Elliptic integrals, the arithmetic-geometric mean and the Brent-Salamin algorithm for π

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Contents

1. The integrals $I(a,b)$ and $K(b)$

For a, $b > 0$, define

$$I(a,b) = \int_0^{\pi/2} \frac{1}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}}\, d\theta. \tag{1}$$

An important equivalent form is:

1.1. *We have*

$$I(a,b) = \int_0^{\infty} \frac{1}{(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}\, dx. \tag{2}$$

Proof. With the substitution $x = b\tan\theta$, the stated integral becomes

$$\begin{aligned}\int_0^{\pi/2} \frac{b\sec^2\theta}{(a^2+b^2\tan^2\theta)^{1/2} b\sec\theta}\, d\theta &= \int_0^{\pi/2} \frac{1}{(a^2+b^2\tan^2\theta)^{1/2}\cos\theta}\, d\theta \\ &= \int_0^{\pi/2} \frac{1}{(a^2\cos^2\theta+b^2\sin^2\theta)^{1/2}}\, d\theta. \qquad \square\end{aligned}$$

We list a number of facts that follow immediately from (1) or (2):

(I1) $I(a,a) = \dfrac{\pi}{2a}$;

(I2) $I(b,a) = I(a,b)$;

(I3) $I(ca,cb) = \frac{1}{c} I(a,b)$, hence $aI(a,b) = I(1, b/a)$;

(I4) $I(a,b)$ decreases with a and with b;

(I5) if $a \ge b$, then $\dfrac{\pi}{2a} \le I(a,b) \le \dfrac{\pi}{2b}$.

Note that the substitution $x = 1/y$ in (2) gives

$$I(a,b) = \int_0^\infty \frac{1}{(1+a^2x^2)^{1/2}(1+b^2x^2)^{1/2}}\,dx. \tag{3}$$

1.2. *We have*

$$I(a,b) \le \frac{\pi}{2(ab)^{1/2}}.$$

Proof. This follows from the integral Cauchy-Schwarz inequality, applied to (2). □

Note further that, by the inequality of the means,

$$\frac{\pi}{2(ab)^{1/2}} \le \frac{\pi}{4}\left(\frac{1}{a}+\frac{1}{b}\right).$$

This upper bound for $I(a,b)$ is also obtained directly by applying the inequality of the means to the integrand in (2).

Clearly, $I(a,0)$ is undefined, and one would expect that $I(a,b) \to \infty$ as $b \to 0^+$. This is, indeed, true. In fact, the behaviour of $I(1,b)$ for b close to 0 is quite interesting; we return to it in section 4.

The "complete elliptic integral of the first kind" is defined by

$$K(b) = \int_0^{\pi/2} \frac{1}{(1-b^2\sin^2\theta)^{1/2}}\,d\theta. \tag{4}$$

For $0 \le b \le 1$, the "conjugate" b^* is defined by $b^* = (1-b^2)^{1/2}$, so that $b^2 + b^{*2} = 1$. Then $\cos^2\theta + b^2\sin^2\theta = 1 - b^{*2}\sin^2\theta$, so that $I(1,b) = K(b^*)$.

(An "incomplete" elliptic integral is one in which the integral is over a general interval $[0,\alpha]$. We do not consider such integrals in these notes.)

Clearly, $K(0) = \frac{\pi}{2}$ and $K(b)$ increases with b. Another equivalent form is:

1.3. *We have*

$$K(b) = \int_0^1 \frac{1}{(1-t^2)^{1/2}(1-b^2t^2)^{1/2}}\,dt. \tag{5}$$

Proof. Denote this integral by I. Substituting $t = \sin\theta$, we have

$$I = \int_0^{\pi/2} \frac{1}{\cos\theta(1-b^2\sin^2\theta)^{1/2}}\cos\theta\,d\theta = K(b). \qquad \square$$

1.4 PROPOSITION. *For* $0 \le b < 1$,

$$\frac{2}{\pi}K(b) = \sum_{n=0}^{\infty} d_{2n}b^{2n} = 1 + \tfrac{1}{4}b^2 + \tfrac{9}{64}b^4 + \cdots, \tag{6}$$

where $d_0 = 1$ *and*

$$d_{2n} = \frac{1^2.3^2 \dots (2n-1)^2}{2^2.4^2 \dots (2n)^2}.$$

Proof. By the binomial series,

$$(1 - b^2 \sin^2 \theta)^{-1/2} = 1 + \sum_{n=1}^{\infty} a_{2n} b^{2n} \sin^{2n} \theta,$$

where

$$a_{2n} = (-1)^n \binom{-\frac{1}{2}}{n} = \frac{1}{n!}(\tfrac{1}{2})(\tfrac{3}{2}) \cdots (n - \tfrac{1}{2}) = \frac{1.3.\dots(2n-1)}{2.4.\dots(2n)}.$$

For fixed $b < 1$, the series is uniformly convergent for $0 \le \theta \le \frac{\pi}{2}$. Now $\int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{\pi}{2} a_{2n}$: identity (6) follows. □

In a sense, this series can be regarded as an evaluation of $I(a, b)$, but a more effective one will be described in section 3.

1.5 COROLLARY. *For* $0 \le b < 1$,

$$1 + \tfrac{1}{4}b^2 \le \frac{2}{\pi} K(b) \le 1 + \frac{b^2}{4(1 - b^2)}.$$

Proof. Just note that $1 + \frac{1}{4}b^2 \le \sum_{n=0}^{\infty} d_{2n} b^{2n} \le 1 + \frac{1}{4}(b^2 + b^4 + \dots)$. □

Hence $\frac{2}{\pi} K(b) \le 1 + \frac{1}{2}b^2$ for $0 \le b \le \frac{1}{\sqrt{2}}$. These are convenient inequalities for $K(b)$, but when translated into an inequality for $I(1, b) = K(b^*)$, the right-hand inequality says $\frac{2}{\pi} I(1, b) \le 1 + (1 - b^2)/(4b^2) = \frac{3}{4} + 1/(4b^2)$, and it is easily checked that this is weaker than the elementary estimate $\frac{4}{\pi} I(1, b) \le 1 + 1/b$ given above.

The special case $I(\sqrt{2}, 1)$

The integral $I(\sqrt{2}, 1)$ equates to a beta integral, and consequently to numerous other integrals. We mention some of them, assuming familiarity with the beta and gamma functions.

1.6 PROPOSITION. *We have*

$$\begin{aligned} I(\sqrt{2}, 1) &= \int_0^1 \frac{1}{(1 - x^4)^{1/2}} \, dx && (7) \\ &= \int_1^{\infty} \frac{1}{(x^4 - 1)^{1/2}} \, dx && (8) \\ &= \tfrac{1}{4} B(\tfrac{1}{4}, \tfrac{1}{2}) && (9) \\ &= \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{2\pi}}. && (10) \end{aligned}$$

Proof. For (7), substitute $x = \sin\theta$, so that $1 - x^4 = \cos^2\theta(1 + \sin^2\theta)$. Then

$$\int_0^1 \frac{1}{(1-x^4)^{1/2}}\,dx = \int_0^{\pi/2} \frac{1}{(1+\sin^2\theta)^{1/2}}\,d\theta = \int_0^{\pi/2} \frac{1}{(\cos^2\theta + 2\sin^2\theta)^{1/2}}\,d\theta = I(\sqrt{2}, 1).$$

Substituting $x = 1/y$, we also have

$$\int_0^1 \frac{1}{(1-x^4)^{1/2}}\,dx = \int_1^\infty \frac{1}{(1-y^{-4})^{1/2}}\frac{1}{y^2}\,dy = \int_1^\infty \frac{1}{(y^4-1)^{1/2}}\,dy.$$

Substituting $x = t^{1/4}$ in (7), we obtain

$$I(\sqrt{2}, 1) = \int_0^1 \frac{1}{(1-t)^{1/2}}(\tfrac{1}{4}t^{-3/4})\,dt = \tfrac{1}{4}B(\tfrac{1}{4}, \tfrac{1}{2}).$$

Now using the identities $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ and $\Gamma(a)\Gamma(1-a) = \pi/(\sin a\pi)$, we have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi/(\sin\frac{\pi}{4}) = \pi\sqrt{2}$, hence

$$B(\tfrac{1}{4}, \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}\Gamma(\frac{1}{4})^2}{\pi\sqrt{2}} = \frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}}. \qquad \square$$

1.7. *We have*

$$\int_0^{\pi/2} \frac{1}{(\sin\theta)^{1/2}}\,d\theta = 2I(\sqrt{2}, 1), \tag{11}$$

$$\int_0^1 (1-x^4)^{1/2}\,dx = \int_0^1 (1-x^2)^{1/4}\,dx = \tfrac{2}{3}I(\sqrt{2}, 1). \tag{12}$$

Proof. For (11), substitute $x = (\sin\theta)^{1/2}$ in (7). Denote the two integrals in (12) by I_1, I_2. The substitutions $x = t^{1/4}$ and $x = t^{1/2}$ give $I_1 = \frac{1}{4}B(\frac{1}{4}, \frac{3}{2})$ and $I_2 = \frac{1}{2}B(\frac{1}{2}, \frac{5}{4})$. The identity $(a+b)B(a, b+1) = bB(a,b)$ now gives $I_1 = I_2 = \frac{1}{6}B(\frac{1}{4}, \frac{1}{2})$. $\square$

2. The arithmetic-geometric mean

Given positive numbers a, b, the *arithmetic mean* is $a_1 = \frac{1}{2}(a+b)$ and the *geometric mean* is $b_1 = (ab)^{1/2}$. If $a = b$, then $a_1 = b_1 = a$. If $a > b$, then $a_1 > b_1$, since

$$4(a_1^2 - b_1^2) = (a+b)^2 - 4ab = (a-b)^2 > 0.$$

With $a > b$, consider the iteration given by $a_0 = a$, $b_0 = b$ and

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \qquad b_{n+1} = (a_n b_n)^{1/2}.$$

At each stage, the two new numbers are the arithmetic and geometric means of the previous two. The following facts are obvious:

(M1) $a_n > b_n$;

(M2) $a_{n+1} < a_n$ and $b_{n+1} > b_n$;

(M3) $a_{n+1} - b_{n+1} < a_{n+1} - b_n = \frac{1}{2}(a_n - b_n)$, hence $a_n - b_n < \frac{1}{2^n}(a - b)$.

It follows that (a_n) and (b_n) converge to a common limit, which is called the *arithmetic-geometric mean* of a and b. We will denote it by $M(a, b)$.

Convergence is even more rapid than the estimate implied by (M3), as we will show below. First, we illustrate the procedure by the calculation of $M(\sqrt{2}, 1)$ and $M(100, 1)$:

n	b_n	a_n
0	1	1.414214
1	1.189207	1.207107
2	1.198124	1.198157
3	1.198140	1.198140

n	b_n	a_n
0	1	100
1	10	50.5
2	22.4722	30.25
3	26.0727	26.3611
4	26.2165	26.2169
5	26.2167	26.2167

Of course, a_n and b_n never actually become equal, but more decimal places would be required to show the difference at the last stage shown. In fact, in the first example, $a_3 - b_3 < 4 \times 10^{-10}$, as we will see shortly.

Clearly, $M(a, b) = M(a_n, b_n)$ and $b_n < M(a, b) < a_n$ for all n. Also, $M(ca, cb) = cM(a, b)$ for $c > 0$, so $M(a, b) = aM(1, b/a) = bM(a/b, 1)$.

For further analysis of the process, define c_n by $a_n^2 - b_n^2 = c_n^2$.

2.1. *With c_n defined in this way, we have*

(M4) $c_{n+1} = \frac{1}{2}(a_n - b_n)$,

(M5) $a_n = a_{n+1} + c_{n+1}, \quad b_n = a_{n+1} - c_{n+1}$,

(M6) $c_n^2 = 4a_{n+1}c_{n+1}$,

(M7) $c_{n+1} < \frac{1}{2}c_n$, also $c_{n+1} \le \dfrac{c_n^2}{4b}$.

