DILOG. The dilogarithm function

Notes by G. J. O. Jameson

The "dilogarithm" function Li_2 is defined for $|x| \leq 1$ by

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots. \tag{1}$$

It has been called "Spence's function", in tribute to the pioneering study by W. Spence in 1809.

This is actually the case $k = 2$ of the "polylogarithm" $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} x^n/n^k$. In particular, $\mathrm{Li}_0(x) = x/(1-x)$ and $\mathrm{Li}_1(x) = -\log(1-x)$. These formulae serve to extend the definitions beyond the interval $(-1, 1)$.

Here we concentrate on $\mathrm{Li}_2(x)$. The series (1) is uniformly convergent on $[-1, 1]$, defining a function that is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Clearly, $\mathrm{Li}_2(0) = 0$ and $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$.

By termwise integration of the series

$$-\frac{\log(1-t)}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n},$$

we obtain the integral expression

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt \tag{2}$$

for $|x| \leq 1$. (Termwise integration of the series is justified by uniform convergence for $|x| < 1$, and then by continuity of $\mathrm{Li}_2(x)$ at 1 and -1.) In particular,

$$\zeta(2) = \mathrm{Li}_2(1) = -\int_0^1 \frac{\log(1-t)}{t}\, dt = -\int_0^1 \frac{\log u}{1-u}\, du. \tag{3}$$

The formula (2) is meaningful for all $x \leq 1$, and we now use it to extend the definition of $\mathrm{Li}_2(x)$ to such x. An immediate consequence is

$$\mathrm{Li}_2'(x) = -\frac{\log(1-x)}{x}. \tag{4}$$

for all non-zero $x < 1$. In turn, this implies:

DILOG1. *The function* $\mathrm{Li}_2(x)$ *is strictly increasing for all* $x < 1$.

Proof. By (4), $\mathrm{Li}_2'(x) > 0$ both for $0 < x < 1$ and for $x < 0$. Also, from the series, $\mathrm{Li}_2'(0) = 1$. □

Properties of $\mathrm{Li}_2(x)$ may be derived from the series definition (1), the integral definition (2), or a combination of both. We record next some facts that follow easily from the series.

DILOG2. *For $|x| \le 1$, we have*

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \tfrac{1}{2}\mathrm{Li}_2(x^2). \tag{5}$$

In particular, $\mathrm{Li}_2(-1) = -\frac{1}{2}\zeta(2)$.

Proof. By cancellation of the odd powers,

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = 2\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)^2} = \tfrac{1}{2}\mathrm{Li}_2(x^2). \quad \square$$

DILOG3. *For $0 \le x \le 1$,*

$$x + \tfrac{1}{4}x^2 \le \mathrm{Li}_2(x) \le \zeta(2)x$$

and

$$-x \le \mathrm{Li}_2(-x) \le -x + \tfrac{1}{4}x^2 \le -\tfrac{3}{4}x.$$

Proof. The upper bound for $\mathrm{Li}_2(x)$ follows from $x^n \le x$ for $0 \le x \le 1$. The inequalities for $\mathrm{Li}_k(-x)$ follows from the fact that the series is alternating with terms decreasing in magnitude. □

These inequalities can easily be strengthened by considering more terms. Also, using the geometric series, we can derive the following upper bound for the tail of the series:

$$\sum_{r=n+1}^{\infty} \frac{x^r}{r^2} \le \frac{x^{n+1}}{(n+1)^2(1-x)}.$$

Calculation of $\mathrm{Li}_2(x)$ is reasonably quick for $|x| \le \frac{1}{2}$. For example, one finds $\mathrm{Li}_2(\frac{1}{2}) \approx 0.58224$ to five d.p.

DILOG4. *We have*

$$\int_0^1 \mathrm{Li}_2(x)\, dx = \zeta(2) - 1.$$

Proof. Integrating termwise, we obtain

$$\int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{n^2}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)} \right) = \zeta(2) - 1.$$

This can also be derived easily from the integral (2). $\square$

The series definition applies equally to complex arguments. For example, substitution in (1) gives

$$\mathrm{Li}_2(i) = -\tfrac{1}{8}\zeta(2) + iG,$$

where

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$$

is Catalan's constant. However, our results here will be confined to the real case.

Most of our further results will be derived from the integral definition.

DILOG5. *The function* $\mathrm{Li}_2(x)$ *is convex.*

Proof. It is clear from the series that $\mathrm{Li}_2''(x) > 0$ for $0 \le x \le 1$. For $x < 0$, we have

$$\mathrm{Li}_2''(x) = \frac{\log(1-x)}{x^2} + \frac{1}{x(1-x)},$$

so we require $\log(1-x) + x/(1-x) > 0$, equivalently $\log(1+y) > y/(1+y)$ for $y > 0$: this is a well-known inequality, seen by comparison with the integral $\int_1^{1+y} \frac{1}{t}\,dt$. $\square$

Formula (2) can be rewritten in various ways. For $x \ge -1$, we have

$$\mathrm{Li}_2(-x) = -\int_0^{-x} \frac{\log(1-t)}{t}\,dt = -\int_0^{x} \frac{\log(1+t)}{t}\,dt. \tag{6}$$

Also, substituting $t = 1-u$ in (2), we have for all $x \ge 0$,

$$\mathrm{Li}_2(1-x) = -\int_0^{1-x} \frac{\log(1-t)}{t}\,dt = -\int_x^{1} \frac{\log u}{1-u}\,du. \tag{7}$$

We now establish some identities relating the values at x, $1-x$, $-x$ and $-\frac{1}{x}$.

