Double zeta sums

Notes by G.J.O. Jameson

Recall that $\zeta(j) = \sum_{n=1}^{\infty} \frac{1}{n^j}$ for $j > 1$. The *double zeta sum* $\zeta(j,k)$ is defined for integers $j \geq 2$ and $k \geq 1$ by

$$\zeta(j,k) = \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{n=m+1}^{\infty} \frac{1}{n^j}. \tag{1}$$

(Unfortunately, the notation $\zeta(k,j)$ is sometimes used instead.) Clearly, $\zeta(j,k)$ can be rewritten as

$$\zeta(j,k) = \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{n=1}^{\infty} \frac{1}{(m+n)^j}. \tag{2}$$

Also, reversing the order in (1), we have

$$\zeta(j,k) = \sum_{n=2}^{\infty} \frac{1}{n^j} \sum_{m=1}^{n-1} \frac{1}{m^k}. \tag{3}$$

Double zeta sums were first investigated by Euler. There is now a massive repertoire of known identities and special values, some of which can be seen, without proof, in compilations such as Wikipedia. Here we present a small selection of some of the simpler results, with proofs (including alternative proofs in some cases). An extensive bibliography (as well as specific references to Euler's work) is given in [BBr]; we will not even attempt to select from it here.

We start with an elegant identity known as Euler's reflection formula (one of several results named this way!).

THEOREM 1. *For $j, k \geq 2$,*

$$\zeta(j,k) + \zeta(k,j) = \zeta(j)\zeta(k) - \zeta(j+k). \tag{4}$$

In particular,

$$\zeta(j,j) = \tfrac{1}{2}\zeta(j)^2 - \tfrac{1}{2}\zeta(2j). \tag{5}$$

Proof. By (1), with m and n reversed,

$$\zeta(k,j) = \sum_{n=1}^{\infty} \frac{1}{n^j} \sum_{m=n+1}^{\infty} \frac{1}{m^k}.$$

Adding this to (1), we obtain

$$\begin{aligned}\zeta(j,k)+\zeta(k,j) &= \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{1}{m^k n^j} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^k n^j} - \sum_{n=1}^{\infty} \frac{1}{n^{j+k}} \\ &= \zeta(j)\zeta(k) - \zeta(j+k). \qquad \square\end{aligned}$$

The case $j = 2$, with the well-known values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, gives

$$\zeta(2,2) = \frac{\pi^4}{2}\left(\frac{1}{36} - \frac{1}{90}\right) = \frac{\pi^4}{120} = \frac{3}{4}\zeta(4). \tag{6}$$

We now focus on sums of the form $\zeta(j,1)$. Write

$$H_n = \sum_{m=1}^{n} \frac{1}{m}$$

(also $H_0 = 0$). By (3), we have

$$\zeta(j,1) = \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^j} = \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^j}, \tag{7}$$

which is worth re-stating in the following form:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^j} = \zeta(j,1) + \zeta(j+1). \tag{8}$$

We mention that sums of the form $\sum_{n=1}^{\infty} H_n^r/n^j$ were also investigated by Euler, and are called *Euler sums*. Again, numerous identities are known, but here we confine attention to the case where $r = 1$.

Values of $\zeta(j,k)$, in terms of ordinary (single) zeta values, are known for assorted small values of j and k. Here we present the two most basic ones, $\zeta(2,1)$ (one of Euler's results) and $\zeta(3,1)$.

THEOREM 2. *We have*

$$\zeta(2,1) = \zeta(3) \tag{9}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn(m+n)} = 2\zeta(3). \tag{10}$$

Proof. By (2),

$$\zeta(2,1) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m(m+n)^2}.$$

Interchanging m and n and adding the two expressions, we have

$$\begin{aligned} 2\zeta(2,1) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^2}\left(\frac{1}{m}+\frac{1}{n}\right) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn(m+n)}. \end{aligned}$$

Now

$$\frac{1}{mn(m+n)} = \frac{1}{m^2}\,\frac{m}{n(m+n)} = \frac{1}{m^2}\left(\frac{1}{n}-\frac{1}{m+n}\right)$$

and by cancellation

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{m+n}\right) = 1+\frac{1}{2}+\cdots+\frac{1}{m} = H_m,$$

so by (8)

$$2\zeta(2,1) = \sum_{m=1}^{\infty}\frac{H_m}{m^2} = \zeta(2,1)+\zeta(3). \qquad \square$$

This pleasantly simple proof is given in [BBr, p. 7–8], where it is attributed to R. Steinberg [St]. Other writers have rediscovered it independently, for example [Jam]. Various alternative proofs and generalisations will emerge below.