Proof. For (M4), we have

$$c_{n+1}^2 = a_{n+1}^2 - b_{n+1}^2 = \tfrac{1}{4}(a_n + b_n)^2 - a_n b_n = \tfrac{1}{4}(a_n - b_n)^2.$$

(M5) follows, since $a_{n+1} = \frac{1}{2}(a_n + b_n)$, and (M6) is given by $c_n^2 = (a_n + b_n)(a_n - b_n) = (2a_{n+1})(2c_{n+1})$. The inequalities in (M7) follow from (M3), (M4) and (M6). □

(M5) shows how to derive a_n and b_n from a_{n+1} and b_{n+1}, reversing the agm iteration.

(M7) shows that c_n (hence also $a_n - b_n$) converges to 0 *quadratically*, hence very rapidly. We can derive an explicit bound for c_n as follows:

2.2. *Let $R > 0$ and let k be such that $c_k \le 4b/R$. Then $c_{n+k} \le 4b/R^{2^n}$ for all $n \ge 0$. The same holds with b replaced by b_k.*

Proof. The hypothesis equates to the statement for $n = 0$. Assuming it now for a general n, we have by (M7)

$$c_{n+k+1} < \frac{1}{4b}\frac{(4b)^2}{R^{2^{n+1}}} = \frac{4b}{R^{2^{n+1}}}. \qquad \square$$

In the case $M(\sqrt{2}, 1)$, we have $c_0 = b = 1$. Taking $R = 4$, we deduce $c_n \le 4/4^{2^n}$ for all n. For a slightly stronger variant, note that $c_2 = a_1 - a_2 < \frac{1}{100}$, hence $c_{n+2} \le 4/(400^{2^n}) = 4/(20^{2^{n+1}})$ for $n \ge 0$, or $c_n \le 4/(20^{2^{n-1}})$ for $n \ge 2$. In particular, $c_4 \le 4 \times 20^{-8} < 2 \times 10^{-10}$.

We finish this section with a variant of the agm iteration in the form of a single sequence instead of a pair of sequences.

2.3. *Let $b > 0$. Define a sequence (k_n) by: $k_0 = b$ and*

$$k_{n+1} = \frac{2k_n^{1/2}}{1 + k_n}.$$

Then

$$M(1, b) = \prod_{n=0}^{\infty} \tfrac{1}{2}(1 + k_n).$$

Proof. Let (a_n), (b_n) be the sequences generated by the iteration for $M(1, b)$ (with $a_0 = 1$, $b_0 = b$, whether or not $b < 1$), and let $k_n = b_n/a_n$. Then $k_0 = b$ and

$$k_{n+1} = \frac{b_{n+1}}{a_{n+1}} = \frac{2(a_n b_n)^{1/2}}{a_n + b_n} = \frac{2k_n^{1/2}}{1 + k_n}.$$

So the sequence (k_n) defined by this iteration is (b_n/a_n). Also,

$$\tfrac{1}{2}(1 + k_n) = \frac{a_n + b_n}{2a_n} = \frac{a_{n+1}}{a_n},$$

so

$$\prod_{r=0}^{n-1} \tfrac{1}{2}(1 + k_r) = a_n \to M(1, b) \quad \text{as } n \to \infty. \qquad \square$$

Quadratic convergence of (k_n) to 1 is easily established directly:

$$1 - k_{n+1} = \frac{1 - 2k_n^{1/2} + k_n}{1 + k_n} < 1 - 2k_n^{1/2} + k_n = (1 - k_n^{1/2})^2 < (1 - k_n)^2.$$

Another way to present this iteration is as follows. Assume now that $b < 1$. Recall that for $0 \le k \le 1$, the conjugate k^* is defined by $k^2 + k^{*2} = 1$. Clearly, if $t = 2k^{1/2}/(1+k)$ (where $0 \le k \le 1$), then $t^* = (1-k)/(1+k)$. So we have $k^*_{n+1} = (1-k_n)/(1+k_n)$, hence $1 + k^*_{n+1} = 2/(1+k_n)$, so that

$$M(1,b) = \prod_{n=1}^{\infty} \frac{1}{1+k_n^*}.$$

3. The evaluation of $I(a,b)$: Gauss's theorem

We now come to the basic theorem relating $I(a,b)$ and $M(a,b)$. It was established by Gauss in 1799. There are two distinct steps, which we present as separate theorems. In the notation of section 2, we show first that $I(a,b) = I(a_1,b_1)$. It is then a straightforward deduction that $I(a,b) = \pi/[2M(a,b)]$. Early proofs of the first step, including that of Gauss, were far from simple. Here we present a proof due to Newman [New1] based on an ingenious substitution.

3.1 THEOREM. *Let $a \ge b > 0$, and let $a_1 = \frac{1}{2}(a+b)$, $b_1 = (ab)^{1/2}$. Then*

$$I(a,b) = I(a_1,b_1). \tag{1}$$

Proof. We write $I(a_1,b_1)$ as an integral on $(-\infty,\infty)$, using the fact that the integrand is even:

$$I(a_1,b_1) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a_1^2)^{1/2}(x^2+b_1^2)^{1/2}}\, dx.$$

Substitute

$$2x = t - \frac{ab}{t}.$$

Then $x \to \infty$ as $t \to \infty$ and $x \to -\infty$ as $t \to 0^+$. Also, $\frac{dx}{dt} = \frac{1}{2}(1 + ab/t^2)$ and

$$4(x^2+b_1^2) = 4x^2 + 4ab = t^2 + 2ab + \frac{a^2b^2}{t^2} = \left(t + \frac{ab}{t}\right)^2,$$

hence

$$(x^2+b_1^2)^{1/2} = \tfrac{1}{2}\left(t + \frac{ab}{t}\right).$$

Further,

$$4(x^2+a_1^2) = 4x^2 + (a+b)^2 = t^2 + a^2 + b^2 + \frac{a^2b^2}{t^2} = \frac{1}{t^2}(t^2+a^2)(t^2+b^2).$$

Hence

$$\begin{aligned} I(a_1, b_1) &= \frac{1}{2}\int_0^\infty \frac{t}{\frac{1}{2}(t^2+a^2)^{1/2}(t^2+b^2)^{1/2}\frac{1}{2}(t+\frac{ab}{t})}\,\frac{1}{2}\left(1+\frac{ab}{t^2}\right)\,dt \\ &= \int_0^\infty \frac{1}{(t^2+a^2)^{1/2}(t^2+b^2)^{1/2}}\,dt \\ &= I(a,b). \qquad \square \end{aligned}$$

Lord [Lo2] gives an interesting alternative proof based on the area in polar coordinates.

3.2 THEOREM. *We have*

$$I(a,b) = \frac{\pi}{2M(a,b)} \tag{2}$$

Proof. Let (a_n), (b_n) be the sequences generated by the iteration for $M(a,b)$. By Theorem 3.1, $I(a,b) = I(a_n, b_n)$ for all n. Hence

$$\frac{\pi}{2a_n} \le I(a,b) \le \frac{\pi}{2b_n}$$

for all n. Since both a_n and b_n converge to $M(a,b)$, the statement follows. $\square$

We describe some immediate consequences; more will follow later.

3.3. *We have*

$$\frac{\pi}{a+b} \le I(a,b) \le \frac{\pi}{2(ab)^{1/2}}. \tag{3}$$

Proof. This follows from Theorem 3.1, since $\pi/(2a_1) \le I(a_1,b_1) \le \pi/(2b_1)$. $\square$

The right-hand inequality in (3) repeats 1.2, but I do not know an elementary proof of the left-hand inequality.

From the expressions equivalent to $I(\sqrt{2},1)$ given in 1.6, we have:

3.4. *Let* $M_0 = M(\sqrt{2},1)$ *(*≈ 1.198140*). Then*

$$\int_0^1 \frac{1}{(1-x^4)^{1/2}}\,dx = I(\sqrt{2},1) = \frac{\pi}{2M_0} \quad (\approx 1.311029),$$

$$B(\tfrac{1}{4},\tfrac{1}{2}) = \frac{2\pi}{M_0} \quad (\approx 5.244116),$$

$$\Gamma(\tfrac{1}{4}) = \frac{(2\pi)^{3/4}}{M_0^{1/2}} \quad (\approx 3.625610). \qquad \square$$

Example. From the value $M(100,1) \approx 26.2167$ found in section 2, we have $I(100,1) \approx 0.059916$.

Finally, we present several ways in which Theorem 3.1 can be rewritten in terms of $K(b)$. Recall that $I(1,b) = K(b^*)$, where $b^2 + b^{*2} = 1$, and that if $k = (1-b)/(1+b)$, then $k^* = 2k^{1/2}/(1+b)$.

3.5. *For* $0 \le b < 1$,

$$K(b) = I(1+b, 1-b). \tag{4}$$

Proof. By Theorem 3.1,

$$I(1+b, 1-b) = I[1, (1-b^2)^{1/2}] = I(1, b^*) = K(b). \qquad \square$$

3.6. *For* $0 < b < 1$,

$$K(b) = \frac{1}{1+b}\, K\left(\frac{2b^{1/2}}{1+b}\right), \tag{5}$$

$$K(b^*) = \frac{2}{1+b}\, K\left(\frac{1-b}{1+b}\right). \tag{6}$$

Proof. By (4) and the property $I(ca, cb) = \frac{1}{c} I(a,b)$,

$$K(b) = I(1+b, 1-b) = \frac{1}{1+b}\, I\left(1, \frac{1-b}{1+b}\right) = \frac{1}{1+b}\, K\left(\frac{2b^{1/2}}{1+b}\right).$$

Also, by Theorem 3.1,

$$K(b^*) = I(1,b) = I\left(\frac{1+b}{2}, b^{1/2}\right) = \frac{2}{1+b}\, I\left(1, \frac{2b^{1/2}}{1+b}\right) = \frac{2}{1+b}\, K\left(\frac{1-b}{1+b}\right). \quad \square$$

Conversely, any of (4), (5), (6) implies (1) (to show this for (4), start by choosing c and k so that $ca = 1+k$ and $cb = 1-k$). For another proof of Theorem 3.1, based on this observation and the series expansion of $(1 + ke^{2i\theta})^{-1/2})(1 + ke^{-2i\theta})^{-1/2}$, see [Bor2, p. 12].

4. Estimates for $I(a,1)$ for large a and $I(1,b)$ for small b

In this section, we will show that $I(1,b)$ is closely approximated by $\log(4/b)$ for b close to 0. Equivalently, $aI(a,1)$ is approximated by $\log 4a$ for large a, and consequently $M(a,1)$ is approximated by

$$F(a) =: \frac{\pi}{2} \frac{a}{\log 4a}.$$

By way of illustration, $F(100) = 26.2172$ to four d.p., while $M(100,1) \approx 26.2167$.

This result has been known for a long time, with varying estimates of the accuracy of the approximation. For example, a version appears in [WW, p. 522], published in 1927.

