DABOUSSI'S VERSION OF VINOGRADOV'S BOUND FOR THE EXPONENTIAL SUM OVER PRIMES (slightly simplified)

Notes by Tim Jameson

(recalled on a busman's holiday in Oslo, 2010)

Vinogradov's method to estimate an exponential sum over primes was introduced in 1937 [Vin], and simplified by Vaughan in 1977 [Vau]. An exposition of Vaughan's method is given in [Dav, chapter 24]. Here we describe a variant devised by Daboussi [Dab] as a step in his elementary proof of the prime number theorem.

Let $f(n) = e(\alpha n)$ where $\alpha = a/q$, $(a,q) = 1$, $q > 0$ (actually this all works with minor alterations with $\alpha = a/q + \beta$ where $|\beta| \le 1/q^2$, but we can instead generalise to such a case afterwards using partial summation). We will assume $q, z, y \le x$ in the following, since otherwise our resulting bound would be trivial. Let

$$\begin{aligned} S &:= \sum_{n \le x} \Lambda(n) f(n) \\ &= T_1 + T_2, \end{aligned}$$

where $P^-(n) > z$ in T_1 and $P^-(n) \le z$ in T_2, with z being a small parameter whose value is at our disposal. We have

$$T_2 = \sum_{p \le z} (\log p) \sum_{k \le \log x/\log p} f(p^k),$$

$$|T_2| \le \sum_{p \le z} (\log p) \frac{\log x}{\log p} = \pi(z) \log x \le z \log x,$$

$$\begin{aligned} S_1 &:= \sum_{\substack{n \le x \\ P^-(n) > z}} f(n) \log n \\ &= \sum_{\substack{dm \le x \\ P^-(dm) > z}} \Lambda(m) f(dm) \\ &= T_1 + S_2, \end{aligned}$$

where

$$S_2 = \sum_{\substack{dm \le x \\ d > z,\ P^-(dm) > z}} \Lambda(m) f(dm).$$

Thus

$$T_1 = S_1 - S_2.$$

We treat S_2 as a "type II" sum in Montgomery and Vaughan's parlance (a bilinear form with both sequences of coefficients being non-smooth). We will bound S_1 using the sieve of Eratosthenes-Legendre, leading to a "type I" sum (a bilinear form with only one of the sequnces being non-smooth).

$$S_1 = \sum_{\substack{n\le x\\ P^-(n)>z}} f(n)\int_1^n \frac{dw}{w} = \int_1^x S_1^*(w)\frac{dw}{w},$$

where

$$\begin{aligned}
S_1^*(w) &:= \sum_{\substack{w<n\le x\\ P^-(n)>z}} f(n)\\
&= \sum_{w<n\le x}\left(\sum_{\substack{d\mid n\\ P^+(d)\le z}}\mu(d)\right) f(n)\\
&= \sum_{P^+(d)\le z}\mu(d)\sum_{\substack{w<n\le x\\ n\equiv 0 \bmod d}} f(n)\\
&= \sum_{P^+(d)\le z}\mu(d)\sum_{w/d<n\le x/d} f(dn).
\end{aligned}$$

Since

$$\left|\sum_{n=1}^{N} e(\alpha n)\right| \le \min\left(N, \frac{1}{2\|\alpha\|}\right),$$

we have

$$|S_1^*(w)| \le S_{1,1}^*(w) + S_{1,2}^*(w),$$

where (introducing another free parameter y)

$$\begin{aligned}
S_{1,1}^*(w) &:= \sum_{d\le y}\min\left(\frac{x}{d}, \frac{1}{2\|ad/q\|}\right)\\
&\le \frac{x}{q}\sum_{n\le y/q}\frac{1}{n} + \left(\frac{y}{q}+1\right)\sum_{\substack{n \bmod q\\ n\not\equiv 0}}\frac{1}{2\|n/q\|}\\
&\ll \left(\frac{x}{q}+y+q\right)\log x,
\end{aligned}$$

and (where $\Psi(t,z)$ counts the integers $n\le t$ with $P^+(n)\le z$)

$$\begin{aligned}
x^{-1}S_{1,2}^*(w) &:= \sum_{\substack{d>y\\ P^+(d)\le z}}\frac{1}{d}\\
&= \int_{t=y^+}^{t=\infty}\frac{d\Psi(t,z)}{t}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\Psi(y,z)}{y} + \int_y^\infty \frac{\Psi(t,z)}{t^2}dt \\
&\le \int_y^\infty \frac{\Psi(t,z)}{t^2}dt \\
&\ll \int_y^\infty t^{-1-\frac{1}{2\log z}}dt \quad \text{(by Rankin's method)} \\
&= 2y^{-\frac{1}{2\log z}} \log z.
\end{aligned}
$$

For the simple version of Rankin's bound used here (Daboussi re-does it with more care) see the one page proof in [TMF, section 4.5], where it is done for the application to Daboussi's elementary proof of the PNT (which is closely related to his work that we are describing here). Adding these two bounds together gives our bound for $S_1^*(w)$, which is independent of w, so that the resulting bound for S_1 is merely weaker by a log factor, i.e.