COROLLARY 2.1. *We have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\max(m,n)} = 3\zeta(3)$$

Proof. Denote the sum by S. The terms with $m=n$ contribute $\zeta(3)$. The terms with $n>m$ contribute the same as those with $m>n$, so we have

$$S = \zeta(3) + 2\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m+1}^{\infty}\frac{1}{n^2} = \zeta(3)+2\zeta(2,1) = 3\zeta(3). \qquad \square$$

The following evaluation of $\zeta(3,1)$ was stated by Goldbach in a letter to Euler in 1742. Our proof is along the same lines as Theorem 2; however, the reader is at liberty to omit it because the result is a special case of both Theorem 4 and Theorem 5 below.

THEOREM 3. *We have*

$$\zeta(3,1) = \tfrac{1}{4}\zeta(4) = \frac{\pi^4}{360}. \tag{11}$$

Proof. We will show that $\zeta(4) = \zeta(3,1) + \zeta(2,2)$. By (6), $\zeta(2,2) = \frac{3}{4}\zeta(4)$, so (11) follows.

By (2), added to the same expression with m and n interchanged,

$$\begin{aligned} 2\zeta(3,1) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^3}\left(\frac{1}{m}+\frac{1}{n}\right) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn(m+n)^2}. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{mn(m+n)^2} &= \frac{1}{m^2(m+n)}\left(\frac{1}{n}-\frac{1}{m+n}\right) \\ &= \frac{1}{m^2 n(m+n)} - \frac{1}{m^2(m+n)^2} \\ &= \frac{1}{m^3}\left(\frac{1}{n}-\frac{1}{m+n}\right) - \frac{1}{m^2(m+n)^2}, \end{aligned}$$

so, as in the proof of Theorem 2,

$$\begin{aligned} 2\zeta(3,1) &= \sum_{m=1}^{\infty}\frac{H_m}{m^3} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2(m+n)^2} \\ &= \zeta(4) + \sum_{m=2}^{\infty}\frac{H_{m-1}}{m^3} - \zeta(2,2) \\ &= \zeta(4) + \zeta(3,1) - \zeta(2,2), \end{aligned}$$

so $\zeta(3,1) = \zeta(4) - \zeta(2,2)$. □

We now show that $\zeta(k,1)$ can be expressed in terms of ordinary zeta values for all k, another result of Euler. First, a lemma.

LEMMA 1. *For fixed* $n \geq 1$,

$$\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left(\frac{1}{m-n}-\frac{1}{m}\right) = -H_{n-1} + \frac{1}{n}. \tag{12}$$

Proof. Clearly,

$$\sum_{m=1}^{n-1}\left(\frac{1}{m-n}-\frac{1}{m}\right) = -2H_{n-1}.$$

Also, by cancellation,

$$\sum_{m=n+1}^{\infty}\left(\frac{1}{m-n}-\frac{1}{m}\right) = \sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{n+r}\right) = H_n.$$

Combine these identities to obtain (12). $\square$

THEOREM 4. *For $k \geq 2$,*

$$2\zeta(k,1) = k\zeta(k+1) - \sum_{j=1}^{k-2} \zeta(k-j)\zeta(j+1). \tag{13}$$

Note. The case $k = 2$ says $\zeta(2,1) = \zeta(3)$, with the right-hand sum empty. The case $k = 3$ says $2\zeta(3,1) = 3\zeta(4) - \zeta(2)^2$, which equates to (11).

Proof (cf. [BBr, p. 10–11]. Assume that $k \geq 3$ and write $S = \sum_{j=1}^{k-2} \zeta(k-j)\zeta(j+1)$. Then $S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{m,n}$, where

$$S_{m,n} = \sum_{j=1}^{k-2} \frac{1}{m^{k-j} n^{j+1}}.$$

Then $S_{n,n} = (k-2)/n^{k+1}$, so

$$S = (k-2)\zeta(k+1) + T, \tag{14}$$

where

$$T = \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} S_{m,n}.$$

For $m \neq n$, by the geometric series,

$$\begin{aligned} S_{m,n} &= \frac{1}{m^{k-1} n^{k-1}} \sum_{j=1}^{k-2} m^{j-1} n^{k-j-2} \\ &= \frac{1}{m^{k-1} n^{k-1}} \frac{m^{k-2} - n^{k-2}}{m-n} \\ &= A_{m,n} + B_{m,n}, \end{aligned}$$

where

$$A_{m,n} = \frac{1}{mn^{k-1}(m-n)}, \qquad B_{m,n} = \frac{1}{m^{k-1} n(n-m)} = A_{n,m}.$$