A highly effective method was introduced by Newman in [New2], but only sketched rather briefly, with the statement given in the form $aI(a,1) = \log 4a + O(1/a^2)$. Following [Jam], we develop Newman's method in more detail to establish a more precise estimation. The method requires nothing more than some elementary integrals and the binomial and logarithmic series, and a weaker form of the result is obtained with very little effort. The starting point is:

4.1. *We have*

$$\int_0^{(ab)^{1/2}} \frac{1}{(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}\,dx = \int_{(ab)^{1/2}}^{\infty} \frac{1}{(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}\,dx.$$

Proof. With the substitution $x = ab/y$, the left-hand integral becomes

$$\int_{(ab)^{1/2}}^{\infty} \frac{1}{(\frac{a^2b^2}{y^2}+a^2)^{1/2}(\frac{a^2b^2}{y^2}+b^2)^{1/2}}\frac{ab}{y^2}\,dy = \int_{(ab)^{1/2}}^{\infty} \frac{1}{(b^2+y^2)^{1/2}(a^2+y^2)^{1/2}}\,dy. \qquad \square$$

Hence

$$I(1,b) = 2\int_0^{b^{1/2}} H(x)\,dx, \tag{1}$$

where

$$H(x) = \frac{1}{(x^2+1)^{1/2}(x^2+b^2)^{1/2}}. \tag{2}$$

Now $(1+y)^{-1/2} > 1 - \frac{1}{2}y$ for $0 < y < 1$, as is easily seen by multiplying out $(1+y)(1-\frac{1}{2}y)^2$, so for $0 < x < 1$,

$$(1-\tfrac{1}{2}x^2)\frac{1}{(x^2+b^2)^{1/2}} < H(x) < \frac{1}{(x^2+b^2)^{1/2}}. \tag{3}$$

Now observe that the substitution $x = by$ gives

$$\int_0^{b^{1/2}} \frac{1}{(x^2+b^2)^{1/2}}\,dx = \int_0^{1/b^{1/2}} \frac{1}{(y^2+1)^{1/2}}\,dy = \sinh^{-1}\frac{1}{b^{1/2}}.$$

4.2 LEMMA. *We have*

$$\log\frac{2}{b^{1/2}} < \sinh^{-1}\frac{1}{b^{1/2}} < \log\frac{2}{b^{1/2}} + \tfrac{1}{4}b. \tag{4}$$

Proof. Recall that $\sinh^{-1} x = \log[x + (x^2+1)^{1/2}]$, which is clearly greater than $\log 2x$. Also, since $(1+b)^{1/2} < 1 + \frac{1}{2}b$,

$$\frac{1}{b^{1/2}} + \left(\frac{1}{b}+1\right)^{1/2} = \frac{1}{b^{1/2}}[1+(1+b)^{1/2}] < \frac{1}{b^{1/2}}(2+\tfrac{1}{2}b) = \frac{2}{b^{1/2}}(1+\tfrac{1}{4}b).$$

The right-hand inequality in (4) now follows from the fact that $\log(1+x) \le x$. $\square$

This is already enough to prove the following weak version of our result:

4.3 PROPOSITION. *For* $0 < b \le 1$ *and* $a \ge 1$,

$$\log\frac{4}{b} - \tfrac{1}{2}b < I(1,b) < \log\frac{4}{b} + \tfrac{1}{2}b, \tag{5}$$

$$\log 4a - \frac{1}{2a} < aI(a,1) < \log 4a + \frac{1}{2a}. \tag{6}$$

Proof. The two statements are equivalent, because $aI(a,1) = I(1, \frac{1}{a})$. By (1) and (3),

$$2\sinh^{-1}\frac{1}{b^{1/2}} - R_1(b) < I(1,b) < 2\sinh^{-1}\frac{1}{b^{1/2}},$$

where

$$R_1(b) = \int_0^{b^{1/2}} \frac{x^2}{(x^2+b^2)^{1/2}}\,dx. \tag{7}$$

Since $(x^2+b^2)^{1/2} > x$, we have

$$R_1(b) < \int_0^{b^{1/2}} \frac{x^2}{x}\,dx = \int_0^{b^{1/2}} x\,dx = \tfrac{1}{2}b.$$

Both inequalities in (5) now follow from (4), since $2\log\frac{2}{b^{1/2}} = \log\frac{4}{b}$. □

This result is sufficient to show that $M(a,1) - F(a) \to 0$ as $a \to \infty$. To see this, write $aI(a,1) = \log 4a + r(a)$. Then

$$\frac{1}{I(a,1)} - \frac{a}{\log 4a} = -\frac{ar(a)}{\log 4a[\log 4a + r(a)]},$$

which tends to 0 as $a \to \infty$, since $|ar(a)| \le \frac{1}{2}$.

Some readers may be content with this version, but for those with the appetite for it, we now refine the method to establish closer estimates. All we need to do is improve our estimations by the insertion of further terms. The upper estimate in (3) only used $(1+y)^{-1/2} < 1$. Instead, we now use the stronger bound

$$\frac{1}{(1+y)^{1/2}} < 1 - \tfrac{1}{2}y + \tfrac{3}{8}y^2$$

for $0 < y < 1$. These are the first three terms of the binomial expansion, and the stated inequality holds because the terms of the expansion alternate in sign and decrease in magnitude. So we have instead of (3):

$$(1 - \tfrac{1}{2}x^2)\frac{1}{(x^2+b^2)^{1/2}} < H(x) < (1 - \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4)\frac{1}{(x^2+b^2)^{1/2}}. \tag{8}$$

We also need a further degree of accuracy in the estimates for $\sinh^{-1}\frac{1}{b^{1/2}}$ and $R_1(b)$.

4.4 LEMMA. *For $0 < b \le 1$, we have*

$$\log \frac{2}{b^{1/2}} + \tfrac{1}{4}b - \tfrac{3}{32}b^2 < \sinh^{-1} \frac{1}{b^{1/2}} < \log \frac{2}{b^{1/2}} + \tfrac{1}{4}b - \tfrac{1}{16}b^2 + \tfrac{1}{32}b^3. \tag{9}$$

Proof. Again by the binomial series, we have

$$1 + \tfrac{1}{2}b - \tfrac{1}{8}b^2 < (1+b)^{1/2} < 1 + \tfrac{1}{2}b - \tfrac{1}{8}b^2 + \tfrac{1}{16}b^3. \tag{10}$$

So, as in Lemma 4.2,

$$\frac{1}{b^{1/2}} + \left(\frac{1}{b} + 1\right)^{1/2} < \frac{1}{b^{1/2}}(2 + \tfrac{1}{2}b - \tfrac{1}{8}b^2 + \tfrac{1}{16}b^3) = \frac{2}{b^{1/2}}(1 + \tfrac{1}{4}b - \tfrac{1}{16}b^2 + \tfrac{1}{32}b^3).$$

The right-hand inequality in (9) now follows from $\log(1+x) \le x$. Also,

$$\frac{1}{b^{1/2}} + \left(\frac{1}{b} + 1\right)^{1/2} > \frac{1}{b^{1/2}}(2 + \tfrac{1}{2}b - \tfrac{1}{8}b^2) = \frac{2}{b^{1/2}}(1 + B),$$

where $B = \frac{1}{4}b - \frac{1}{16}b^2$ (so $B < \frac{1}{4}b$). By the log series, $\log(1+x) > x - \frac{1}{2}x^2$ for $0 < x < 1$, so

$$\log(1+B) > B - \tfrac{1}{2}B^2 > \tfrac{1}{4}b - \tfrac{1}{16}b^2 - \tfrac{1}{32}b^2 = \tfrac{1}{4}b - \tfrac{3}{32}b^2. \qquad \square$$

4.5 LEMMA. *For $0 < b \le 1$, we have:*

$$R_1(b) < \tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 \log \frac{2}{b^{1/2}}, \tag{11}$$

$$R_1(b) > \tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 \log \frac{2}{b^{1/2}} - \tfrac{3}{16}b^3, \tag{12}$$

$$R_1(b) < \tfrac{1}{2}b - \tfrac{1}{5}b^2. \tag{13}$$

Proof. We can evaluate $R_1(b)$ explicitly by the substitution $x = b \sinh t$:

$$R_1(b) = \int_0^{c(b)} b^2 \sinh^2 t \, dt = \tfrac{1}{2}b^2 \int_0^{c(b)} (\cosh 2t - 1)\, dt = \tfrac{1}{4}b^2 \sinh 2c(b) - \tfrac{1}{2}b^2 c(b),$$

where $c(b) = \sinh^{-1}(1/b^{1/2})$. Now

$$\sinh 2c(b) = 2 \sinh c(b) \cosh c(b) = \frac{2}{b^{1/2}} \left(\frac{1}{b} + 1\right)^{1/2} = \frac{2}{b}(1+b)^{1/2},$$

so

$$R_1(b) = \tfrac{1}{2}b(1+b)^{1/2} - \tfrac{1}{2}b^2 c(b).$$

Statement (11) now follows from $(1+b)^{1/2} < 1 + \frac{1}{2}b$ and $c(b) > \log \frac{2}{b^{1/2}}$. Statement (12) follows from the left-hand inequality in (10) and the right-hand one in (4).

Unfortunately, (13) does not quite follow from (11). We prove it directly from the integral, as follows. Write $x^2/(x^2+b^2)^{1/2} = g(x)$. Since $g(x) < x$, we have $\int_b^{b^{1/2}} g(x)\,dx < \frac{1}{2}(b-b^2)$. For $0 < x < b$, we have by the binomial expansion again

$$g(x) = \frac{x^2}{b(1+x^2/b^2)^{1/2}} \le \frac{x^2}{b}\left(1 - \frac{x^2}{2b^2} + \frac{3x^4}{8b^4}\right),$$

and hence

$$\int_0^b g(x)\,dx < \left(\tfrac{1}{3} - \tfrac{1}{10} + \tfrac{3}{56}\right)b^2 < \tfrac{3}{10}b^2.$$

Together, these estimates give $R_1(b) < \frac{1}{2}b - \frac{1}{5}b^2$. □

We can now state the full version of our Theorem.

4.6 THEOREM. *For $0 < b \le 1$ and $a \ge 1$, we have*

$$\log\frac{4}{b} < I(1,b) < (1 + \tfrac{1}{4}b^2)\log\frac{4}{b}, \tag{14}$$

$$\log 4a < aI(a,1) < \left(1 + \frac{1}{4a^2}\right)\log 4a. \tag{15}$$

Moreover, the following lower bounds also apply:

$$I(1,b) > (1 + \tfrac{1}{4}b^2)\log\frac{4}{b} - \tfrac{7}{16}b^2, \tag{16}$$

$$aI(a,1) > \left(1 + \frac{1}{4a^2}\right)\log 4a - \frac{7}{16a^2}. \tag{17}$$

Hence

$$I(1,b) = (1 + \tfrac{1}{4}b^2)\log\frac{4}{b} + O(b^2) \quad \textit{for } 0 < b < 1,$$

$$aI(a,1) = \left(1 + \frac{1}{4a^2}\right)\log 4a + O(1/a^2) \quad \textit{for } a > 1.$$

Note. As this shows, the estimation $\log 4a + O(1/a^2)$ in [New2] is not quite correct.

Proof: lower bounds. By (8), (9) and (13),

$$\begin{aligned} I(1,b) &> 2\sinh^{-1}\frac{1}{b^{1/2}} - R_1(b) \\ &> \log\frac{4}{b} + \tfrac{1}{2}b - \tfrac{3}{16}b^2 - (\tfrac{1}{2}b - \tfrac{1}{5}b^2) \\ &> \log\frac{4}{b}, \end{aligned}$$

(with a spare term $\frac{1}{80}b^2$). Of course, the key fact is the cancellation of the term $\frac{1}{2}b$. Also, using (11) instead of (13), we have

$$\begin{aligned} I(1,b) &> \log\frac{4}{b} + \tfrac{1}{2}b - \tfrac{3}{16}b^2 - (\tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{4}b^2\log\frac{4}{b}) \\ &= (1 + \tfrac{1}{4}b^2)\log\frac{4}{b} - \tfrac{7}{16}b^2. \end{aligned}$$

Upper bound: By (8),

$$I(1,b) < 2\sinh^{-1}\frac{1}{b^{1/2}} - R_1(b) + \tfrac{3}{4}R_2(b),$$

where

$$R_2(b) = \int_0^{b^{1/2}} \frac{x^4}{(x^2+b^2)^{1/2}}\,dx < \int_0^{b^{1/2}} x^3\,dx = \tfrac{1}{4}b^2.$$

So by (9) and (12),

$$\begin{aligned} I(1,b) &< \log\frac{4}{b} + \tfrac{1}{2}b - \tfrac{1}{8}b^2 + \tfrac{1}{16}b^3 - (\tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{4}b^2\log\frac{4}{b} - \tfrac{3}{16}b^3) + \tfrac{3}{16}b^2 \\ &= (1+\tfrac{1}{4}b^2)\log\frac{4}{b} - \tfrac{3}{16}b^2 + \tfrac{1}{4}b^3. \end{aligned}$$

If $b < \frac{3}{4}$, then $\frac{3}{16}b^2 \geq \frac{1}{4}b^3$, so $I(1,b) < (1+\frac{1}{4}b^2)\log\frac{4}{b}$.

