Using double integrals to solve single integrals

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Consider the integral

$$I_1 = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx,$$

where $b > a > 0$. First, let us clarify why it even exists. Of course, convergence at infinity is ensured by the exponential terms, but the integrals of e^{-ax}/x and e^{-bx}/x, taken separately, are divergent at 0, since these integrands equate asymptotically to $1/x$ as $x \to 0$. However, $e^{-ax} - e^{-bx} = (b-a)x + \frac{1}{2}(b^2 - a^2)x^2 + \cdots$, so $(e^{-ax} - e^{-bx})/x$ tends to the finite limit $b - a$ as $x \to 0$ and there is no problem integrating it on intervals of the form $[0, r]$.

A neat way to evaluate I_1 starts by expressing the integrand itself as an integral:

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy}\, dy. \tag{1}$$

Inserting this into I_1 converts it into a double integral. One might have thought that this is simply making things worse, but reversal of the double integral now delivers the solution in elegant style:

$$\begin{aligned} I_1 &= \int_0^\infty \int_a^b e^{-xy}\, dy\, dx \\ &= \int_a^b \int_0^\infty e^{-xy}\, dx\, dy \\ &= \int_a^b \frac{1}{y}\, dy \\ &= \log b - \log a. \end{aligned} \tag{2}$$

There are other ways to evaluate I_1. It is a special case of the "Frullani integral"

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = (c_0 - c_\infty)(\log b - \log a),$$

where $f(x)$ tends to c_0 as $x \to 0^+$ and to c_∞ as $x \to \infty$ [Fer, p. 134–135]. Another quite simple method is described in [Jam1, p. 280].

However, our topic here is the double-integral method. It has certainly been in circulation for a long time. An example of its use occurs in the 1909 note [Har] by G. H. Hardy: I am grateful to the *Math. Gazette* referee for drawing my attention to this reference. It seems highly likely the method was already well established by then.

We will describe a number of other integrals for which the double-integral method is effective, comparing it with any other available methods. We will not suppress one or two cases where it compares unfavourably with other methods, thereby demonstrating its limitations. Most of our examples are variations of I_1 in one way or another.

First, consider

$$I_2 = \int_0^\infty \frac{\tan^{-1} bx - \tan^{-1} ax}{x}\, dx.$$

This is again of the Frullani type. Taken separately, the two terms would give integrals that diverge at infinity. Since $\frac{d}{dy} \tan^{-1} xy = x/(1 + x^2y^2)$, we have

$$\frac{\tan^{-1} bx - \tan^{-1} ax}{x} = \int_a^b \frac{1}{1 + x^2y^2}\, dy.$$

So after reversal of the double integral as before, we obtain

$$\begin{aligned} I_2 &= \int_a^b \int_0^\infty \frac{1}{1 + x^2y^2}\, dx\, dy \\ &= \int_a^b \frac{\pi}{2y}\, dy \\ &= \frac{\pi}{2}(\log b - \log a). \end{aligned} \tag{3}$$

Another natural variation of I_1 is

$$I_3 = \int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx.$$

This converges at 0, since $\cos ax - \cos bx = \frac{1}{2}(b^2 - a^2)x^2 + O(x^4)$. Both the Frullani method, and the method of [Jam1], establish quite easily that (again) $I_3 = \log b - \log a$. How does the double-integral method perform? The analogue of (1) is

$$\frac{\cos ax - \cos bx}{x} = \int_a^b \sin xy\, dy,$$

so $I_3 = \int_0^\infty \int_a^b \sin xy\, dy\, dx$. Formal reversal now fails completely at the first hurdle: it serves up the obviously divergent integral $\int_0^\infty \sin xy\, dx$! Clearly, we are outside the rules for the procedure (of which more later). However, there is no problem about reversal if we reduce the interval for x-integration to a bounded interval $[0, X]$, to obtain

