^{-t} f(t)\,dt$. Integrating by parts, we have

$$I_f(x) = \Big[-e^{-t} f(t) \Big]_x^\infty + \int_x^\infty e^{-t} f'(t)\,dt = e^{-x} f(x) + I_{f'}(x).$$

Applying this to $f'(t)$ and substituting, and repeating the process, we obtain for all $k \geq 1$

$$I_f(x) = e^{-x}[f(x) + f'(x) + \cdots + f^{(k-1)}(x)] + I_{f^{(k)}}(x).$$

If also $(-1)^k f^{(k)}(t)$ is non-negative and decreasing, then $0 \leq (-1)^k I_{f^{(k)}}(x) \leq (-1)^k e^{-x} f^{(k)}(x)$. For $E(x)$, we apply this with $f(x) = 1/x$. The conditions are satisfied, so we obtain

$$E(x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \cdots + (-1)^{k-1}\frac{(k-1)!}{x^k} + (-1)^k r_k(x)\right), \tag{29}$$

where

$$0 \leq r_k(x) \leq \frac{k!}{x^{k+1}}.$$

For large x, we can derive a fairly accurate estimation of $E(x)$ by choosing a suitable number of terms. For example $E(10) = \rho e^{-10}$, where $0.0915 < \rho < 0.0917$. However, what this generates is an *asymptotic expression*, not a convergent series, because for any particular x, the successive derivatives ultimately grow large.

Some integrals involving $E(x)$

We record some integrals involving $E(x)$. None of them require the fact that $c = -\gamma$. For the first one, recall that $\int_0^\infty t^{a-1} e^{-t}\,dt$ defines $\Gamma(a)$ for $a > 0$. We compare two methods.

PROPOSITION 7. *For $a > 0$,*

$$\int_0^\infty x^{a-1} E(x)\,dx = \frac{\Gamma(a)}{a}. \tag{30}$$

In particular,

$$\int_0^\infty E(x)\,dx = 1. \tag{31}$$

Proof 1. Reversing the double integral, we obtain

$$\begin{aligned}
\int_0^\infty x^{a-1} E(x)\,dx &= \int_0^\infty x^{a-1} \int_x^\infty \frac{e^{-t}}{t}\,dt\,dx \\
&= \int_0^\infty \frac{e^{-t}}{t} \int_0^t x^{a-1}\,dx\,dt \\
&= \frac{1}{a}\int_0^\infty t^{a-1} e^{-t}\,dt \\
&= \frac{\Gamma(a)}{a}.
\end{aligned}$$

Proof 2. Integrate by parts:

$$\int_0^\infty x^{a-1} E(x)\,dx = \Big[\frac{x^a}{a} E(x)\Big]_0^\infty + \int_0^\infty \frac{x^a}{a}\,\frac{e^{-x}}{x}\,dx.$$

Now $x^a E(x) \to 0$ as $x \to \infty$. Also, since $E(x) \sim -\log x$ for x close to 0, $x^a E(x) \to 0$ as $x \to 0$. So the integral equates to $\frac{1}{a}\int_0^\infty x^{a-1} e^{-x}\,dx = \Gamma(a)/a$. □

So for positive integers n, $\int_0^\infty x^n E(x)\,dx = n!/(n+1)$.

PROPOSITION 8. *For $a > 0$,*

$$\int_0^\infty e^{-ax} E(x)\,dx = \frac{1}{a}\log(a+1). \tag{32}$$

Proof. The integral is

$$\int_0^\infty e^{-ax} \int_x^\infty \frac{e^{-t}}{t}\,dt\,dx = \int_0^\infty \frac{e^{-t}}{t} \int_0^t e^{-ax}\,dx\,dt = \int_0^\infty \frac{e^{-t}(1-e^{-at})}{at}\,dt.$$

The stated value now follows, by (16). □

PROPOSITION 9. *We have*

$$\int_0^\infty E(x)^2\,dx = 2\log 2. \tag{33}$$

Proof. Integrate by parts:

$$\begin{aligned}\int_0^\infty 1.E(x)^2\,dx &= \Big[xE(x)^2\Big]_0^\infty + 2\int_0^\infty xE(x)\frac{e^{-x}}{x}\,dx \\ &= 2\int_0^\infty e^{-x}E(x)\,dx \\ &= 2\log 2,\end{aligned}$$

in which we used (32) and the fact that $\lim_{x\to 0^+}[xE(x)^2] = 0$. $\square$