DILOG6 THEOREM ("Euler's reflection formula"). *For* $0 < x < 1$,

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log x \log(1-x). \tag{8}$$

Proof. Integrating by parts in (2), we have

$$-\mathrm{Li}_2(x) = \Big[\log t \log(1-t)\Big]_0^x + \int_0^x \frac{\log t}{1-t}\,dt.$$

Since $|\log(1-t)| \le 2t$ for $0 < t \le \frac{1}{2}$, we have $\log t \log(1-t) \to 0$ as $t \to 0^+$, so, by (7) and (3),

$$\begin{aligned} -\mathrm{Li}_2(x) &= \log x \log(1-x) + \int_0^1 \frac{\log t}{1-t}\,dt - \int_x^1 \frac{\log t}{1-t}\,dt \\ &= \log x \log(1-x) - \zeta(2) + \mathrm{Li}_2(1-x). \qquad \square \end{aligned}$$

In particular, the case $x = \frac{1}{2}$ gives:

DILOG7 COROLLARY. *We have*

$$\mathrm{Li}_2(\tfrac{1}{2}) = \tfrac{1}{2}(\zeta(2) - \log^2 2) = \frac{\pi^2}{12} - \tfrac{1}{2}\log^2 2 \tag{9}$$

We can rewrite (9) as an expression for $\zeta(2)$:

$$\zeta(2) = \log^2 2 + 2\mathrm{Li}_2(\tfrac{1}{2}) = \log^2 2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}.$$

Taking $\log 2$ as known, this can be regarded as a series for $\zeta(2)$ that converges much more rapidly than $\sum_{n=1}^{\infty} 1/n^2$.

Also, for the purpose of calculation of $\mathrm{Li}_2(x)$ when $\frac{1}{2} < x < 1$, we can use (8) to replace $\mathrm{Li}_2(x)$ by $\mathrm{Li}_2(1-x)$, which is given by a more rapidly convergent series. Using (5), we can similarly accelerate the calculation of $\mathrm{Li}_2(-x)$.

DILOG8 THEOREM. *For $x > 0$,*

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2(-\tfrac{1}{x}) = -\zeta(2) - \tfrac{1}{2}\log^2 x. \tag{10}$$

Proof. By (6), we have

$$\mathrm{Li}_2(-x) = \mathrm{Li}_2(-1) + F(x) = -\tfrac{1}{2}\zeta(2) + F(x)$$

where

$$F(x) = \int_x^1 \frac{\log(1+t)}{t}\, dt.$$

Combined with the same statement for $1/x$, this gives

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2(-\tfrac{1}{x}) = -\zeta(2) + F(x) + F(\tfrac{1}{x}).$$

But the substitution $t = 1/u$ gives

$$\begin{aligned} F(x) &= \int_1^{1/x} u \log\left(1 + \frac{1}{u}\right) \frac{1}{u^2}\, du \\ &= \int_1^{1/x} \frac{\log(1+u) - \log u}{u}\, du \\ &= -F(\tfrac{1}{x}) - \tfrac{1}{2}\log^2 x, \end{aligned}$$

so (10) follows. □

So for $x > 1$, we have the following series expression for $\mathrm{Li}_2(-x)$ in powers of $\frac{1}{x}$:

$$\mathrm{Li}_2(-x) = -\tfrac{1}{2}\log^2 x - \zeta(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 x^n}. \tag{11}$$

and we can deduce

$$\mathrm{Li}_2(-x) = -\tfrac{1}{2}\log^2 x - \zeta(2) + r(x),$$

where $0 < r(x) \leq 1/x$.

The next identity is attributed to Landen.

DILOG9 THEOREM. *For all $x > 0$, we have*

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-\tfrac{1}{x}) = -\tfrac{1}{2}\log^2 x. \tag{12}$$

. *Proof.* By (7),

$$\begin{aligned}
\mathrm{Li}_2(1-x) &= -\int_x^1 \frac{\log t}{1-t}\,dt \\
&= \int_1^{\frac{1}{x}} \frac{\log u}{1-1/u}\frac{1}{u^2}\,du \\
&= \int_1^{\frac{1}{x}} \log u\left(\frac{1}{u-1} - \frac{1}{u}\right) du \\
&= \int_0^{\frac{1}{x}-1} \frac{\log(1+v)}{v}\,dv - \tfrac{1}{2}\log^2 x \\
&= -\mathrm{Li}_2(1-\tfrac{1}{x}) - \tfrac{1}{2}\log^2 x, \qquad \text{by (6)} \qquad \square
\end{aligned}$$

The case $x = \frac{1}{2}$ gives (9) again.

With the substitution $y = x - 1$ (where $y > -1$), (12) can be restated as follows:

$$\mathrm{Li}_2(-y) + \mathrm{Li}_2\left(\frac{y}{y+1}\right)) = -\tfrac{1}{2}\log^2(y+1). \tag{13}$$

Note that (12), unlike (8) and (10), relates the values at two points of opposite sign. It is possible, but not really any easier, to deduce (10) from (12) and (8).

The only other explicitly known values of $\mathrm{Li}_2(x)$ relate to the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$. By a judicious combination of the identities above, one finds that $\mathrm{Li}_2(\phi^{-1}) = \frac{1}{10}\pi^2 - \log^2\phi$, together with values for $-\phi$, $-\phi^{-1}$ and ϕ^{-2}. See [Jam], [Lew], [AAR].

Expositions of analagous results for the trilogarithm $\mathrm{Li}_3(x)$ can be seen in [Jam] and [Lew].

References

[AAR] George Andrews, Richard Askey and Ranjan Roy, *Special Functions*, Cambridge Univ. Press (1999).

[Jam] Tim Jameson, Polylogarithms, multiple zeta values and the series of Hjortnaes and Comtet, at www.maths.lancs.ac.uk/~jameson.

[Lew] Leonard Lewin, *Polylogarithms and Associated Functions*, North Holland (1981).

updated 13 November 2018