THE GAUSS CIRCLE PROBLEM (the classical exponent $\frac{1}{3}$)

Notes by Tim Jameson

Notation

We write $\lfloor x \rfloor$ for the integer part of x, the unique integer such that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$, and $\{x\}$ for the fractional part $x - \lfloor x \rfloor$. On average $\{x\}$ is $\frac{1}{2}$, motivating us to define $\rho(x) = \frac{1}{2} - \{x\}$. Thus we have $\lfloor x \rfloor = x - \frac{1}{2} + \rho(x)$. We use this notation to write the number of integers in $(a, b]$ as $\lfloor b \rfloor - \lfloor a \rfloor = b - a + \rho(b) - \rho(a)$. And (as is totally usual in any sensible kind of maths) we write $e(x) = e^{2\pi i x}$. Note in particular that $e(x) = e(x)e(y)$ and $e(n) = 1$ for integers n, and so also of course $e(x + n) = e(x)$ (e is a periodic function with period 1).

Formula for the number of integer points inside the circle

Let $N(r)$ denote the number of pairs of integers (m, n) satisfying $m^2 + n^2 \le r^2$, $m > 0$, $n \ge 0$. Clearly then $1 + 4N(r)$ is the number of integer points inside (or on) the circle with centre at $(0, 0)$ and radius r. Since we can make each integer point correspond to a unit square, we see that $N(r)$ is approximately equal to the area of a quadrant of the circle, $\pi r^2/4$. Indeed it is not hard to see that the absolute error in this approximation is $\ll r$. We call the problem of determining the true order of magnitude of the error the 'Gauss Circle Problem'.

In 1906, Sierpiński proved that the error is in fact $\ll r^{2/3}$ [Sier]. We shall give a proof of this estimate here. Over the last century, after a great deal of work, the bound has been improved repeatedly, the latest bound being $r^{131/208} \log^A r$ (for some positive A), due to Huxley [Hux]. In 1915, Hardy proved that the exponent in any such result must be at least $\frac{1}{2}$ [Har]. [Note added by Graham Jameson: Hardy's result was beautifully generalised and simplified by Erdös and Fuchs [EF].]

We are aiming for an accuracy of smaller order of magnitude than r, so we must be careful that we count each integer point on the axes and ray of slope 1 exactly once. Let $M = \lfloor r/\sqrt{2} \rfloor$, and

$$f(x) = \sqrt{r^2 - x^2} - x.$$

Then

$$N(r) = 2\sum_{m=1}^{M} \lfloor f(m) \rfloor + M + \lfloor r \rfloor$$

$$
\begin{aligned}
&= 2\sum_{m=1}^{M}(f(m)-\tfrac12+\rho(f(m)))+M+\lfloor r\rfloor \\
&= 2\sum_{m=1}^{M} f(m)+2\sum_{m=1}^{M}\rho(f(m))+\lfloor r\rfloor .
\end{aligned}
$$

We estimate the sum of the $f(m)$ using the trapezium rule. To this end, let

$$
\begin{aligned}
E &:= \int_0^M f'(x)\rho(x)\,dx \\
&= \sum_{m=0}^{M-1}\int_0^1 f'(m+x)(\tfrac12-x)\,dx \\
&= \sum_{m=0}^{M-1}\left([f(m+x)(\tfrac12-x)]_0^1+\int_0^1 f(m+x)\,dx\right) \\
&= -\tfrac12\sum_{m=0}^{M-1}((f(m)+f(m+1))+\int_0^M f(x)\,dx \\
&= -\tfrac12 f(0)-\tfrac12 f(M)-\sum_{m=1}^{M-1} f(m)+\int_0^M f(x)\,dx .
\end{aligned}
$$

Thus E is the error in the trapezium rule approximation to the integral $\int_0^M f$ using strips of unit width.