For $b \geq \frac{3}{4}$, we reason as follows. By 1.2, $I(1,b) \leq \frac{\pi}{4}(1+\frac{1}{b})$. Write $h(b) = (1+\frac{1}{4}b^2)\log\frac{4}{b} - \frac{\pi}{4}(1+\frac{1}{b})$. We find that $h(\frac{3}{4}) \approx 1.9094 - 1.8326 > 0$. One verifies by differentiation that $h(b)$ is increasing for $0 < b \leq 1$ (we omit the details), hence $h(b) > 0$ for $\frac{3}{4} \leq b \leq 1$. □

By Gauss's theorem and the inequality $1/(1+x) > 1-x$, applied with $x = 1/(4a^2)$, statement (15) translates into the following pair of inequalities for $M(a,1)$:

4.7 COROLLARY. *Let $F(a) = (\pi a)/(2\log 4a)$. Then for $a \geq 1$,*

$$\left(1 - \frac{1}{4a^2}\right)F(a) < M(a,1) < F(a). \tag{18}$$

Since $M(a,b) = b\,M(a/b,\,1)$ we can derive the following bounds for $M(a,b)$ (where $a > b$) in general:

$$\frac{\pi}{2}\frac{a}{\log(4a/b)}\left(1 - \frac{b^2}{4a^2}\right) < M(a,b) < \frac{\pi}{2}\frac{a}{\log(4a/b)}. \tag{19}$$

Note: We mention very briefly an older, perhaps better known, approach (cf. [Bow, p. 21–22]). Use the identity $I(1,b) = K(b^*)$, and note that $b^* \to 1$ when $b \to 0$. Define

$$A(b^*) = \int_0^{\pi/2} \frac{b^*\sin\theta}{(1-b^{*2}\sin^2\theta)^{1/2}}\,d\theta$$

and $B(b^*) = K(b^*) - A(b^*)$. One shows by direct integration that $A(b^*) = \log[(1+b^*)/b]$ and $B(1) = \log 2$, so that when $b \to 0$, we have $A(b^*) - \log\frac{2}{b} \to 0$ and hence $K(b^*) - \log\frac{4}{b} \to 0$. This method can be developed to give an estimation with error term $O(b^2\log\frac{1}{b})$, [Bor1, p. 355–357], but the details are distinctly more laborious, and (even with some refinement) the end result does not achieve the accuracy of our Theorem 4.6.

By a different method, we now establish exact upper and lower bounds for $I(1,b) - \log(1/b)$ on $(0,1]$. The lower bound amounts to a second proof that $I(1,b) > \log(4/b)$.

4.8. *The expression* $I(1,b) - \log(1/b)$ *increases with* b *for* $0 < b \le 1$. *Hence*

$$\log 4 \le I(1,b) - \log\frac{1}{b} \le \frac{\pi}{2} \qquad \text{for } 0 < b \le 1; \tag{20}$$

$$\log 4 \le aI(a,1) - \log a \le \frac{\pi}{2} \qquad \text{for } a \ge 1; \tag{21}$$

$$\frac{\pi}{2}\,\frac{a}{\log a + \frac{\pi}{2}} \le M(a,1) \le \frac{\pi}{2}\,\frac{a}{\log a + \log 4} \qquad \text{for } a \ge 1. \tag{22}$$

Proof. We show that $K(b) - \log(1/b^*)$ decreases with b, so that $I(1,b) - \log(1/b) = K(b^*) - \log(1/b)$ increases with b. Once this is proved, the inequalities follow, since $I(1,b) - \log(1/b)$ takes the value $\pi/2$ at $b = 1$ and (by 4.3) tends to $\log 4$ as $b \to 0^+$.

By 1.4, for $0 \le b < 1$, we have $K(b) = \frac{\pi}{2}\sum_{n=0}^{\infty} d_{2n} b^{2n}$, where $d_0 = 1$ and

$$d_{2n} = \frac{1^2.3^2 \dots (2n-1)^2}{2^2.4^2 \dots (2n)^2}.$$

Hence $K'(b) = \pi \sum_{n=1}^{\infty} n d_{2n} b^{2n-1}$. Meanwhile, $\log(1/b^*) = -\frac{1}{2}\log(1-b^2)$, so

$$\frac{d}{db}\log\frac{1}{b^*} = \frac{b}{1-b^2} = \sum_{n=1}^{\infty} b^{2n-1}.$$

By well-known inequalities for the Wallis product, $n\pi d_{2n} < 1$, so $K'(b) < \frac{d}{db}\log(1/b^*)$. □

Application to the calculation of logarithms and π. Theorem 4.6 has pleasing applications to the calculation of logarithms (assuming π known) and π (assuming logarithms known), exploiting the rapid convergence of the agm iteration. Consider first the problem of calculating $\log x$ (where $x > 1$). Choose n, in a way to be discussed below, and let $4a = x^n$, so that $n \log x = \log 4a$, which is approximated by $aI(a,1)$. We calculate $M = M(a,1)$, and hence $I(a,1) = \pi/(2M)$. Then $\log x$ is approximated by $(a/n)I(a,1)$.

How accurate is this approximation? By Theorem 4.6, $\log 4a = aI(a,1) - r(a)$, where $0 < r(a) < \frac{1}{4a^2}\log 4a$. Hence

$$\log x = \frac{a}{n}I(a,1) - \frac{r(a)}{n},$$

in which

$$0 < \frac{r(a)}{n} < \frac{\log x}{4a^2} = \frac{4\log x}{x^{2n}}.$$

We choose n so that this is within the desired degree of accuracy. It might seem anomalous that $\log x$, which we are trying to calculate, appears in this error estimation, but a rough estimate for it is always readily available.

For illustration, we apply this to the calculation of $\log 2$, taking $n = 12$, so that $a = 2^{10} = 1024$. We find that $M = M(1024, 1) \approx 193.38065$, so our approximation is

$$\frac{\pi}{2} \frac{1024}{12M} \approx 0.6931474,$$

while in fact $\log 2 = 0.6931472$ to seven d.p.. The discussion above (just taking $\log 2 < 1$) shows that the approximation overestimates $\log 2$, with error no more than $4/2^{24} = 1/2^{20}$.

We now turn to the question of approximating π. The simple-minded method is as follows. Choose a suitably large a. By Theorems 3.2 and 4.6,

$$\frac{\pi}{2} = M(a,1)I(a,1) = M(a,1)\frac{\log 4a}{a} + s(a),$$

where

$$s(a) = M(a,1)\frac{r(a)}{a} < M(a,1)\frac{\log 4a}{4a^3}.$$

So an approximation to π is $\frac{2}{a}M(a,1)\log 4a$, with error no more than $2s(a)$. Using our original example with $a = 100$, this gives 3.14153 to five d.p., with the error estimated as no more than 8×10^{-5} (the actual error is about 6.4×10^{-5}).

For a better approximation, one would repeat with a larger a. However, there is a way to generate a sequence of increasingly close approximations from a single agm iteration, which we now describe (cf. [Bor1, Proposition 3]).

In the original agm iteration, let $c_n = (a_n^2 - b_n^2)^{1/2}$. Note that c_n converges rapidly to 0. By Gauss's theorem, $\frac{\pi}{2} = M(a_0, c_0)I(a_0, c_0)$. To apply Theorem 4.6, we need to equate $I(a_0, c_0)$ to an expression of the form $I(d_n, e_n)$, where e_n/d_n tends to 0. This is achieved as follows. Recall from 2.1 that $a_n = a_{n+1} + c_{n+1}$ and $c_n^2 = 4a_{n+1}c_{n+1}$, so the arithmetic mean of $2a_{n+1}$ and $2c_{n+1}$ is a_n, while the geometric mean is c_n. So by Theorem 3.1, $I(a_n, c_n) = I(2a_{n+1}, 2c_{n+1})$, and hence

$$I(a_0, c_0) = I(2^n a_n, 2^n c_n) = \frac{1}{2^n a_n} I\left(1, \frac{c_n}{a_n}\right).$$

Since c_n/a_n tends to 0, Theorem 4.6 (or even 4.3) applies to show that

$$I(a_0, c_0) = \lim_{n\to\infty} \frac{1}{2^n a_n} \log \frac{4a_n}{c_n}.$$

Now take $a_0 = 1$ and $b_0 = \frac{1}{\sqrt{2}}$. Then $c_0 = b_0$, so $M(a_0, b_0) = M(a_0, c_0)$, and (a_n) converges to this value. Since $\pi = 2M(a_0, c_0)I(a_0, c_0)$, we have established the following:

4.9 PROPOSITION. *With a_n, b_n, c_n defined in this way, we have*

$$\pi = \lim_{n\to\infty} \frac{1}{2^{n-1}} \log \frac{4a_n}{c_n}. \qquad \square$$

Denote this approximation by p_n. With enough determination, one can derive an error estimation, but we will just illustrate the speed of convergence by performing the first three steps. Values are rounded to six significant figures. To calculate c_n, we use $c_n = c_{n-1}^2/(4a_n)$ rather than $\frac{1}{2}(a_{n-1} - b_{n-1})$, since this would require much greater accuracy in the values of a_{n-1} and b_{n-1}.

n	a_n	b_n	c_n	p_n
0	1	0.707106	0.707106	
1	0.853553	0.840896	0.146447	3.14904
2	0.847225	0.847201	0.00632849	3.14160
3	0.847213	0.847213	0.0000118181	3.14159

More accurate calculation gives the value $p_3 = 3.14159265$, showing that it actually agrees with π to eight decimal places.

While this expression does indeed converge rapidly to π, it has the disadvantage that it requires the calculation of logarithms at each stage. This is overcome by the Gauss-Brent-Salamin algorithm, which we describe in section 7.

5. The integrals $J(a, b)$ and $E(b)$

For a, $b \geq 0$, define

$$J(a, b) = \int_0^{\pi/2} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{1/2} \, d\theta. \tag{1}$$

Note that $4J(a, b)$ is the perimeter of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Some immediate facts:

(E1) $J(a, a) = \int_0^{\pi/2} a \, d\theta = \dfrac{\pi a}{2}$,

(E2) $J(a, 0) = \int_0^{\pi/2} a \cos \theta \, d\theta = a$,

(E3) for $c > 0$, $J(ca, cb) = cJ(a, b)$;

(E4) $J(b, a) = J(a, b)$ (substitute $\theta = \frac{\pi}{2} - \phi$);

(E5) $J(a, b)$ increases with a and with b;

(E6) if $a \geq b$, then $\dfrac{\pi b}{2} \leq J(a, b) \leq \dfrac{\pi a}{2}$.

(E6) follows from (E1) and (E5), since $J(b, b) \leq J(a, b) \leq J(a, a)$.

Since $x + \frac{1}{x} \geq 2$ for $x > 0$, we have $I(a, b) + J(a, b) \geq \pi$. For $0 < b \leq 1$, we have $J(1, b) \leq \frac{\pi}{2} \leq I(1, b)$.