Hence

$$T = 2 \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} A_{m,n}.$$

Now

$$A_{m,n} = \frac{1}{n^k} \left(\frac{1}{m-n} - \frac{1}{m} \right).$$

By (12) and (7), we now have

$$\begin{aligned} T &= -2 \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} + 2 \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} \\ &= -2\zeta(k,1) + 2\zeta(k+1). \end{aligned}$$

This, together with (14), gives (13). $\square$

We now present a result in the opposite direction, expressing single zeta values as a combination of double zeta values. The geometric series is again a key element of the proof..

THEOREM 5. *For $k \geq 3$,*

$$\sum_{j=1}^{k-2} \zeta(k-j, j) = \zeta(k). \tag{15}$$

Note. The case $k = 3$ says $\zeta(2,1) = \zeta(3)$, and the case $k = 4$ says $\zeta(4) = \zeta(3,1) + \zeta(2,2)$, as seen in Theorem 3.

Proof (cf. [BBr, Theorem 3]). By (2), we have

$$\zeta(k-j, j) = \sum_{m=1}^{\infty} \frac{1}{m^j} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{k-j}},$$

so

$$\sum_{j=1}^{k-2} \zeta(k-j, j) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{m,n},$$

where

$$S_{m,n} = \sum_{j=1}^{k-2} \frac{1}{m^j (m+n)^{k-j}} = \frac{1}{(m+n)^k} \sum_{j=1}^{k-2} \left(\frac{m+n}{m} \right)^j.$$

Now

$$\sum_{j=1}^{k-2} x^j = \frac{x}{x-1}(x^{k-2} - 1)$$

for $x > 1$. Applying this with $x = (m+n)/m$, so that $x/(x-1) = (m+n)/n$, we obtain

$$\begin{aligned} S_{m,n} &= \frac{1}{(m+n)^k} \frac{m+n}{n} \left[\left(\frac{m+n}{m} \right)^{k-2} - 1 \right] \\ &= \frac{1}{m^{k-2} n(m+n)} - \frac{1}{n(m+n)^{k-1}}. \end{aligned}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{m^{k-2} n(m+n)} = \frac{1}{m^{k-1}} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{m+n} \right) = \frac{H_m}{m^{k-1}},$$

hence

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{m,n} = \sum_{m=1}^{\infty} \frac{H_m}{m^{k-1}} - \zeta(k-1, 1).$$

By (8), this equates to $\zeta(k)$. $\square$

Integral representations: alternative proof of Theorem 2

There are numerous integral representations of zeta sums, both single and double. Here we confine ourselves to $\zeta(2)$, $\zeta(3)$ and $\zeta(2,1)$, thereby obtaining another proof of (9). We freely assume the validity of termwise integration for the series that arise. This is justified by the monotone convergence theorem, since the terms are all positive. It can also be justified in more elementary terms by considering integrals on $[0,x]$, applying estimates for the tails of the series in question, and taking the limit as $x \to 1$.

First, from the power series for $\log(1-x)$, we have

$$-\int_0^1 \frac{\log(1-x)}{x} = \sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2). \tag{16}$$

THEOREM 6. *We have*

$$\zeta(3) = \zeta(2,1) = \int_0^1 \frac{\log x \log(1-x)}{x}\,dx. \tag{17}$$

Proof. Denote this integral by I. Recall that for $n \geq 1$,

$$\int_0^1 x^{n-1}(-\log x)\,dx = \frac{1}{n^2}.$$

Substituting the series for $\log(1-x)/x$ as in (16) and integrating termwise, we obtain

$$I = \sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n}(-\log x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3).$$

Meanwhile, the substitution $x = 1-y$ gives

$$I = \int_0^1 \frac{\log(1-y)\log y}{1-y}\,dy.$$

Now

$$\frac{-\log(1-y)}{1-y} = \left(\sum_{m=1}^{\infty} \frac{y^m}{m}\right)\left(\sum_{n=0}^{\infty} y^n\right) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{\infty} y^{m+n}$$

for $|y| < 1$. Multiplying by $-\log y$ and integrating termwise, we obtain

$$I = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{\infty} \frac{1}{(m+n+1)^2} = \zeta(2,1). \qquad \square$$

A minor variation, proved similarly or deduced from (17) on integrating by parts, is

$$\zeta(3) = \zeta(2,1) = \frac{1}{2}\int_0^1 \frac{\log^2(1-x)}{x}\,dx.$$

We mention briefly some double integral representations obtained by similar methods, although they do not comfortably deliver another proof of (9).