Since $f(r/\sqrt{2})=0$ and (clearly) $f'(x)\in[-2,-1]$ for $x\in[0,r/\sqrt{2}]$, we see that $|f(x)|$ is bounded by an absolute constant for $x\in[M,r/\sqrt{2}]$. Hence only a bounded error is introduced by replacing $f(M)$ by zero and changing the upper limit of integration to $r/\sqrt{2}$ in the RHS of this formula for E. The integral then becomes the area of an octant of the circle, namely $\pi r^2/8$. Since also $f(0)=r$, we have now proved that

$$
\sum_{m=1}^{M} f(m)=\frac{\pi r^2}{8}-\tfrac12 r-E+O(1),
$$

and hence

$$
N(r)=\frac{\pi r^2}{4}+2\sum_{m=1}^{M}\rho(f(m))-2E+O(1).
$$

Now we show that E is also bounded by integrating by parts the other way round:

$$
\begin{aligned}
\left|\int_0^1 f'(m+x)(\tfrac12-x)\,dx\right| &\le \left|[f'(m+x)(\tfrac12 x-\tfrac12 x^2)]_0^1-\int_0^1 f''(m+x)\tfrac12 x(1-x)\,dx\right| \\
&\le \frac18\int_m^{m+1}|f''(x)|\,dx .
\end{aligned}
$$

Now $f'(x)=-x/\sqrt{r^2-x^2}-1$ clearly decreases and is negative on $[0,M]$, so that $|f''(x)|=-f''(x)$ and

$$
|E| \le -\frac18\int_0^M f''(x)\,dx
$$

$$
\begin{aligned}
&= \frac{1}{8}(f'(0) - f'(M)) \\
&\ll 1,
\end{aligned}
$$

so finally we have our fundamental formula

$$N(r) = \frac{\pi r^2}{4} + 2\sum_{m=1}^{M} \rho(f(m)) + O(1).$$

An averaging trick

Write $R(x) = N(\sqrt{x}) = \pi x/4 + P(x)$. Our aim is to show that $P(x) \ll x^{1/3}$. Since R is non-decreasing, we have

$$0 \leq R(X+y) - R(X) = \frac{\pi y}{4} + P(X+y) - P(X),$$

i.e.

$$P(X) \leq P(X+y) + \frac{\pi y}{4}.$$

Integrating this from $y = 0$ to $y = Y$, then dividing by Y gives

$$P(X) \leq \frac{1}{Y}\int_X^{X+Y} P(x)\, dx + \frac{\pi Y}{8}.$$

And for a lower bound for $P(X)$, we merely replace X by $X - y$ in the first line of this section and rearrange it as

$$P(X) \geq P(X-y) - \frac{\pi y}{4},$$

to deduce

$$P(X) \geq \frac{1}{Y}\int_{X-Y}^{X} P(x)\, dx - \frac{\pi Y}{8}.$$

Combining both bounds to form a bound for the modulus gives

$$|P(X)| \leq \frac{1}{Y}\max\left(\left|\int_{X-Y}^{X} P(x)\, dx\right|, \left|\int_{X}^{X+Y} P(x)\, dx\right|\right) + \frac{\pi Y}{8}.$$

This yields a bound for $P(X)$, given a bound for the average of $P(x)$ over a short interval. That's a 'mean-to-max' trick –although perhaps Huxley only wants to use that phrase when we're talking about 'mean modulus'-to-'max modulus' theorem, whereas here we actually have a (stronger) 'signed mean'-to-'max modulus' result. Conversely, we can obviously just integrate any given bound for $P(x)$.

The point of all this is that integration of $P(x)$ smooths it out into a continuous function which can be easily bounded using the absolutely convergent Fourier series for $\int \rho$ and our bound for exponential sums, as we now show.

Formula for the average value of $P(x)$ over short intervals

In the all the folllowing we shall assume that $Y \ll \sqrt{X}$. Then for $x = X + O(Y)$, we have $\sqrt{x} = \sqrt{X} + O(Y/\sqrt{X}) = \sqrt{X} + O(1)$,

$$P(x) = 2 \sum_{m \le \sqrt{X/2}} \rho(\sqrt{x - m^2}) + O(1),$$

and

$$\int_{X-Y}^{X+Y} P(x)\, dx = 2 \sum_{m \le \sqrt{X/2}} \int_{X-Y}^{X+Y} \rho(\sqrt{x - m^2})\, dx + O(Y).$$