We start with some further inequalities for $J(a,b)$. Given that $J(a,b)$ is the length of the quarter-ellipse, it would seem geometrically more or less obvious that

$$(a^2+b^2)^{1/2} \le J(a,b) \le a+b, \tag{2}$$

since this says that the curve is longer than the straight-line path between the same points, but shorter than two sides of the rectangle. An analytic proof of the right-hand inequality is very easy: since $(x^2+y^2)^{1/2} \le x+y$ for positive x, y, we have $(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2} \le a\cos\theta + b\sin\theta$ for each θ. Integration on $[0,\pi/2]$ gives $J(a,b) \le a+b$. We will improve this inequality below.

For the left-hand inequality in (2), and some further estimations, we use the Cauchy-Schwarz inequality, in both its discrete and its integral form.

5.1. *We have*

$$J(a,b) \ge (a^2+b^2)^{1/2}. \tag{3}$$

Proof. By the Cauchy-Schwarz inequality,

$$a^2\cos\theta + b^2\sin\theta = a(a\cos\theta) + b(b\sin\theta) \le (a^2+b^2)^{1/2}(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}.$$

Integrating on $[0,\pi/2]$, we obtain

$$a^2+b^2 \le (a^2+b^2)^{1/2} J(a,b),$$

hence (3). □

A slight variation of this reasoning gives a second lower bound, improving upon (E6).

5.2. *We have*

$$J(a,b) \ge \frac{\pi}{4}(a+b). \tag{4}$$

Proof. By the Cauchy-Schwarz inequality,

$$\begin{aligned} a\cos^2\theta + b\sin^2\theta &= (a\cos\theta)\cos\theta + (b\sin\theta)\sin\theta \\ &\le (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}(\cos^2\theta + \sin^2\theta)^{1/2} \\ &= (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}. \end{aligned}$$

Integration gives (4), since $\int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{\pi}{4}$. □

Note that equality holds in (3) when $b=0$, and in (4) when $b=a$.

5.3. *We have*

$$J(a,b) \le \frac{\pi}{2\sqrt{2}}(a^2+b^2)^{1/2}. \tag{5}$$

Equality holds when $a = b$.

Proof. By the Cauchy-Schwarz inequality for integrals (with both sides squared),

$$J(a,b)^2 \le \left(\int_0^{\pi/2} 1\right) \int_0^{\pi/2} (a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta = \frac{\pi}{2}\,\frac{\pi}{4}\,(a^2+b^2). \qquad \square$$

Since $\pi/(2\sqrt{2}) \approx 1.1107$, $J(a,b)$ is actually modelled fairly well by $(a^2+b^2)^{1/2}$.

For $u = (a,b)$, write $\|u\| = (a^2+b^2)^{1/2}$: this is the *Euclidean norm* on $\mathbb{R}^2$. In this notation, we have

$$J(a,b) = \int_0^{\pi/2} \|(a\cos\theta, b\sin\theta)\|\, d\theta.$$

The triangle inequality for vectors says that $\|u_1+u_2\| \le \|u_1\| + \|u_2\|$. We show that $J(a,b)$ also satisfies the triangle inequality (so in fact is also a norm on $\mathbb{R}^2$).

5.4. *Let $u_j = (a_j, b_j)$ ($j = 1,2$), where a_j, $b_j \ge 0$. Then*

$$J(u_1+u_2) \le J(u_1) + J(u_2). \tag{6}$$

Proof. For each θ, we have

$$\|[(a_1+a_2)\cos\theta, (b_1+b_2)\sin\theta]\| \le \|(a_1\cos\theta, b_1\sin\theta)\| + \|(a_2\cos\theta, b_2\sin\theta)\|.$$

Integrating on $[0, \frac{\pi}{2}]$, we deduce that

$$J(a_1+a_2, b_1+b_2) \le J(a_1,b_1) + J(a_2,b_2). \qquad \square$$

By suitable choices of u_1 and u_2, we can read off various inequalities for $J(a,b)$. Firstly, $J(a,b) \le J(a,0) + J(0,b) = a+b$, as seen in (2). Secondly:

Second proof of (4). Since $(a+b, a+b) = (a,b) + (b,a)$, we have

$$(a+b)\frac{\pi}{2} = J(a+b, a+b) \le J(a,b) + J(b,a) = 2J(a,b). \qquad \square$$

Thirdly, we can compare $J(a,b)$ with the linear function of b that agrees with $J(a,b)$ at $b = 0$ and $b = a$. The resulting upper bound improves simultaneously on those in (E6) and (2).

5.5. *For* $0 \le b \le a$, *we have*

$$J(a,b) \le a + \left(\frac{\pi}{2} - 1\right) b. \tag{7}$$

Equality holds when $b = 0$ *and when* $b = a$.

Proof. Since $(a,b) = (a-b,0) + (b,b)$,

$$J(a,b) \le J(a-b,0) + J(b,b) = (a-b) + \frac{\pi}{2}b. \qquad \square$$

Inequality (7) reflects the fact that $J(a,b)$ is a *convex* functon of b for fixed a; this also follows easily from (6).

We now consider equivalent and related integrals.

5.6. *We have*

$$J(a,b) = b^2 \int_0^\infty \frac{(x^2+a^2)^{1/2}}{(x^2+b^2)^{3/2}}\, dx. \tag{8}$$

Proof. Substituting $x = b\tan\theta$, we obtain

$$\begin{aligned}
\int_0^\infty \frac{(x^2+a^2)^{1/2}}{(x^2+b^2)^{3/2}}\, dx. &= \int_0^{\pi/2} \frac{(a^2+b^2\tan^2\theta)^{1/2}}{b^3\sec^3\theta}\, b\sec^2\theta\, d\theta \\
&= \frac{1}{b^2}\int_0^{\pi/2} (a^2+b^2\tan^2\theta)^{1/2}\cos\theta\, d\theta \\
&= \frac{1}{b^2} J(a,b). \qquad \square
\end{aligned}$$

The "complete elliptic integral of the second kind" is

$$E(b) = \int_0^{\pi/2} (1-b^2\sin^2\theta)^{1/2}\, d\theta. \tag{9}$$

Since $\cos^2\theta + b^2\sin^2\theta = 1 - b^{*2}\sin^2\theta$, we have $J(1,b) = E(b^*)$.

Clearly, $E(0) = \pi/2$, $E(1) = 1$ and $E(b)$ decreases with b. Also,

$$K(b) - E(b) = \int_0^{\pi/2} \frac{1-(1-b^2\sin^2\theta)}{(1-b^2\sin^2\theta)^{1/2}}\, d\theta = \int_0^{\pi/2} \frac{b^2\sin^2\theta}{(1-b^2\sin^2\theta)^{1/2}}\, d\theta. \tag{10}$$

We use the usual mathematical notation $E'(b)$ for the derivative $\frac{d}{db}E(b)$. We alert the reader to the fact that some of the literature on this subject, including [Bor2], writes b' where we have b^* and then uses $E'(b)$ to mean $E(b')$.

5.7. *We have*

$$E'(b) = \frac{1}{b}[E(b) - K(b)] \quad \text{for } 0 < b < 1. \tag{11}$$

Proof. This follows at once by differentiating under the integral sign in (9) and comparing with (10). □

5.8. *We have*

$$E(b) = \int_0^1 \frac{(1-b^2t^2)^{1/2}}{(1-t^2)^{1/2}}\,dt. \tag{12}$$

Proof. Denote this integral by I. The substitution $t = \sin\theta$ gives

$$I = \int_0^{\pi/2} \frac{(1-b^2\sin^2\theta)^{1/2}}{\cos\theta}\cos\theta\,d\theta = E(b). \qquad \square$$

5.9. *For* $0 \le b \le 1$,

$$1 - \tfrac{1}{2}b^2 \le \frac{2}{\pi}E(b) \le 1 - \tfrac{1}{4}b^2. \tag{13}$$

Proof. For $0 \le x \le 1$, we have $1 - x \le (1-x)^{1/2} \le 1 - \frac{1}{2}x$. Hence

$$E(b) \ge \int_0^{\pi/2} (1 - b^2\sin^2\theta)\,d\theta = \frac{\pi}{2}(1 - \tfrac{1}{2}b^2),$$

and similarly for the upper bound. □

When applied to $J(1,b) = E(b^*)$, (13) gives weaker inequalities than (4) and (5). We now give the power series expression. Recall from 1.4 that $\frac{2}{\pi}K(b) = \sum_{n=0}^{\infty} d_{2n}b^{2n}$, where $d_0 = 1$ and

$$d_{2n} = \frac{1^2.3^2\ldots(2n-1)^2}{2^2.4^2\ldots(2n)^2}.$$

5.10 PROPOSITION. *For* $0 \le b < 1$,

$$\frac{2}{\pi}E(b) = 1 - \sum_{n=1}^{\infty} \frac{d_{2n}}{2n-1}b^{2n} = 1 - \tfrac{1}{4}b^2 - \tfrac{3}{64}b^4 + \cdots \tag{14}$$

Proof. By the binomial series,

$$(1 - b^2\sin^2\theta)^{1/2} = 1 + \sum_{n=1}^{\infty} a_{2n}b^{2n}\sin^{2n}\theta,$$

where

$$a_{2n} = (-1)^n\binom{\frac{1}{2}}{n} = \frac{(-1)^n}{n!}(\tfrac{1}{2})(-\tfrac{1}{2})\cdots(\tfrac{1}{2}-n+1) = -\frac{1.3.\ldots(2n-3)}{2.4.\ldots(2n)}.$$

For fixed $b < 1$, the series is uniformly convergent for $0 \le \theta \le \frac{\pi}{2}$. The statement follows, since

$$\int_0^{\pi/2} \sin^{2n}\theta\,d\theta = \frac{\pi}{2}\frac{1.3.\ldots(2n-1)}{2.4.\ldots(2n)}. \qquad \square$$

6. The integral $L(a, b)$

Most of the material in this section follows the excellent exposition in [Lo1].

Further results on $I(a, b)$ and $J(a, b)$ are most conveniently derived and expressed with the help of another integral. For $a > 0$ and $b \geq 0$, define

$$L(a, b) = \int_0^{\pi/2} \frac{\cos^2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}}\, d\theta. \tag{1}$$

The first thing to note is that $L(b, a) \neq L(a, b)$. In fact, we have:

6.1. *For $a \geq 0$ and $b > 0$,*

$$L(b, a) = \int_0^{\pi/2} \frac{\sin^2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}}\, d\theta. \tag{2}$$

Proof. The substitution $\theta = \frac{\pi}{2} - \phi$ gives

$$L(b, a) = \int_0^{\pi/2} \frac{\cos^2\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}}\, d\theta = \int_0^{\pi/2} \frac{\sin^2\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{1/2}}\, d\phi. \quad \square$$

Hence, immediately:

6.2. *For a, $b > 0$,*

$$L(a, b) + L(b, a) = I(a, b), \tag{3}$$

$$a^2 L(a, b) + b^2 L(b, a) = J(a, b). \tag{4}$$

6.3 COROLLARY. *For a, $b > 0$,*

$$(a^2 - b^2)L(a, b) = J(a, b) - b^2 I(a, b). \qquad \square \tag{5}$$

Some immediate facts:

(L1) $L(a, 0) = \dfrac{1}{a}$; $L(0, b)$ is undefined;

(L2) $L(a, a) = \dfrac{\pi}{4a}$;

(L3) $L(ca, cb) = \frac{1}{c}L(a, b)$ for $c > 0$;

(L4) $L(a, b)$ decreases with a and with b;

(L5) $L(a, b) \leq \dfrac{1}{a}$ for all b;

(L6) if $a \geq b$, then $\dfrac{\pi}{4a} \leq L(a, b) \leq \dfrac{\pi}{4b}$, and similarly for $L(b, a)$ (by (L5) and (L2)).

6.4. *We have*

$$\frac{\partial}{\partial b} J(a,b) = bL(b,a). \tag{6}$$

Proof. Differentiate under the integral sign in the expression for $J(a,b)$. $\square$

So if we write $J_1(b)$ for $J(1,b)$, then $J_1'(b) = bL(b,1)$. In particular, $J_1'(1) = L(1,1) = \pi/4$.