PROPOSITION 7. *We have*

$$\zeta(2) = \int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy, \tag{18}$$

$$\zeta(3) = -\frac{1}{2}\int_0^1 \int_0^1 \frac{\log xy}{1-xy}\, dx\, dy. \tag{19}$$

Proof. By the geometric series and termwise integration,

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 x^n y^n\, dx\, dy == \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \zeta(2),$$

also

$$\begin{aligned} -\int_0^1 \int_0^1 \frac{\log x}{1-xy}\, dx\, dy. &= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n(-\log x)\, dx\, dy \\ &= \sum_{n=0}^{\infty} \int_0^1 x^n(-\log x)\, dx \int_0^1 y^n\, dy \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} \\ &= \zeta(3). \end{aligned}$$

The same applies with $\log x$ replaced by $\log y$. Add the two expressions to obtain (19). $\square$

PROPOSITION 8. *We have*

$$\zeta(2,1) = -\int_0^1 \int_0^1 \frac{\log(1-xy)}{1-xy}\, dx\, dy. \tag{20}$$

Proof. As in Theorem 6,

$$\frac{-\log(1-xy)}{1-xy} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{\infty} (xy)^{m+n}$$

for $|xy| < 1$. Hence the integral in (20) equates to

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^{m+n}\, dx\, dy = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{\infty} \frac{1}{(m+n+1)^2} = \zeta(2,1). \qquad \square$$

Zeta sums with alternating signs

A variation is to introduce alternating signs. As in [BBr], we use the following notation: if j is replaced by $\overline{j}$, then $1/n^j$ is replaced by $(-1)^n/n^j$ where it occurs. So for single zeta sums, we have

$$\zeta(\overline{j}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^j}.$$

Clearly,

$$\zeta(j) + \zeta(\overline{j}) = 2\sum_{m=1}^{\infty} \frac{1}{(2m)^j} = \frac{\zeta(j)}{2^{j-1}},$$

so

$$\zeta(\overline{j}) = -\left(1 - \frac{1}{2^{j-1}}\right)\zeta(j). \tag{21}$$

In particular, $\zeta(\overline{2}) = -\frac{1}{2}\zeta(2)$ and $\zeta(\overline{3}) = -\frac{3}{4}\zeta(3)$. Also, $\zeta(\overline{1}) = -\log 2$.

For double sums, we have

$$\begin{aligned}
\zeta(\overline{j}, k) &= \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{n=m+1}^{\infty} \frac{(-1)^n}{n^j} & (22)\\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{m^k} \sum_{n=1}^{\infty} \frac{(-1)^n}{(m+n)^j} & (23)\\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{n^j} \sum_{m=1}^{n-1} \frac{1}{m^k}, & (24)
\end{aligned}$$

and corresponding expressions for $\zeta(j, \overline{k})$ and $\zeta(\overline{j}, \overline{k})$. Among these, we note

$$\zeta(\overline{j}, \overline{k}) = \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{n=1}^{\infty} \frac{(-1)^n}{(m+n)^j}. \tag{25}$$

In the same way as (4), we have for $j \geq 2$,

$$\zeta(\overline{j}, \overline{j}) = \zeta(\overline{j})^2 - \zeta(2j).$$

Restricting now to $k = 1$, we exhibit the three expressions corresponding to (7). Write

$$\overline{H}_n = \sum_{m=1}^{n} \frac{(-1)^m}{m}.$$

By (24) and the corresponding formulae, we have

$$\zeta(\overline{j}, 1) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^j} H_{n-1}, \qquad \zeta(j, \overline{1}) = \sum_{n=2}^{\infty} \frac{1}{n^j} \overline{H}_{n-1}, \qquad \zeta(\overline{j}, \overline{1}) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^j} \overline{H}_{n-1}.$$