We now evaluate the integral in the sum on the RHS:

$$\int_{X-Y}^{X+Y} \rho(\sqrt{x - m^2})\, dx = 2 \int_{\sqrt{X-Y-m^2}}^{\sqrt{X+Y-m^2}} \rho(u) u\, du.$$

$$\begin{aligned}
\int_0^N \rho(u) u\, du &= \sum_{n=0}^{N-1} \int_0^1 (\tfrac{1}{2} - u)(u + n)\, du \\
&= \sum_{n=0}^{N-1} \int_0^1 (\tfrac{1}{2} - u) u\, du \\
&= N \int_0^1 (\tfrac{1}{2} u - u^2)\, du \\
&= \tfrac{1}{4} N - \tfrac{1}{3} N \\
&= -\tfrac{1}{12} N.
\end{aligned}$$

$$\begin{aligned}
\int_{\lfloor w \rfloor}^{w} \rho(u) u\, du &= \int_0^{\{w\}} (\tfrac{1}{2} - u)(u + \lfloor w \rfloor)\, du \\
&= \left[\tfrac{1}{4} u^2 - \tfrac{1}{3} u^3\right]_0^{\{w\}} + \lfloor w \rfloor \left[\tfrac{1}{2} u - \tfrac{1}{2} u^2\right]_0^{\{w\}} \\
&= \tfrac{1}{4}\{w\}^2 - \tfrac{1}{3}\{w\}^3 + \lfloor w \rfloor (\tfrac{1}{2}\{w\} - \tfrac{1}{2}\{w\}^2) \\
&= w\sigma(w) + O(1),
\end{aligned}$$

where

$$\sigma(w) = \int_0^w \rho(u)\, du = \tfrac{1}{2}\{w\}(1 - \{w\}).$$

So (using $N = \lfloor w \rfloor$)

$$\begin{aligned}
\int_0^w \rho(u) u\, du &= -\frac{w}{12} + w\sigma(w) + O(1) \\
&= -\frac{w}{2} \tilde{B}_2(w) + O(1),
\end{aligned}$$

$$2\int_a^b \rho(u)u\,du = a\tilde{B}_2(a) - b\tilde{B}_2(b) + O(1),$$

$$\begin{aligned}\int_{X-Y}^{X+Y} \rho(\sqrt{x-m^2})\,dx &= \sqrt{X-Y-m^2}\,\tilde{B}_2(\sqrt{X-Y-m^2}) \\ &\quad -\sqrt{X+Y-m^2}\,\tilde{B}_2(\sqrt{X+Y-m^2}) + O(1) \\ &= \sqrt{X-m^2}\,\Big(\tilde{B}_2(\sqrt{X-Y-m^2}) - \tilde{B}_2(\sqrt{X+Y-m^2})\Big) + O(1),\end{aligned}$$

$$\begin{aligned}\int_{X-Y}^{X+Y} P(x)\,dx &= 2\sum_{m\le\sqrt{X/2}} \sqrt{X-m^2}\,\Big(\tilde{B}_2(\sqrt{X-Y-m^2}) - \tilde{B}_2(\sqrt{X+Y-m^2})\Big) + O(\sqrt{X}) \\ &= \frac{2}{\pi^2}\Re\sum_{h=1}^{\infty}\frac{1}{h^2}\sum_{m\le\sqrt{X/2}} \sqrt{X-m^2}\,\Big(e(\sqrt{X-Y-m^2}) - e(\sqrt{X+Y-m^2})\Big) \\ &\quad +O(\sqrt{X}),\end{aligned}$$

where we have used case $k=2$ of the Fourier series for the k^{th} periodified Bernoulli polynomial

$$\tilde{B}_k(x) = B_k(\{x\}) = -k!\sum_{\substack{h=-\infty \\ h\neq 0}} \frac{e(hx)}{(2\pi i h)^k}.$$