6.5. *We have*

$$L(a,b) = \int_0^\infty \frac{b^2}{(x^2+a^2)^{1/2}(x^2+b^2)^{3/2}}\, dx, \tag{7}$$

$$L(b,a) = \int_0^\infty \frac{x^2}{(x^2+a^2)^{1/2}(x^2+b^2)^{3/2}}\, dx, \tag{8}$$

Proof. With the substitution $x = b\tan\theta$, the integral in (7) becomes

$$\begin{aligned}\int_0^{\pi/2} \frac{b^3\sec^2\theta}{(a^2+b^2\tan^2\theta)^{1/2}b^3\sec^3\theta}\, d\theta &= \int_0^{\pi/2} \frac{\cos\theta}{(a^2+b^2\tan^2\theta)^{1/2}}\, d\theta \\ &= \int_0^{\pi/2} \frac{\cos^2\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{1/2}}\, d\theta.\end{aligned}$$

The integral in (8) now follows from (3) and the integral (1.2) for $I(a,b)$. $\square$

6.6. *If $a > b$, then $L(a,b) < L(b,a)$.*

Proof. By (7), applied to both $L(a,b)$ and $L(b,a)$,

$$L(b,a) - L(a,b) = \int_0^\infty \frac{a^2(x^2+b^2) - b^2(x^2+a^2)}{(x^2+a^2)^{3/2}(x^2+b^2)^{3/2}}\, dx.$$

This is positive, since the numerator is $(a^2-b^2)x^2$. $\square$

6.7. *We have*

$$\frac{\partial}{\partial b} I(a,b) = -\frac{1}{b}L(a,b), \tag{9}$$

$$b(a^2-b^2)\frac{\partial}{\partial b} I(a,b) = b^2 I(a,b) - J(a,b). \tag{10}$$

Proof. (9) is obtained by differentiating the integral expression (1.2) for $I(a,b)$ under the integral sign and comparing with (7). (The integral expressions in terms of θ do not deliver this result so readily.) Using (5) to substitute for $L(a,b)$, we obtain (10). $\square$

Hence, writing $I_1(b)$ for $I(1,b)$, we have $I_1'(1) = -L(1,1) = -\pi/4$.

There is a corresponding expression for $K'(b)$, but we defer it to section 8.

Identities (9) and (10) are an example of a relationship that is expressed more pleasantly in terms of $L(a,b)$ rather than $J(a,b)$. More such examples will be encountered below.

The analogue of the functions $K(b)$ and $E(b)$ is the pair of integrals

$$L(1,b^*) = \int_0^{\pi/2} \frac{\cos^2\theta}{(1-b^2\sin^2\theta)^{1/2}}\,d\theta, \qquad L(b^*,1) = \int_0^{\pi/2} \frac{\sin^2\theta}{(1-b^2\sin^2\theta)^{1/2}}\,d\theta.$$

Corresponding results apply. For example, the substitution $t = \sin\theta$ gives

$$L(1,b^*) = \int_0^1 \frac{(1-t^2)^{1/2}}{(1-b^2t^2)^{1/2}}\,dt$$

(compare 1.3 and 5.8), and power series in terms of b are obtained by modifying 1.5. However, we will not introduce a special notation for these functions.

Recall Theorem 4.6, which stated: $\log\frac{4}{b} < I(1,b) < (1+\frac{1}{4}b^2)\log\frac{4}{b}$ for $0<b<1$. We now derive a corresponding estimation for $J(1,b)$ (actually using the upper bound from 4.8).

6.8 LEMMA. *For* $0<b<1$,

$$\log\frac{1}{b} + c_1 \le L(b,1) \le \log\frac{1}{b} + c_2, \tag{11}$$

where $c_1 = \log 4 - 1$, $c_2 = \pi/4$.

Proof. By 4.8, we have $\log\frac{1}{b} + \log 4 \le I(1,b) \le \log\frac{1}{b} + \frac{\pi}{2}$. Also, $\frac{\pi}{4} \le L(1,b) \le 1$. Statement (11) follows, by (3). □

6.9 PROPOSITION. *For* $0<b<1$,

$$J(1,b) = 1 + \tfrac{1}{2}b^2\log\frac{1}{b} + r(b), \tag{12}$$

where $c_3b^2 \le r(b) < c_4b^2$, *with* $c_3 = \log 2 - \frac{1}{4}$ *and* $c_4 = \frac{1}{4} + \frac{\pi}{8}$.

Proof. Write $J_1(t) = J(1,t)$. By (6), $J_1'(t) = tL(t,1)$, so $J_1(b) - 1 = \int_0^b tL(t,1)\,dt$. Now

$$\begin{aligned}\int_0^b (-t\log t)\,dt &= \left[-\tfrac{1}{2}t^2\log t\right]_0^b + \int_0^b \tfrac{1}{2}t\,dt \\ &= \tfrac{1}{2}b^2\log\frac{1}{b} + \tfrac{1}{4}b^2.\end{aligned}$$

By (11), a lower bound for $J(1,b)$ is obtained by adding $\int_0^b c_1t\,dt = \frac{1}{2}c_1b^2$, and an upper bound by adding $\frac{1}{2}c_2b^2$. □

Since $J(a,1) = aJ(1,\frac{1}{a})$, we can derive the following estimation for $a>1$,

$$J(a,1) = a + \frac{\log a}{2a} + r_1(a),$$

where $c_3/a \le r_1(a) \le c_4/a$.

It follows from (12) that $J_1'(0) = 0$ (this is not a case of (6)). Note that $\frac{1}{b}J_1'(b) = L(b,1) \to \infty$ as $b \to 0^+$, so J_1 does not have a second derivative at 0.

By (4), we have $L(1,b) + b^2 L(b,1) = J(1,b)$. By (11) and (12), it follows that $L(1,b) = 1 - \frac{1}{2}b^2 \log \frac{1}{b} + O(b^2)$. A corresponding expression for $L(b,1)$ can be derived by combining this with Theorem 4.6.

Values at $(\sqrt{2}, 1)$

Recall from 1.6 that

$$I(\sqrt{2},1) = \int_0^1 \frac{1}{(1-x^4)^{1/2}}\,dx = \tfrac{1}{4}B(\tfrac{1}{4},\tfrac{1}{2}).$$

6.10. *We have*

$$L(\sqrt{2},1) = \int_0^1 \frac{x^2}{(1-x^4)^{1/2}}\,dx = \tfrac{1}{4}B(\tfrac{1}{2},\tfrac{3}{4}). \tag{13}$$

Proof. Substitute $x = \sin\theta$, so that $1 - x^4 = \cos^2\theta(1+\sin^2\theta) = \cos^2\theta(\cos^2\theta + 2\sin^2\theta)$. Then

$$\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}\,dx = \int_0^{\pi/2} \frac{\sin^2\theta}{(\cos^2\theta + 2\sin^2\theta)^{1/2}}\,d\theta = L(\sqrt{2},1).$$

Also, the substitution $x = t^{1/4}$ gives

$$\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}\,dx = \int_0^1 \frac{t^{1/2}}{(1-t)^{1/2}}\tfrac{1}{4}t^{-3/4}\,dt = \tfrac{1}{4}\int_0^1 \frac{1}{t^{1/4}(1-t)^{1/2}}\,dt = \tfrac{1}{4}B(\tfrac{1}{2},\tfrac{3}{4}). \quad \square$$

Now using the identities $\Gamma(a+1) = a\Gamma(a)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, we deduce:

6.11. *We have*

$$L(\sqrt{2},1)I(\sqrt{2},1) = \frac{\pi}{4}. \tag{14}$$

Proof. By the identities just quoted,

$$B(\tfrac{1}{2},\tfrac{1}{4})B(\tfrac{1}{2},\tfrac{3}{4}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} = \frac{\Gamma(\frac{1}{2})^2\Gamma(\frac{1}{4})}{\Gamma(\frac{5}{4})} = 4\pi,$$

hence (14). $\square$

Now $I(\sqrt{2},1) = \pi/(2M_0)$, where $M_0 = M(\sqrt{2},1)$. We deduce:

6.12. *Let* $M_0 = M(\sqrt{2},1)$. *Then*

$$L(\sqrt{2},1) = \frac{M_0}{2} \quad (\approx 0.599070), \tag{15}$$

$$L(1,\sqrt{2}) = \frac{\pi}{2M_0} - \frac{M_0}{2} \quad (\approx 0.711959), \tag{16}$$

$$J(\sqrt{2},1) = \frac{\pi}{2M_0} + \frac{M_0}{2} \quad (\approx 1.910099). \tag{17}$$

Proof. (15) is immediate. Then (16) follows from (3), and (17) from (4), in the form $J(\sqrt{2},1) = 2L(\sqrt{2},1) + L(1,\sqrt{2})$. □

Two related integrals

We conclude with a digression on two related integrals that can be evaluated explicitly.

6.13. *For* $0 < b < 1$,

$$\int_0^{\pi/2} \frac{\cos\theta}{(1-b^2\sin^2\theta)^{1/2}}\,d\theta = \frac{\sin^{-1} b}{b},$$

$$\int_0^{\pi/2} \frac{\sin\theta}{(1-b^2\sin^2\theta)^{1/2}}\,d\theta = \frac{1}{2b}\log\frac{1+b}{1-b}.$$

Proof. Denote these two integrals by $F_1(b)$, $F_2(b)$ respectively. The substitution $x = b\sin\theta$ gives

$$F_1(b) = \frac{1}{b}\int_0^b \frac{1}{(1-x^2)^{1/2}}\,dx = \frac{\sin^{-1} b}{b}.$$

Now let $b^* = (1-b^2)^{1/2}$, so that $b^2 + b^{*2} = 1$. Then $1 - b^2\sin^2\theta = b^{*2} + b^2\cos^2\theta$. Substitute $x = b\cos\theta$ to obtain

$$F_2(b) = \frac{1}{b}\int_0^b \frac{1}{(b^{*2}+x^2)^{1/2}}\,dx = \frac{1}{b}\sinh^{-1}\frac{b}{b^*}.$$

Now $\sinh^{-1} u = \log[u + (1+u^2)^{1/2}]$ and $1 + b^2/b^{*2} = 1/b^{*2}$, so

$$F_2(b) = \frac{1}{b}\log\frac{1+b}{b^*}.$$

Now observe that

$$\frac{1+b}{b^*} = \frac{1+b}{(1-b^2)^{1/2}} = \frac{(1+b)^{1/2}}{(1-b)^{1/2}}. \qquad \square$$

The integral $F_2(b)$ can be used as the basis for an alternative method for the results section 4. Another application is particularly interesting. Writing t for b, note that

$$F_2(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1},$$

so that

$$\int_0^1 F_2(t)\,dt = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

By reversing the implied double integral, one obtains an attractive proof of the well-known identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$ [JL].

7. Further evaluations in terms of the agm; the Brent-Salamin algorithm

We continue to follow the exposition of [Lo1]. Recall that Gauss's theorem (Theorem 3.1) says that $I(a_1, b_1) = I(a, b)$, where $a_1 = \frac{1}{2}(a+b)$ and $b_1 = (ab)^{1/2}$. There are corresponding identities for L and J, but they are not quite so simple. We now establish them, using the same substitution as in the proof of Gauss's theorem.