$$\begin{aligned} \int_0^X \frac{\cos ax - \cos bx}{x}\, dx &= \int_0^X \int_a^b \sin xy\, dy\, dx \\ &= \int_a^b \int_0^X \sin xy\, dx\, dy \\ &= \int_a^b \frac{1}{y}(1 - \cos Xy)\, dy \\ &= \log b - \log a - J(X), \end{aligned}$$

where $J(X) = \int_a^b \frac{1}{y} \cos Xy \, dy$. We can now deduce the result by invoking the Riemann-Lebesgue lemma, which says that for a function $f(y)$ with continuous derivative on $[a, b]$, we have $\int_a^b f(y) \cos Xy \, dy \to 0$ as $X \to \infty$ (this is quite easily proved by integration by parts). Hence $J(X) \to 0$ as $X \to \infty$, and $I_3 = \log b - \log a$. However, we must concede that the other methods deliver this integral more easily.

Our next example is the integral of a product of two terms of the type (1): substitution now results in a *triple* integral. This example appeared in [95F], as part of the evaluation of another double integral. Let

$$I_4 = \int_0^\infty \frac{1}{x^2}(1 - e^{-ax})(1 - e^{-bx}) \, dx.$$

By (1), $\frac{1}{x}(1 - e^{-ax}) = \int_0^a e^{-xy} \, dy$. Substituting similarly for the second factor, we obtain

$$\begin{aligned}
I_4 &= \int_0^\infty \left(\int_0^a e^{-xy} \, dy \right) \left(\int_0^b e^{-xz} \, dz \right) \, dx \\
&= \int_0^a \int_0^b \left(\int_0^\infty e^{-x(y+z)} \, dx \right) \, dz \, dy \\
&= \int_0^a \int_0^b \frac{1}{y+z} \, dz \, dy \\
&= \int_0^a [\log(y+b) - \log y] \, dy \\
&= \Big[(y+b) \log(y+b) - y \log y \Big]_0^a \\
&= (a+b) \log(a+b) - a \log a - b \log b. \qquad (4)
\end{aligned}$$

An alternative method for (4) is to integrate by parts, with $1/x^2$ as one factor, and then deduce the result using (2). The amount of work is comparable. The reader may care to work through the details.

Now consider the following integrals, in which I_1 is modified by the insertion of $\sin x$ or $\cos x$:

$$\begin{aligned}
I_5 &= \int_0^\infty (e^{-ax} - e^{-bx}) \frac{\sin x}{x} \, dx, \\
I_6 &= \int_0^\infty (e^{-ax} - e^{-bx}) \frac{\cos x}{x} \, dx,
\end{aligned}$$

where $b > a > 0$. These integrals are not readily solved by either of the alternative methods mentioned for I_1: they are perhaps among the best applications of the double-integral method.

We recall the well-known integrals

$$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}, \qquad \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \qquad (5)$$

for $a > 0$. Substituting (1) and applying (5), we have

$$\begin{aligned} I_5 &= \int_0^\infty \sin x \int_a^b e^{-xy}\,dy\,dx \\ &= \int_a^b \int_0^\infty e^{-xy} \sin x\,dx\,dy \\ &= \int_a^b \frac{1}{y^2+1}\,dy \\ &= \tan^{-1} b - \tan^{-1} a, \end{aligned} \tag{6}$$

and similarly

$$I_6 = \int_a^b \frac{y}{y^2+1}\,dy = \tfrac{1}{2}\log(b^2+1) - \tfrac{1}{2}\log(a^2+1). \tag{7}$$

An interesting question now presents itself. In the proof of (6), can't we just put $a = 0$ and $b = \infty$, apparently obtaining a very quick and neat proof of the "sine integral"

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \quad ? \tag{8}$$

This would amount to using (1) in the especially simple form $\int_0^\infty e^{-xy}\,dy = 1/x$. The problem is that there are conditions for the reversal of double integrals to be valid, even when (unlike I_3) they both converge! The proper terminology, when distinguishing between the two possible orders of integration, is "repeated" (or "iterated") integrals. For bounded, continuous functions on a bounded region of the plane, all is well: the two repeated integrals are equal. But where an unbounded integrand $f(x,y)$ or an unbounded region is involved, further conditions are needed. There are various ways to state these conditions (e.g. see [Ti, p. 53–55]), but for continuous functions, the following version is sufficient: (i) both functions delivered by the first stage of integration are continuous, and (ii) at least one of the repeated integrals remains finite when applied to $|f(x,y|$. Later, to drive this point home, we will present an example - of a quite similar type to I_1 - in which the two repeated integrals exist but are unequal.