The Kusmin-Landau theorem

Let $f_A,\ldots,f_B$ be real numbers (where A,B are integers with $A<B$) and let $\delta_n = f_n - f_{n-1}$. Suppose that δ_n is a monotone function of n (i.e. it is either a non-decreasing function of n or a non-increasing one). Suppose also that there is some integer K and some $\delta\in(0,\frac{1}{2}]$ such that $\delta_n\in[K+\delta, K+1-\delta]$ for all n. We will prove that we then have

$$S := \left|\sum_{n=A}^{B} e(f_n)\right| \le \frac{2}{\sin\ \pi\delta}.$$

We call various sums such as S 'exponential' or 'trigonometric' sums. This result, by the way, is 'sharp', in the sense that we can can choose the f_n in such a way that $|S| = 2\csc\pi\delta$ (for this extremal case just let the f_n be in arithmetic progression, so that S becomes a geometric sum, and choose the parameters appropriately). Thus if we wanted to show that S were any smaller, we would have to allow ourselves to use more information about the f_n.

If δ_n is decreasing then we negate all the f_n, which simply turns S into $\bar{S}$, which has the same size as S. Thus we may assume without loss of generality that δ_n is non-decreasing. We may also assume that $K=0$, by replacing f_n by $f_n - Kn$.

The proof of the Kusmin-Landau theorem rests on a cunning little trick to express $e(f_n)$ as a difference. We will then perform summation by parts, obtaining a new sum $\sum a_n$. We will then apply the triangle inequality and find that the sum $\sum |a_n|$ telescopes.

Thus we begin by writing (for $A < n \leq B$)

$$e(f_n) - e(f_{n-1}) = e(f_n)(1 - e(-\delta_n)).$$

Now we have (noting that $\delta_n \notin \mathbb{Z}$, so that $e(\delta_n) \neq 1$)

$$\begin{aligned} g_n &:= \frac{1}{1 - e(-\delta_n)} \\ &= \frac{e(\delta_n/2)}{e(\delta_n/2) - e(-\delta_n/2)} = \frac{i \sin\ \pi\delta_n + \cos\ \pi\delta_n}{2i \sin\ \pi\delta_n} \\ &= \tfrac{1}{2} - \tfrac{i}{2} \cot\ \pi\delta_n, \end{aligned}$$

giving the expression we will use for $e(f_n)$ in S:

$$e(f_n) = g_n e(f_n) - g_n e(f_{n-1}).$$

Notice also that

$$|1 - g_n| = |\bar{g}_n| = |g_n| = \frac{1}{|e(\delta_n) - e(-\delta_n)|} = \tfrac{1}{2} \csc\ \pi\delta_n.$$

The expression on the RHS is positive because from our assumptions above we have $0 < \delta_n < 1$.

So now we may write S as

$$\begin{aligned} S &= e(f_A) + \sum_{n=A+1}^{B} (g_n e(f_n) - g_n e(f_{n-1})) \\ &= e(f_A) + \sum_{n=A+1}^{B} g_n e(f_n) - \sum_{n=A}^{B-1} g_{n+1} e(f_n) \\ &= (1 - g_{A+1}) e(f_A) + g_B e(f_B) + \sum_{n=A+1}^{B-1} (g_n - g_{n+1}) e(f_n). \end{aligned}$$

Notice that if $B = A + 1$ then the sum from $n = A + 1$ to $n = B - 1$ is empty, and when interpreted as zero the reasoning here remains valid. This gives

$$\begin{aligned} |S| &\leq |1 - g_{A+1}| + |g_B| + \sum_{n=A+1}^{B-1} |g_n - g_{n+1}| \\ &= \tfrac{1}{2} \csc\ \pi\delta_{A+1} + \tfrac{1}{2} \csc\ \pi\delta_B + \sum_{n=A+1}^{B-1} \tfrac{1}{2}(\cot\ \pi\delta_n - \cot\ \pi\delta_{n+1}) \\ &= \tfrac{1}{2} \csc\ \pi\delta_{A+1} + \tfrac{1}{2} \cot\ \pi\delta_{A+1} + \tfrac{1}{2} \csc\ \pi\delta_B - \tfrac{1}{2} \cot\ \pi\delta_B \\ &\leq \csc\ \pi\delta_{A+1} + \csc\ \pi\delta_B \\ &\leq 2 \csc\ \pi\delta, \end{aligned}$$

as required. Notice that (crucially to the proof) the summand in the last sum was non-negative without us having to write modulus bars around it, because cot is a decreasing function on $(0, \pi)$, and the δ_n are non-decreasing.