7.1 THEOREM. *Let $a \neq b$ and $a_1 = \frac{1}{2}(a+b), \quad b_1 = (ab)^{1/2}$. Then*

$$L(b_1, a_1) = \frac{a+b}{a-b}\,[L(b,a) - L(a,b)]. \tag{1}$$

$$L(a_1, b_1) = \frac{2}{a-b}\,[aL(a,b) - bL(b,a)]. \tag{2}$$

Proof. By (6.7), written as an integral on $(-\infty, \infty)$,

$$L(b_1, a_1) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{a_1^2}{(x^2+a_1^2)^{3/2}(x^2+b_1^2)^{1/2}}\,dx.$$

As in Theorem 3.1, we substitute

$$2x = t - \frac{ab}{t}.$$

As before, we find

$$(x^2+b_1^2)^{1/2} = \tfrac{1}{2}\left(t + \frac{ab}{t}\right),$$

$$4(x^2+a_1^2) = 4x^2 + (a+b)^2 = t^2 + a^2 + b^2 + \frac{a^2b^2}{t^2} = \frac{1}{t^2}(t^2+a^2)(t^2+b^2),$$

so that

$$\begin{aligned} L(b_1, a_1) &= \frac{1}{8}(a+b)^2 \int_0^{\infty} \frac{8t^3}{(t^2+a^2)^{3/2}(t^2+b^2)^{3/2}\frac{1}{2}(t+\frac{ab}{t})}\,\tfrac{1}{2}\left(1+\frac{ab}{t^2}\right)\,dt \\ &= (a+b)^2 \int_0^{\infty} \frac{t^2}{(t^2+a^2)^{3/2}(t^2+b^2)^{3/2}}\,dt. \end{aligned}$$

But by (6.7) again,

$$\begin{aligned} L(b,a) - L(a,b) &= \int_0^{\infty} \frac{a^2(t^2+b^2) - b^2(t^2+a^2)}{(t^2+a^2)^{3/2}(t^2+b^2)^{3/2}}\,dt \\ &= (a^2-b^2)\int_0^{\infty} \frac{t^2}{(t^2+a^2)^{3/2}(t^2+b^2)^{3/2}}\,dt. \end{aligned}$$

This proves (1). To deduce (2), write $I(a,b) = I$, $L(a,b) = L$ and $L(b,a) = L^*$. Then $I = L + L^*$, also $I = I(a_1, b_1) = L(a_1, b_1) + L(b_1, a_1)$, so

$$\begin{aligned} (a-b)L(a_1,b_1) &= (a-b)[I - L(b_1,a_1)] \\ &= (a-b)(L+L^*) + (a+b)(L-L^*) \\ &= 2aL + 2bL^*. \qquad \square \end{aligned}$$

7.2 COROLLARY. *We have*

$$2J(a_1,b_1) = abI(a,b) + J(a,b). \tag{3}$$

Proof. By (6.4), $J(a_1,b_1) = a_1^2 L(a_1,b_1) + b_1^2 L(b_1,a_1)$. Continue to write I, L, L^* as above. Substituting (1) and (2), we obtain

$$\begin{aligned} 2(a-b)J(a_1,b_1) &= \tfrac{1}{2}(a+b)^2 2(aL - bL^*) + 2ab(a+b)(L^* - L) \\ &= (a+b)[a(a+b) - 2ab]L + (a+b)[2ab - b(a+b)]L^* \\ &= a(a+b)(a-b)L + b(a+b)(a-b)L^*, \end{aligned}$$

hence

$$2J(a_1,b_1) = a(a+b)L + b(a+b)L^* = ab(L+L^*) + (a^2L + b^2L^*) = abI + J. \quad \square$$

It is possible, but no shorter, to prove (3) directly from the integral expression in 5.6, without reference to $L(a,b)$.

It follows from (1) and (2) that if $a > b$, then $L(a,b) < L(b,a)$ (as already seen in 6.6) and $aL(a,b) > bL(b,a)$.

Note. If a and b are interchanged, then a_1 and b_1 remain the same: they are *not* interchanged! The expressions in (1), (2) and (3) are not altered.

We now derive expressions for $L(a,b)$, $L(b,a)$ and $J(a,b)$ in terms of the agm iteration. In this iteration, recall that c_n is defined by $c_n^2 = a_n^2 - b_n^2$: for this to make sense when $n = 0$, we require $a > b$. We saw in 2.2 that for a certain k, we have $c_n \le 4b/4^{2^{n-k}}$ for $n \ge k$, from which it follows that $\lim_{n\to\infty} 2^n c_n^2 = 0$ and the series $\sum_{n=0}^{\infty} 2^n c_n^2$ is convergent.

7.3 THEOREM. *Let $a > b > 0$, and let a_n, b_n, c_n be defined by the agm iteration for $M(a,b)$, so that $c^2 = a^2 - b^2$. Let $S = \sum_{n=0}^{\infty} 2^{n-1} c_n^2$. Then*

$$c^2 L(b,a) = SI(a,b), \tag{4}$$

$$c^2 L(a,b) = (c^2 - S)I(a,b), \tag{5}$$

$$J(a,b) = (a^2 - S)I(a,b). \tag{6}$$

Proof. Note first that (5) and (6) follow from (4), because $L(a,b) + L(b,a) = I(a,b)$ and $J(a,b) = a^2 I(a,b) - c^2 L(b,a)$ (see 6.3).

Write $I(a,b) = I$, and recall that $I(a_n, b_n) = I$ for all n. Also, write $L(b_n, a_n) = L_n$. Note that $L(b_n, a_n) - L(a_n, b_n) = 2L_n - I$. Recall from 2.2 that $c_{n+1} = \frac{1}{2}(a_n - b_n)$. So by (1), applied to a_n, b_n, we have (for each $n \geq 0$

$$4c_{n+1}^2 L_{n+1} = (a_n - b_n)^2 L_{n+1} = (a_n^2 - b_n^2)(2L_n - I) = c_n^2(2L_n - I),$$

hence

$$2^n c_n^2 L_n - 2^{n+1} c_{n+1}^2 L_{n+1} = 2^{n-1} c_n^2 I.$$

Adding these differences for $0 \leq n \leq N$, we have

$$c_0^2 L_0 - 2^{N+1} c_{N+1}^2 L_{N+1} = \left(\sum_{n=0}^{N} 2^{n-1} c_n^2 \right) I.$$

Taking the limit as $N \to \infty$, we have $c_0^2 L_0 = SI$, i.e. (4). $\square$

Since $I(a,b) = \pi/(2M(a,b))$, this expresses $L(a,b)$ and $J(a,b)$ in terms of the agm iteration.

Recall from 6.11 that $L(\sqrt{2}, 1) I(\sqrt{2}, 1) = \pi/4$. With Theorem 7.3, we can deduce the following expressions for π:

7.4 THEOREM. *Let $M(\sqrt{2}, 1) = M_0$ and let a_n, b_n, c_n be defined by the agm iteration for M_0. Let $S = \sum_{n=0}^{\infty} 2^{n-1} c_n^2$. Then*

$$\pi = \frac{M_0^2}{1 - S}. \tag{7}$$

Hence $\pi = \lim_{n \to \infty} \pi_n$, where

$$\pi_n = \frac{a_{n+1}^2}{1 - S_n}, \tag{8}$$

where $S_n = \sum_{r=0}^{n} 2^{r-1} c_r^2$.

Proof. By (5), $L(\sqrt{2}, 1) = (1 - S) I(\sqrt{2}, 1)$. So, by 6.11,

$$\frac{\pi}{4} = (1 - S) I(\sqrt{2}, 1)^2 = (1 - S) \frac{\pi^2}{4M_0^2},$$

hence $\pi = M_0^2/(1 - S)$. Clearly, (8) follows. $\square$

The sequence in (8) is called the *Brent-Salamin iteration* for π. However, the idea was already known to Gauss.

A minor variant is to use a_n rather than a_{n+1}. Another, favoured by some writers (including [Bor2]) is to work with $M_0' = M(1, \frac{1}{\sqrt{2}}) = M_0/\sqrt{2}$. The resulting sequences are a_n', b_n', c_n', where $a_n' = a_n/\sqrt{2}$ (etc.). Then $S = \sum_{n=0}^{\infty} 2^n c_n'^2$ and $\pi = 2M_0'^2/(1-S)$.

Of course, the rapid convergence of the agm iteration ensures rapid convergence of (8). We illustrate this by performing the first three steps. This repeats the calculation of M_0 in section 2 (with slightly greater accuracy), but this time we record c_n and c_n^2. To calculate c_n, we use the identity $c_n = c_{n-1}^2/(4a_n)$.

n	a_n	b_n	c_n	c_n^2
0	1.41421356	1	1	1
1	1.20710678	1.18920712	0.20710678	0.04289322
2	1.19815695	1.19812352	0.00894983	0.00008010
3	1.19814023	1.19814023	0.00001671	2.79×10^{-10}

From this, we calculate $S_3 = 0.54305342$, hence $1 - S_3 = 0.45694658$ and $\pi_3 = 3.14159264$ to eight d.p. (while $\pi \approx 3.14159265$).

We now give a reasonably simple error estimation, which is essentially equivalent to the more elaborate one in [Bor2, p. 48]. Recall from 2.2 that $c_{n+1} \leq 4/(20^{2^n})$ for all $n \geq 1$.

7.5. *With π_n defined by (8), we have $\pi_n < \pi$ and*

$$\pi - \pi_n < \frac{2^{n+9}}{20^{2^{n+1}}}. \tag{9}$$

Proof. We have

$$\pi - \pi_n = \frac{M_0^2}{1-S} - \frac{a_{n+1}^2}{1-S_n} = \frac{X_n}{(1-S)(1-S_n)},$$

where

$$X_n = M_0^2(1-S_n) - a_{n+1}^2(1-S) = (S-S_n)M_0^2 - (1-S)(a_{n+1}^2 - M_0^2).$$

Now $S_n < S$, so $1 - S_n > 1 - S$. From the calculations above, it is clear that $1 - S > \frac{2}{5}$, so

$$\frac{1}{(1-S)(1-S_n)} < \tfrac{25}{4} < 2^3.$$

Now consider X_n. We have $0 < a_{n+1}^2 - M_0^2 < a_{n+1}^2 - b_{n+1}^2 = c_{n+1}^2$. Meanwhile, $1 < M_0^2 < 2$ and

$$S - S_n = \sum_{r=n+1}^{\infty} 2^{r-1} c_r^2 > 2^n c_{n+1}^2.$$

At the same time, since $2^r c_{r+1}^2 < \frac{1}{2} 2^{r-1} c_r^2$, we have $S - S_n < 2^{n+1} c_{n+1}^2$. Hence $X_n > 0$ (so $\pi_n < \pi$) and

$$X_n < M_0^2 (S - S_n) < 2(S - S_n) < 2^{n+2} c_{n+1}^2 < \frac{2^{n+6}}{20^{2^{n+1}}}.$$

Inequality (9) follows. □

A slight variation of the proof that $X_n > 0$ shows that π_n increases with n.

8. Further results on $K(b)$ and $E(b)$

In this section, we translate some results from sections 6 and 7 into statements in terms of $K(b)$ and $E(b)$. First we give expressions for the derivatives, with some some applications.

We have already seen in 5.7 that $E'(b) = \frac{1}{b}[E(b) - K(b)]$. The expression for $K'(b)$ is not quite so simple:

8.1. *For* $0 < b < 1$,

$$bb^{*2} K'(b) = E(b) - b^{*2} K(b). \tag{1}$$

Proof. Recall that $b^* = (1 - b^2)^{1/2}$. By 6.7, with $a = 1$,

$$bb^{*2} \frac{d}{db} I(1, b) = b^2 I(1, b) - J(1, b).$$

With b replaced by b^*, this says

$$b^* b^2 \frac{d}{db^*} K(b) = b^{*2} K(b) - E(b).$$

Now $db^*/db = -b/b^*$, so by the chain rule

$$K'(b) = -\frac{b}{b^*} \frac{d}{db^*} K(b) = -\frac{b}{b^*} \frac{1}{b^2 b^*} [b^{*2} K(b) - E(b)] = -\frac{1}{bb^{*2}} [b^{*2} K(b) - E(b)]. \quad \square$$

Alternatively, 8.1 can be proved from the power series for $K(b)$ and $E(b)$. Note that these power series also show that $K'(0) = E'(0) = 0$.