All our repeated integrals so far have satisfied these conditions. Condition (i) is no problem: in fact, one of the first-stage integrals is simply the original integrand. Condition (ii) is satisfied whenever $f(x,y)$ is non-negative and one repeated integral is finite, and (6) and (7) are justified by reference to (2), just using the fact that $|\sin x|$ and $|\cos x|$ are not greater than 1.

However, if we attempt (6) with $[a,b]$ replaced by $(0,\infty)$, condition (ii) fails, because

$$\int_0^\infty |\sin x| \int_0^\infty e^{-xy}\,dy\,dx = \int_0^\infty \frac{|\sin x|}{x}\,dx$$

is divergent. Most frustrating, when we can see that the end result is actually correct!

In fact, it is quite easy to derive the case $b = \infty$ in (6), while retaining $a > 0$. Let

$$I_7 = \int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx.$$

Since $|\sin x|/x \le 1$ for all $x > 0$, we have

$$\left| \int_0^\infty e^{-bx} \frac{\sin x}{x}\, dx \right| \le \int_0^\infty e^{-bx}\, dx = \frac{1}{b}.$$

Now letting b tend to infinity in (6), we deduce that (for $a > 0$)

$$I_7 = \frac{\pi}{2} - \tan^{-1} a. \tag{9}$$

We pause here to give a second proof of (9). This, in fact, is the example of the double-integral method which appeared in the 1909 note [Har], where it is called "Mr. Berry's first proof" (by which Hardy means a proof of (8)). Instead of (1), we use the substitution

$$\frac{\sin x}{x} = \int_0^1 \cos xy\, dy.$$

Inserting this, reversing the repeated integral and applying (5), we obtain

$$\begin{aligned} I_7 &= \int_0^\infty e^{-ax} \int_0^1 \cos xy\, dy\, dx \\ &= \int_0^1 \int_0^\infty e^{-ax} \cos xy\, dx\, dy \\ &= \int_0^1 \frac{a}{y^2 + a^2}\, dy \\ &= \tan^{-1} \frac{1}{a} \\ &= \frac{\pi}{2} - \tan^{-1} a. \end{aligned}$$

Using the fact that $|\cos xy| \le 1$, one checks easily that reversal is within the rules.

There is a companion result for $\cos x$. Note that $\int_0^1 \sin xy\, dy = (1 - \cos x)/x$. Let

$$I_8 = \int_0^\infty e^{-ax} \frac{1 - \cos x}{x}\, dx,$$

where $a > 0$. Copying the second proof of (9), we have

$$\begin{aligned} I_8 &= \int \int_0^\infty e^{-ax} \int_0^1 \sin xy\, dy\, dx \\ &= \int_0^1 \int_0^\infty e^{-ax} \sin xy\, dx\, dy \\ &= \int_0^1 \frac{y}{y^2 + a^2}\, dy \\ &= \tfrac{1}{2} \log(a^2 + 1) - \log a. \end{aligned} \tag{10}$$

Alternatively, one can prove (10) by taking the difference between (2) and (7) and considering the limit as $b \to \infty$.