An estimate of van der Corput

Suppose we have a differentiable function, $f(x)$, with $f'(x)$ monotone and $f'(x) \in [K+\delta, K+1-\delta]$ for all $x \in [A, B]$. Then with $f_n = f(n)$ we have in the notation above,

$$\delta_n = f(n) - f(n-1) = \int_{n-1}^{n} f'(x)\, dx,$$

which clearly satisfies the conditions for the applicability of the Kusmin-Landau theorem. Since also we have $\sin \pi\delta \geq 2\delta$ for $\delta \in (0, \frac{1}{2}]$, we may write

$$\left| \sum_{n=A}^{B} e(f(n)) \right| \leq \frac{1}{\delta}.$$

Also, if $A = B$ then this sum contains only one term and has modulus 1, so that this bound then holds trivially (since $1/\delta \geq 2$).

Suppose now that we have a function twice differentiable function, $f(x)$, defined for $x \in [a, b]$, and for all such x we have $0 < \lambda \leq f''(x) \leq h\lambda$. Here we have $a < b$ and a, b need not be integers. This implies that $h \geq 1$, and that $f'(x)$ is strictly increasing. From the Kusmin-Landau theorem we will derive a bound for

$$S := \sum_{a \leq n \leq b} e(f(n)).$$

Let $f'(a) = \alpha$, and $f'(b) = \beta$. For a free parameter $\delta \in (0, \frac{1}{2}]$, we partition the interval $[\alpha, \beta]$ into subintervals of the form $I_n = (n-\delta, n+\delta)$ (containing the reals close to the integer n), and of the form $J_n = [n+\delta, n+1-\delta]$ (containing reals a distance at least δ away from the integers). Accordingly, we split the sum S into subsums over ranges for x corresponding to these ranges for $f'(x)$. In the following, note that the number of integers in an interval $[u, v]$ is at most $v - u + 1$. The condition that $I_n \subseteq [\alpha, \beta]$ is equivalent to the condition $n \in (\alpha - \delta, \beta + \delta)$, and the number of n satisfying this is $\leq \beta - \alpha + 2\delta + 1 \leq \beta - \alpha + 2$. Similarly, the condition that $J_n \subseteq [\alpha, \beta]$ is equivalent to the condition $n \in (\alpha - 1 + \delta, \beta - \delta)$, and the number of n satisfying this is $\leq \beta - \alpha - 2\delta + 2 \leq \beta - \alpha + 2$.

$$|S| \leq (\beta - \alpha + 2) \left(\frac{1}{\delta} + \frac{2\delta}{\lambda} + 1 \right).$$

This expression is minimised by choosing $\delta = \sqrt{\lambda/2}$, giving

$$|S| \le (\beta - \alpha + 2)\left(\sqrt{\frac{8}{\lambda}} + 1\right).$$

A partial summation lemma

We have

$$\begin{aligned}
\left|\sum_{a<n\le b} g(n)c_n\right| &= \left|\int_{a^+}^{b^+} g(x) d\sum_{a<n\le x} c_n\right| \\
&= \left|g(b)\sum_{a<n\le b} c_n - \int_a^b g'(x)\sum_{a<n\le x} c_n\, dx\right| \\
&\le \left(|g(b)| + \int_a^b |g'(x)|\, dx\right)\max_{a<x\le b}\left|\sum_{a<n\le x} c_n\right|.
\end{aligned}$$

We often use this to remove (or 'insert' –depending which way round you read things) a smooth slowly varying weight $g(x)$, like with the $|g(x)|$ and $|g'(x)|$ bounded above and below I mean, usually with $g(x) > 0$ real (in which case the integral on the RHS may clearly be scribbled out). Our first use of it will just like that, but in the second use of it our $g(x)$ will have oscillating sign.