Adding the term $1/n^{j+1}$ or $(-1)^n/n^{j+1}$ as appropriate, we obtain:

$$\zeta(\overline{j},1)+\zeta(\overline{j+1})=\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^j}, \tag{26}$$

$$\zeta(j,\overline{1})+\zeta(\overline{j+1})=\sum_{n=1}^{\infty}\frac{\overline{H}_n}{n^j}, \tag{27}$$

$$\zeta(\overline{j},\overline{1})+\zeta(j+1)=\sum_{n=1}^{\infty}(-1)^n\frac{\overline{H}_n}{n^j}. \tag{28}$$

Suitable variants of the previous integral exprsssions apply, for example

$$\zeta(\overline{2},1)=\int_0^1\frac{\log(1+y)(-\log y)}{1+y}\,dy=\int_0^1\int_0^1\frac{\log(1+xy)}{1+xy}\,dx\,dy. \tag{29}$$

We finish with a companion to Theorem 2, with a proof that is recognisably a development of the original one; it is a somewhat simplified version of the method in [BBr, p. 8–9],

THEOREM 9. *We have*

$$\zeta(\overline{2},1)=\tfrac{1}{8}\zeta(3). \tag{30}$$

$$\zeta(2,\overline{1})+\zeta(\overline{2},\overline{1})=-\tfrac{5}{8}\zeta(3). \tag{31}$$

Proof. Writing $\zeta(\overline{2},1)$ twice, with m and n interchanged, and adding, we have

$$2\zeta(\overline{2},1)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{(m+n)^2}\left(\frac{1}{m}+\frac{1}{n}\right).$$

There is no transparent way to evaluate this directly. Instead, we combine it with the following sum:

$$\zeta(2,\overline{1})+\zeta(\overline{2},\overline{1})=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m}}{(m+n)^2}\left(\frac{1}{m}+\frac{1}{n}\right), \tag{32}$$

in which we have used (25) with m and n interchanged. So

$$\begin{aligned}2\zeta(\overline{2},1)+\zeta(2,\overline{1})+\zeta(\overline{2},\overline{1}) &= \sum_{m=1}^{\infty}(-1)^m\sum_{n=1}^{\infty}[1+(-1)^n]\frac{1}{(m+n)^2}\left(\frac{1}{m}+\frac{1}{n}\right)\\ &= \sum_{n=1}^{\infty}[1+(-1)^n]\sum_{m=1}^{\infty}(-1)^m\frac{1}{mn(m+n)}.\end{aligned}$$

Denote this sum by S. Clearly, the terms with odd n are zero. For even n, we can express the inner sum in terms of $\overline{H}_n$. For a sequence (a_m) that converges to 0, we have $\sum_{m=1}^{n}a_m=\sum_{m=1}^{\infty}(a_m-a_{m+n})$, so for even n,

$$\overline{H}_n=\sum_{m=1}^{\infty}\left(\frac{(-1)^m}{m}-\frac{(-1)^{m+n}}{m+n}\right)=\sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{m}-\frac{1}{m+n}\right)=\sum_{m=1}^{\infty}(-1)^m\frac{n}{m(m+n)}.$$

Hence

$$S = \sum_{n=1}^{\infty} [1 + (-1)^n] \frac{\overline{H}_n}{n^2}.$$

By (27) and (28), we now have

$$S = \zeta(2, \overline{1}) + \zeta(\overline{3}) + \zeta(\overline{2}, \overline{1}) + \zeta(3),$$

and hence, after cancellation

$$2\zeta(\overline{2}, 1) = \zeta(3) + \zeta(\overline{3}) = \tfrac{1}{4}\zeta(3).$$

For this, we did not need the value of $\zeta(2, \overline{1}) + \zeta(\overline{2}, \overline{1})$. However, this (though not the separate values) now follows easily. By (32),

$$\begin{aligned}
\zeta(2, \overline{1}) + \zeta(\overline{2}, \overline{1}) &= \sum_{m=1}^{\infty} (-1)^m \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} \\
&= \sum_{m=1}^{\infty} (-1)^m \frac{H_m}{m^2} \qquad \text{(as in Theorem 2)} \\
&= \zeta(\overline{2}, 1) + \zeta(\overline{3}),
\end{aligned}$$

by (26). This equates to $(\frac{1}{8} - \frac{3}{4})\zeta(3) = -\frac{5}{8}\zeta(3)$. □

References

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