Now let us return to the question of proving (8). We have seen how to replace b by ∞ in (6); we also need to replace a by 0. This appears plausible when expressed in the following way: write $f_a(x) = e^{-ax} \sin x/x$ and $f(x) = \sin x/x$. Then $\lim_{a\to 0} f_a(x) = f(x)$ for each x, and we have to show that $\lim_{a\to\infty} \int_0^\infty f_a(x)\,dx = \int_0^\infty f(x)\,dx$. However, the pointwise convergence of $f_a(x)$ to $f(x)$ is nowhere near sufficient to imply this, and in fact the proof is quite delicate. For the purpose of proving (8), it is at least equally easy to abandon (6) and start instead with $\int_0^X \frac{\sin x}{x}\,dx$: this is the "incomplete sine integral", $\mathrm{Si}(X)$. So

$$\mathrm{Si}(X) = \int_0^X \sin x \int_0^\infty e^{-xy}\,dy\,dx.$$

Since $|\sin x|/x \le 1$, we have $\int_0^X |\sin x|/x\,dx \le X$, which is finite. So we are now entitled to reverse the double integral, obtaining

$$\mathrm{Si}(X) = \int_0^\infty \int_0^X e^{-xy} \sin x\,dx\,dy.$$

Now

$$\int_0^X e^{-xy} \sin x\,dx = \frac{1}{y^2+1} - r(X,y),$$

where $r(X,y) = \int_X^\infty e^{-xy} \sin x\,dx$, hence $\mathrm{Si}(X) = \frac{\pi}{2} - R(X)$, where $R(X) = \int_0^\infty r(X,y)\,dy$. We can evaluate $r(X,y)$ explicitly, most pleasantly by replacing $\sin x$ by e^{ix} to obtain

$$r_C(X,y) = \int_X^\infty e^{-xy} e^{ix}\,dx = \int_X^\infty e^{-(y-i)x}\,dx = \frac{e^{-(y-i)X}}{y-i}.$$

Since $|y-i| \ge 1$ and $|e^{iX}| = 1$, we have $|r_C(X,y)| \le e^{-Xy}$, hence also $|r(X,y)| \le e^{-Xy}$ and

$$|R(X)| \le \int_0^\infty |r(X,y)|\,dy \le \int_0^\infty e^{-Xy}\,dy = \frac{1}{X}.$$

So $\mathrm{Si}(X) \to \frac{\pi}{2}$ as $X \to \infty$, which proves (8). It must be admitted that we have lost the beautiful simplicity of the original non-proof, but even as amended the method still stands comparison with other methods for the sine integral (e.g. [Har], [Wa], [Lo], [Jam1]). It has done more than just evaluate the integral: it has also established the inequality $|R(X)| \le 1/X$, where $R(X) = \frac{\pi}{2} - \mathrm{Si}(X)$, which is of some interest in its own right. (A stronger inequality was proved in [JLM]: $|R(X)| \le \frac{\pi}{2} - \tan^{-1} X$.)

All this flowed from the integral I_5. Again, there is a corresponding result for I_6. In I_6, the integrals involving e^{-ax} and e^{-bx} do not converge separately, but if (7) were valid with $a = 0$, it would say

$$\int_0^\infty (1 - e^{-bx}) \frac{\cos x}{x}\,dx = \tfrac{1}{2}\log(b^2+1). \tag{11}$$

By similar steps to those above, with the x-integration performed on $[0, X]$ and the y-integration on $[0, b]$, one can show that (11) is indeed correct (we spare the reader the details).

Now combining (10) and (11), with $a = b = 1$, we can deduce

$$\int_0^\infty \frac{e^{-x} - \cos x}{x}\, dx = 0. \tag{12}$$

This integral is also mentioned in [Har]. It has an interesting application. Write

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt, \qquad C(x) = \int_x^\infty \frac{\cos t}{t}\, dt.$$

These are the "incomplete" exponential and cosine integrals. By (12),

$$E(x) - C(x) \to 0 \quad \text{as } x \to 0^+.$$

It is well known that $\int_0^\infty e^{-t} \log t\, dt = -\gamma$, where γ is Euler's constant. It is not hard to show that this fact is equivalent to the statement $E(x) + \log x \to -\gamma$ as $x \to 0^+$. By (12), this statement in turn is equivalent to $C(x) + \log x \to -\gamma$ as $x \to 0^+$, the basic property of the cosine integral. A more thorough discussion of these connections, and a proof of (12) by contour integration, can be seen in [Jam1].