Bound for the average value of $P(x)$ over short intervals

If $n = \sqrt{X - m^2}$ then (and one should see this geometrically rather than thinking of the chain rule)

$$\frac{dn}{dm} = -\frac{m}{n},$$

and so

$$\frac{d^2n}{dm^2} = \frac{m}{n^2}\frac{dn}{dm} - \frac{1}{n} = -\frac{m^2}{n^3} - \frac{1}{n} = -\frac{m^2 + n^2}{n^3} = -\frac{X}{n^3},$$

and so for $0 \le m \le n$ (hence $n \asymp \sqrt{X}$) we have

$$-\frac{d^2n}{dm^2} \asymp \frac{1}{\sqrt{X}}.$$

For $h \le X^{1/2}$ (so that $\lambda \ll 1$), multiplying n (as a function of m) by h, and plugging the orders of magnitudes of the 1^{st} 2 derivatives into van der Corput's exponential sum bound gives

$$\sum_{m\le t} e(h\sqrt{X - m^2}) \ll h\left(\frac{h}{\sqrt{X}}\right)^{-1/2} = X^{1/4}h^{1/2}$$

for $t \le \sqrt{X/2} + O(1)$. This bound also holds (trivially) for $h > X^{1/2}$. Then applying partial summation gives

$$\sum_{m \le t} \sqrt{X - m^2}\, e(h\sqrt{X - m^2}) \ll X^{3/4} h^{1/2}.$$

$$\begin{aligned}
\sqrt{X - m^2 \pm Y} &= \sqrt{X - m^2}\, \left(1 \pm \frac{Y}{X - m^2}\right)^{\frac{1}{2}} \\
&= \sqrt{X - m^2}\, \left(1 \pm \tfrac{1}{2} \cdot \frac{Y}{X - m^2} + O\left(\frac{Y^2}{X^2}\right)\right) \\
&= \sqrt{X - m^2} \pm \frac{Y}{2\sqrt{X - m^2}} + O\left(\frac{Y^2}{X^{3/2}}\right),
\end{aligned}$$

implying

$$e(h\sqrt{X - m^2 \pm Y}) = e(h\sqrt{X - m^2}) e\left(\pm \frac{Y}{2\sqrt{X - m^2}}\right) + O\left(\frac{hY^2}{X^{3/2}}\right),$$

and so

$$\begin{aligned}
I &= \sum_{m \le \sqrt{X/2}} \sqrt{X - m^2}\, \left(e(h\sqrt{X - Y - m^2}) - e(h\sqrt{X + Y - m^2})\right) \\
&= \sum_{m \le \sqrt{X/2}} \sqrt{X - m^2}\, e(h\sqrt{X - m^2}) \left(e\left(-\tfrac{hY}{2\sqrt{X - m^2}}\right) - e\left(+\tfrac{hY}{2\sqrt{X - m^2}}\right)\right) + O\left(\frac{hY^2}{\sqrt{X}}\right) \\
&= -2i \sum_{m \le \sqrt{X/2}} \sqrt{X - m^2}\, e(h\sqrt{X - m^2}) \sin \frac{\pi h Y}{\sqrt{X - m^2}} + O\left(\frac{hY^2}{\sqrt{X}}\right) \\
&\ll X^{3/4} h^{1/2} \cdot \sqrt{X} \cdot \frac{hY}{X} \\
&= X^{1/4} h^{3/2} Y.
\end{aligned}$$

So in general we have

$$I \ll X^{3/4} h^{1/2} \min(1, hYX^{-1/2})$$

$$\begin{aligned}
\int_{X-Y}^{X+Y} P(x)dx &\ll \sum_{h \le X^{1/2}/Y} Y X^{1/4} h^{-1/2} + \sum_{h > X^{1/2}/Y} X^{3/4} h^{-3/2} \\
&\ll Y X^{1/4} \left(\frac{X^{1/2}}{Y}\right)^{1/2} + X^{3/4} \left(\frac{X^{1/2}}{Y}\right)^{-1/2} \\
&\ll (YX)^{1/2}.
\end{aligned}$$

The resulting bound for $P(x)$

Using the mean-to-max trick now gives us

$$P(X) \ll \left(\frac{X}{Y}\right)^{1/2} + Y,$$

which is clearly minimised by taking $Y = X^{1/3}$, finally giving

$$P(X) \ll X^{1/3}$$

as promised.