The expressions for $K'(b)$ and $E'(b)$ can be combined to give the following more pleasant identities:

8.2 COROLLARY. *For* $0 < b < 1$,

$$b^{*2}[K'(b) - E'(b)] = bE(b), \tag{2}$$

$$b^{*2} K'(b) - E'(b) = bK(b). \tag{3}$$

Proof. We have $bE'(b) = E(b) - K(b)]$. With (1), this gives

$$bb^{*2}[K'(b) - E'(b)] = (1 - b^{*2})E(b) = b^2E(b),$$

hence (2), and (3) similarly. $\square$

8.3 THEOREM (Legendre's identity). *For* $0 < b < 1$,

$$E(b)K(b^*) + E(b^*)K(b) - K(b)K(b^*) = \frac{\pi}{2}. \tag{4}$$

Proof. Write $K(b) - E(b) = F(b)$. Then (4) is equivalent to

$$E(b)E(b^*) - F(b)F(b^*) = \frac{\pi}{2}.$$

Now $E'(b) = -F(b)/b$ and, by (2), $F'(b) = -(b/b^{*2})E(b)$. Hence, again using the chain rule, we have

$$\frac{d}{db}E(b)E(b^*) = -\frac{1}{b}F(b)E(b^*) + E(b)\frac{b}{b^{*2}}F(b^*)$$

and

$$\begin{aligned}\frac{d}{db}F(b)F(b^*) &= \frac{b}{b^{*2}}E(b)F(b^*) + F(b)\left(-\frac{b}{b^*}\right)\frac{b^*}{b^2}E(b^*)\\ &= \frac{b}{b^{*2}}E(b)F(b^*) - \frac{1}{b}F(b)E(b^*).\end{aligned}$$

Hence $E(b)E(b^*) - F(b)F(b^*)$ is constant (say c) on $(0, 1)$.

To find c, we consider the limit as $b \to 0^+$. Note first that $0 < F(b) < K(b)$ for $0 < b < 1$. Now $E(0) = \frac{\pi}{2}$ and $E(1) = 1$. By 1.5 and 5.9, $F(b) < \frac{\pi}{2}b^2$ for $0 < b < \frac{1}{2}$. Also, by 1.2, $F(b^*) < K(b^*) = I(1, b) \le \pi/(2b^{1/2})$. Hence $F(b)F(b^*) \to 0$ as $b \to 0^+$, so $c = E(0)E(1) = \pi/2$. $\square$

Taking $b = b^* = \frac{1}{\sqrt{2}}$ and writing $E(\frac{1}{\sqrt{2}}) = E_1$, and similarly K_1, M_1, so that $K_1 = \pi/(2M_1)$, this gives $2E_1K_1 - K_1^2 = \frac{\pi}{2}$, so that

$$E_1 = \frac{\pi}{4K_1} + \frac{K_1}{2} = \frac{M_1}{2} + \frac{\pi}{4M_1},$$

as found in 6.12 (note that $M_0 = M(\sqrt{2}, 1) = \sqrt{2}M_1$ and $J(\sqrt{2}, 1) = \sqrt{2}E_1$).

In 3.5 and 3.6, we translated the identity $I(a_1, b_1) = I(a, b)$ into identities in terms of $K(b)$. In particular, $I(1 + b, 1 - b) = K(b)$. The corresponding identity for J was given in 7.2:

$$2J(a_1, b_1) = abI(a, b) + J(a, b). \tag{5}$$

We now translate this into statements involving K and E. The method given here can be compared with [Bor2, p. 12]. Recall that if $k = (1-b)/(1+b)$, then $k^* = 2b^{1/2}/(1+b)$: denote this by $g(b)$.

8.4. *We have*

$$2E(b) - b^{*2}K(b) = (1+b)E[g(b)]. \tag{6}$$

Proof. By (5), with a and b replaced by $1+b$ and $1-b$,

$$b^{*2}I(1+b, 1-b) + J(1+b, 1-b) = 2J(1, b^*) = 2E(b).$$

As mentioned above, $I(1+b, 1-b) = K(b)$. Further

$$J(1+b, 1-b) = (1+b)J\left(1, \frac{1-b}{1+b}\right) = (1+b)E[g(b)]. \qquad \square$$

8.5. *We have*

$$E(b^*) + bK(b^*) = (1+b)E\left(\frac{1-b}{1+b}\right). \tag{7}$$

Proof. By (5),

$$\begin{aligned} E(b^*) + bK(b^*) &= J(1,b) + bI(1,b) \\ &= 2J\big(\tfrac{1}{2}(1+b),\, b^{1/2}\big) \\ &= (1+b)J\left(1, \frac{2b^{1/2}}{1+b}\right) \\ &= (1+b)E\left(\frac{1-b}{1+b}\right). \qquad \square \end{aligned}$$

Note that (5) and (6) can be arranged as expressions for $E(b)$ and $E(b^*)$ respectively.

Outline of further results obtained from differential equations

Let us restate the identities for $E'(b)$ and $K'(b)$, writing out b^{*2} as $1-b^2$:

$$bE'(b) = E(b) - K(b), \tag{8}$$

$$b(1-b^2)K'(b) + (1-b^2)K(b) = E(b). \tag{9}$$

We can eliminate $E(b)$ by differentiating (9), using (8) to substitute for $E'(b)$, and then (9) again to substitute for $E(b)$. The result is the following identity, amounting to a second-order differential equation satisfied by $K(b)$:

$$b(1-b^2)K''(b) + (1-3b^2)K'(b) - bK(b) = 0. \tag{10}$$

Similarly, we find that

$$b(1-b^2)E''(b) + (1-b^2)E'(b) + bE(b) = 0. \tag{11}$$

By following this line of enquiry further, we can obtain series expansions for $K(b^*) = I(1,b)$ and $E(b^*) = J(1,b)$. Denote these functions by $K^*(b)$ and $E^*(b)$. To derive the differential equations satisfied by them, we switch temporarily to notation more customary for differential equations. Suppose we know a second-order differential equation satisfied by $y(x)$ for $0 < x < 1$, and let $z(x) = y(x^*)$, where $x^* = (1-x^2)^{1/2}$. By the chain rule, we find that $y'(x^*) = -(x^*/x)z'(x)$ and

$$y''(x^*) = \frac{x^{*2}}{x^2}z''(x) - \frac{1}{x^3}z'(x).$$

Substituting in the equation satisfied by $y(x^*)$, we derive one satisfied by $z(x)$. In this way, we find that $K^*(x)$ satisfies

$$x(1-x^2)y''(x) + (1-3x^2)y'(x) - xy(x) = 0. \tag{12}$$

Interestingly, this is the same as the equation (10) satisfied by $K(x)$. Meanwhile, $E^*(x)$ satisfies

$$x(1-x^2)y''(x) - (1+x^2)y'(x) + xy(x) = 0 \tag{13}$$

(which is not the same as (11)).

Now look for solutions of (12) of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+\rho}$. We substitute in the equation and equate all the coefficients to zero. The coefficient of $x^{\rho-1}$ is ρ^2. Taking $\rho = 0$, we obtain the solution $y_0(x) = \sum_{n=0}^{\infty} d_{2n}x^{2n}$, where $d_0 = 1$ and

$$d_{2n} = \frac{1^2.3^2.\ldots.(2n-1)^2}{2^2.4^2.\ldots.(2n)^2}.$$

Of course, we knew this solution already: by 1.4, $y_0(x) = \frac{2}{\pi}K(x)$. A second solution can be found by the method of Frobenius. We give no details here, but the solution obtained is

$$y_1(x) = y_0(x)\log x + 2\sum_{n=1}^{\infty} d_{2n}e_{2n}x^{2n},$$

where

$$e_{2n} = \sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r}\right).$$

Now relying on the appropriate uniqueness theorem for differential equations, we conclude that $K^*(x) = c_0y_0(x) + c_1y_1(x)$ for some c_0 and c_1. To determine these constants, we use 4.3, which can be stated as follows: $K^*(x) = \log 4 - \log x + r(x)$, where $r(x) \to 0$ as $x \to 0^+$. By

1.4, we have $0 < y_0(x) - 1 < x^2$ for $x < \frac{1}{2}$. Hence $[y_0(x) - 1]\log x$, therefore also $y_1(x) - \log x$, tends to 0 as $x \to 0^+$. So

$$K^*(x) = c_0 y_0(x) + c_1 y_1(x) = c_0 + c_1 \log x + s(x),$$

where $s(x) \to 0$ as $x \to 0^+$. It follows that $c_1 = -1$ and $c_0 = \log 4$. We summarise the conclusion, reverting to the notation b (compare [Bor2, p. 9]):

8.6 THEOREM. *Let $0 < b < 1$. With d_{2n}, e_{2n} as above, we have*

$$\begin{aligned} I(1,b) = K(b^*) &= \frac{2}{\pi} K(b) \log \frac{4}{b} - 2 \sum_{n=1}^{\infty} d_{2n} e_{2n} b^{2n} && (14) \\ &= \log \frac{4}{b} + \sum_{n=1}^{\infty} d_{2n} \left(\log \frac{4}{b} - 2e_{2n} \right) b^{2n}. && (15) \end{aligned}$$

It is interesting to compare (15) with Theorem 4.6. The first term of the series in (15) is $\frac{1}{4}b^2(\log \frac{4}{b} - 1)$, so we can state

$$I(1,b) = (1 + \tfrac{1}{4}b^2) \log \frac{4}{b} - \tfrac{1}{4}b^2 + O\left(b^4 \log \frac{1}{b} \right)$$

for small b, which is rather more accurate than the corresponding estimate in 4.6. Also, $e_{2n} < \log 2$ for all n, since it is increasing with limit $\log 2$. Hence $\log \frac{4}{b} - 2e_{2n} > 0$ for all n, so (15) gives the lower bound

$$I(1,b) > (1 + \tfrac{1}{4}b^2) \log \frac{4}{b} - \tfrac{1}{4}b^2,$$

slightly stronger than the lower bound in 4.6. However, the upper bound in 4.6 is $(1 + \frac{1}{4}b^2)\log(4/b)$, and some detailed work is needed to recover this from (15).

For the agm, the estimation above translates into

$$M(a,1) = F(a) - \frac{F(a)}{4a^2} \frac{\log 4a - 1}{\log 4a} + O(1/a^3) \quad \text{as } a \to \infty,$$

where $F(a) = (\pi a)/(2 \log 4a)$.

One could go through a similar procedure to find a series expansion of $J(1,b)$. However, it is less laborious (though still fairly laborious!) to use 6.7 in the form

$$J(1,b) = b^2 I(1,b) - b(1 - b^2) \frac{d}{db} I(1,b)$$

and substitute (15). The result is:

8.7 THEOREM. *For* $0 < b < 1$,

$$J(1,b) = 1 + \sum_{n=1}^{\infty} f_{2n} \left(\log \frac{4}{b} - g_{2n} \right) b^{2n}, \tag{16}$$

where

$$f_{2n} = \frac{1^2.3^2.\ldots(2n-3)^2(2n-1)}{2^2.4^2.\ldots(2n-2)^2(2n)},$$

$$g_{2n} = 2\sum_{r=1}^{n-1} \left(\frac{1}{2r-1} - \frac{1}{2r} \right) + \frac{1}{2n-1} - \frac{1}{2n}.$$

The first term in the series in (16) is $(\frac{1}{2}\log\frac{4}{b} - \frac{1}{4})b^2$. Denoting this by $q(b)$, we conclude that $J(1,b) > 1 + q(b)$ (since all terms of the series are positive), and

$$J(1,b) = 1 + q(b) + O(b^4 \log 1/b)$$

for small b. This can be compared with the estimation in 6.9.

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