The method we outlined for the sine integral can be generalised to the case where x^{-1} is replaced by x^{p-1}. To express the result, we need the *gamma function*. Recall that this is defined, for $p > 0$, by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx.$$

The outcome is the following pair of integrals, which are neatly analogous to this formula:

$$\int_0^\infty x^{p-1} \cos x\, dx = \Gamma(p) \cos \tfrac{1}{2}\pi p, \qquad \int_0^\infty x^{p-1} \sin x\, dx = \Gamma(p) \sin \tfrac{1}{2}\pi p \tag{13}$$

for $0 < p < 1$. To start, we observe that the substitution $xy = t$ gives

$$\int_0^\infty y^{-p} e^{-xy}\, dy = x^{p-1}\Gamma(1-p).$$

Where we previously substituted $\int_0^\infty e^{-xy}\, dy$ for $1/x$, we now use this (perverse as it might seem) to substitute for x^{p-1}. The steps are essentially similar to those above, combined with some well-known identities for the gamma function. The full proof was set out in the recent *Gazette* article [Jam2], so we will not reproduce it here. Again it delivers an estimate for the remainder: the absolute value of $\int_X^\infty x^{p-1} \sin x\, dx$ is not greater than X^{p-1} itself.

Are there alternative methods for (13)? These are also discussed in [Jam2]. For the special case $p = \frac{1}{2}$ (the "Fresnel integrals"), there is a rather attractive alternative proof by Fourier series, but for general p, the only viable alternative of which I am aware is by contour integration.

We now give the promised example showing that repeated integrals really can be unequal if the conditions fail. It is a slightly modified version of one in [Ti, p. 61]. Pleasingly, it features our own integral I_1.

Example. Consider the repeated integrals

$$J_1 = \int_0^1 \left(\int_0^\infty (ae^{-axy} - be^{-bxy})\, dy \right) dx,$$

$$J_2 = \int_0^\infty \left(\int_0^1 (ae^{-axy} - be^{-bxy})\, dx \right) dy,$$

where $a > b > 0$. Now

$$\int_0^\infty ae^{-axy}\, dy = \int_0^\infty be^{-bxy}\, dy = \frac{1}{x},$$

so $J_1 = 0$. Meanwhile,

$$\int_0^1 ae^{-axy}\, dx = \left[\frac{1}{y} e^{-axy} \right]_{x=0}^1 = \frac{1}{y}(1 - e^{-ay}),$$

so

$$J_2 = \int_0^\infty \frac{1}{y}(e^{-by} - e^{-ay})\, dy.$$

This is exactly $-I_1$, which we have seen equals $\log a - \log b$. Actually, for the purposes of the example, we don't even need to know this value: it is sufficient to observe that $J_2 > 0$, since $e^{-by} > e^{-ay}$ for all $y > 0$,

A related method: differentiation under the integral sign. The method we have been considering is closely related to the equally venerable technique of differentiation under the integral sign. We illustrate this by our original integral I_1. With b fixed, denote the integral by $F(a)$. Differentiate under the integral sign with respect to a: since $\frac{d}{da}e^{-ax} = -xe^{-ax}$, we obtain

$$F'(a) = -\int_0^\infty e^{-ax}\, dx = -\frac{1}{a}.$$

Hence $F(a) = c - \log a$ for some constant c. Since $F(b) = 0$, we have $c = \log b$, hence (2). Most of our other examples (but not I_3) can be translated similarly; at the end of the day, the working is roughly equivalent. However, where unbounded functions or regions are involved, the process of differentation under the integral sign also comes with conditions which need to be verified (they are framed in terms of uniform convergence: see [Ti, p. 59]).

Final remark. There are, of course, other ways in which a double integral is used in the evaluation of a single integral. For example, in one of the standard methods for the probability integral, the given integral is squared, forming a double integral that can be solved more easily. Our theme in this article has been restricted to the tactic of introducing a double integral by expressing part of the integrand as an integral.

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