To prove (33) by the double-integral method, one would start by expressing $E(x)^2$ as an integral:

$$E(x)^2 = -\int_x^\infty 2E(t)E'(t)\,dt = 2\int_x^\infty E(t)\frac{e^{-t}}{t}\,dt.$$

Application to the cosine integral

Consider the integrals corresponding to (1) with e^{-t} replaced by $\cos t$:

$$C(x) = \int_x^\infty \frac{\cos t}{t}\,dt, \qquad C^*(x) = \int_0^x \frac{1-\cos t}{t}\,dt. \tag{34}$$

Convergence of the integral defining $C(x)$, together with the fact that $\lim_{x\to\infty} C(x) = 0$, is easily established by integration by parts. We also define (following standard notation)

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

We will use a contour integral to relate $C^*(x)$ and $E^*(x)$, enabling us derive a result analogous to Theorem 2 for $C(x)$.

Exactly as for $E(x)$, we have

$$C(x) = C^*(x) - \log x + c', \tag{35}$$

where $c' = C(1) - C^*(1)$. By (12) and (35),

$$C(x) - E(x) = C^*(x) - E^*(x) + c' - c. \tag{36}$$

We will prove:

THEOREM 10. *We have $C^*(x) - E^*(x) \to 0$ as $x \to \infty$, hence $c' = -\gamma$ and*

$$C(x) = C^*(x) - \log x - \gamma, \tag{37}$$

$$\int_0^\infty \frac{\cos t - e^{-t}}{t}\,dt = 0. \tag{38}$$

Once we have shown that $C^*(x) - E^*(x) \to 0$ as $x \to \infty$, it follows from (36) that $c' = c$, so $c' = -\gamma$, hence (37). Also, it follows that $C(x) - E(x) \to 0$ as $x \to 0^+$, hence (38) (for this, we do not need to know that $c = -\gamma$).

LEMMA 2. *We have*

$$0 \le \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{2R}.$$

Proof. Since the function $\sin\theta$ is concave on $[0, \frac{\pi}{2}]$, we have $\sin\theta \ge \frac{2\theta}{\pi}$ on $[0, \frac{\pi}{2}]$. Hence

$$\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \le \int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta = \left[-\frac{\pi}{2R}e^{-2R\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{2R}(1 - e^{-R}). \quad \square$$

Proof of Theorem 10. Let C_R be the circular arc of radius R in the positive quadrant, represented by $z = Re^{i\theta}$ for $0 \le \theta \le \frac{\pi}{2}$. Denote by Γ the closed contour consisting of C_R together with the real interval $[0, R]$, and the imaginary axis from iR to 0. Let

$$f(z) = \frac{1 - e^{iz}}{z}.$$

Then f has no pole at 0, since $f(z) = -i + \frac{1}{2}z + \cdots$. By Cauchy's integral theorem, $\int_\Gamma f(z)\, dz = 0$. The contribution of the real axis is

$$\int_0^R \frac{1 - e^{it}}{t}\, dt = C^*(R) - i\,\mathrm{Si}(R).$$

The contribution of the imaginary axis, taken towards the origin, is

$$-\int_0^R \frac{1 - e^{-t}}{t}\, dt = -E^*(R).$$

Now consider C_R. The contribution of the term $\frac{1}{z}$ is $\frac{\pi}{2}i$. The contribution of the other term is

$$I_R =: \int_{C_R} \frac{e^{iz}}{z}\, dz = \int_0^{\pi/2} ie^{iRe^{i\theta}}\, d\theta.$$

Now $|e^{iRe^{i\theta}}| = e^{-R\sin\theta}$, so by Lemma 2, $|I_R| \le \pi/(2R)$.

Now considering the real part, we deduce

$$|C^*(R) - E^*(R)| \le \frac{\pi}{2R},$$

so $C^*(R) - E^*(R) \to 0$ as $R \to \infty$. $\square$

At the same time, we deduce from the imaginary part that $|\mathrm{Si}(R) - \frac{\pi}{2}| \le \frac{\pi}{2R}$, thereby proving the "sine integral"

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

An alternative proof of Theorem 10 by double integrals is given in [Jam2]. A direct proof of (37) can be seen in [Jam1].

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