$$S_1 \ll \left(\frac{x}{q} + y + q + xy^{-\frac{1}{2\log z}}\right)\log^2 x.$$

Up to order of magnitude, the y that minimises this expression is given by

$$y = x^{1/\left(1+\frac{1}{2\log z}\right)} \asymp x^{1-\frac{1}{2\log z}},$$

giving

$$S_1 \ll \left(\frac{x}{q} + x^{1-\frac{1}{2\log z}} + q\right)\log^2 x.$$

In S_2 we sum over m on the outside, and split the sum into $\ll \log x$ subsums over ranges $M < m \le 2M$, where $M = 2^r z$ and $z \le M < x/z$. Slapping modulus bars around the inner sum over d means we can ignore the condition $P^-(m) > z$ (which in view of the presence of the $\Lambda(m)$ in the sum merely removes the few prime powers $m = p^k$ with $p \le z$ and $k > 2$). We then apply Cauchy's inequality to the sum over m, expand the modulus squared of the sum over d as a double sum (over d_1, d_2), then switch the order of summation so that we sum over m on the inside (resulting in a geometric sum over m, and a sum over d_2 which is much like the sum over d in $S_{1,2}^*$). If there were an intractable coefficient c_d present then at this point in the argument we would be applying the inequality $|c_{d_1}c_{d_2}| \le \frac{1}{2}(|c_{d_1}|^2 + |c_{d_2}|^2)$, which we would then replace by just $|c_{d_1}|^2$ (using the fact that swapping d_1 with d_2 in the sum over m merely conjugates it). This is what happens in the corresponding part of [Dav, chapter 24] (however, the whole process we have just described is then a proof and use of the 'Halasz-Montgomery' inequality, which is redone in Chapter 27 in the language of vectors for one of the proofs of the large sieve inequality).

Actually in [Dab] (which gives an effective bound without $\ll$ notation) the rôles of m and d in this sequence of manipulations are reversed, ending up using a bound for the correlation sum $\sum_m \Lambda(m)\Lambda(m+k)$ (and for $\Phi(x,y) = \sum_{n\le x, P^-(n)>y} 1$) coming from the large sieve (thus ensuring a slightly sharper result). I couldn't get hold of his earlier paper (which I suspect outlined the method with ineffective constants), meaning that I had a bit of work to do to decipher the details of the combinatorics.

Anyway, instead of spending all day describing what we're about to do, we ought to just do things:

$$\begin{aligned}
\left(\sum_{M<m\le 2M} \Lambda(m) \left| \sum_{\substack{z<d\le x/m \\ P^-(d)>z}} f(dm) \right| \right)^2 &\le \left(\sum_{M<m\le 2M} \Lambda(m)^2\right) \sum_{M<m\le 2M} \left| \sum_{\substack{z<d\le x/m \\ P^-(d)>z}} f(dm) \right|^2 \\
&\ll (M\log M) \sum_{z<d_1,d_2\le x/M} \left| \sum_{\substack{M<m\le 2M \\ m\le x/\max(d_1,d_2)}} f(d_1 m)\overline{f(d_2 m)} \right| \\
&\ll (M\log x) \sum_{d_1,d_2\le x/M} \min\left(M, \frac{1}{\|a(d_1-d_2)/q\|}\right) \\
&\ll (M\log x)\cdot \frac{x}{M}\cdot\left(\frac{x}{Mq}+1\right)(M+q\log q) \\
&\ll \left(\frac{x^2}{q}+\frac{x^2}{M}+xM+xq\right)\log^2 x \\
&\ll \left(\frac{x^2}{q}+\frac{x^2}{z}+xq\right)\log^2 x.
\end{aligned}$$

Taking the square root and summing over M gives

$$S_2 \ll \left(\frac{x}{q^{1/2}} + \frac{x}{z^{1/2}} + (xq)^{1/2}\right)\log^2 x.$$

Combining our results, we now see that

$$S \ll \left(\frac{x}{q^{1/2}} + x^{1-\frac{1}{2\log z}} + \frac{x}{z^{1/2}} + z + (xq)^{1/2}\right)\log^2 x.$$

If we ignore the z term, then we minimise this by choosing z such that $z = x^{1/\log z}$, i.e.

$$z = e^{\sqrt{\log x}},$$

which ultimately grows faster than any positive power of $\log x$, but slower than any positive power of x. Thus the z term we ignored is negligible, and finally we have proved

$$S \ll \left(\frac{x}{q^{1/2}} + xe^{-\frac{1}{2}\sqrt{\log x}} + (xq)^{1/2}\right)\log^2 x.$$

Vaughan's result is stronger for larger x (having $x^{4/5}$ in place of our middle term), but in the (circle method) application to Vinogradov's ternary Goldbach theorem (saying that all large odd numbers can be written as a sum of three primes) we have $q \leq \log^A x$, so that this difference in the shape of the bound becomes irrelevant. Indeed, in that case we have a superior result (since Vaughan has $\log x$ raised to the power $7/2$ rather than our 2). It seems that one of Daboussi's main points was that his approach looks considerably cleaner (especially when it comes to explicitly working it out with good constants).

References *(added by Graham Jameson)*

[Dab] H. Daboussi, Sur le théorème des nombres premiers, *Comptes Rendus Acad. Sci. Paris, Sér. A* **2989** (1984), 161–164.

[Dav] H. Davenport, *Multiplicative Number Theory*, 2nd ed., Springer (1980).

[MV] H. L. Montgomery and R. C. Vaughan, Hilbert's inequality, *J. London Math. Soc.* (2) **8** (1974), 73-82.

[TMF] G. Tenenbaum and M. Mendès France, *The Prime Numbers and Their Distribution*, Amer. Math. Soc. (2000).

[Vau] R. C. Vaughan, Sommes trigonométriques sur les nombres premiers, *Comptes Rendus Acad. Sci. Paris, Sér. A* **258** (1977), 981–983.

[Vin] I. M. Vinogradov, A new method in analytic number theory, *Trav. Inst. Steklov* **10** (1937), 1–122.

tpjameson@gmail.com,

facebook.com/tpjameson

youtube.com/ortega24024