Comments concernings variations of this proof

It should not take too much imagination to envisage that there's nothing spectacularly important about us restricting to a circle in all this, so that (using mainly just a little notational gameplay) we can get similar results for simple closed curves satisfying certain conditions (like twice differentiability and radius of curvature (which is constant for the circle) bounded above and below). We've had formulae with square root signs all over the place coming from Pythagoras. But in essence we are just using linearising tricks.

We have given an elementary derivation of the exponent pair $(\frac{1}{2}, \frac{1}{2})$, resting on the Kusmin-Landau theorem, which is perhaps not as well-known as it ought to be. It appears in the first few pages of Graham and Kolesnik's classic book on exponential sums [GK], which went out-of-print too early!

More usually a proof is given using Poisson summation and bounds for the resulting 'exponential' integrals (proved using integration by parts as suffices for simple forms of the Riemann-Lesbesgue lemma). This method really becomes appropriate when instead of merely bounding all the integrals involved, we estimate the most significant ones (the ones in which there is a 'stationary phase') using quadratic approximations to the 'phase' $f(x)$ to yield Fresnel integrals. This process results in van der Corput's 'B step' for transforming an exponential sum into (when used appropriately) a shorter one. In the case of the circle problem, the integrals arising from the Poisson summation (if we had kept the sum over the full range $(0, r)$ rather than reducing to $(0, r/\sqrt{2})$) turn out to be representations of the J-Bessel function. The convergence issues are delicate: Hardy worked out the details in 1915, resulting in a formula for $P(x)$ (analogous to a formula for $\Delta(x)$ due to Voronoi in 1903).

The B step is 'involutary', meaning that if applied twice then one would simply recover (essentially) the original sum.

Van der Corput's very simple (yet clever) 'A step' is based on Weyl's differencing method and Cauchy's inequality. Well-chosen repeated applications of steps of the form A^kB can then give reasonable bounds for various exponential sums. The relevant theory was worked out in detail by mathematicians such as Phillips and Rankin.

Sums of fractional parts

To estimate sums of values of the ρ function, the natural way to proceed would be to use the Fourier series

$$\rho(x) = \lim_{H\to\infty} \sum_{\substack{h=-H\\ h\neq 0}}^{H} \frac{e(hx)}{2\pi i h},$$

where the LHS is to be interpreted as zero when x is an integer. But the slow convergence of this series leads to problems (the series is necessarily conditionally convergent because of the discontinuities at the integers). By careful estimates (essentially involving finite Fourier polynomials which majorise and minorise ρ) one can prove the Erdös-Turan theorem:

$$\sum_{n=1}^{N} \rho(f(n)) \ll \frac{N}{H} + \sum_{h=1}^{H} \frac{1}{h} \left| \sum_{n=1}^{N} e(hf(n)) \right|.$$

One estimates the RHS, then chooses H to minimise the resulting bound. A few decades ago, Vaaler (using the Beurling-Selberg function) worked out how to choose the approximating functions optimally (in a sense), resulting in the Erdös-Turan theorem with good constants. Not only that, but also such careful work yields a similar result, but with the modulus bars around the inner sum (over n) removed -which allows estimates utilising cancellation in the double sum to give sharper bounds (as in the work of Iwaniec and Mozzochi). A simpler argument (see for example Tenenbaum's book – perhaps the reader can work out such an argument herself) majorising and minorising ρ by continuous functions (with absolutely – hence uniformly – convergent Fourier series) yields this result but with the additional sum

$$\sum_{h=H+1}^{\infty} \frac{H}{h^2} \left| \sum_{n=1}^{N} e(hf(n)) \right|$$

on the RHS. In the application to the circle method (using the $(\frac{1}{2}, \frac{1}{2})$ exponential sum bound) our bound for this extra sum works out to look just the same as the bound for the one in the Erdös-Turan theorem.

References *(added by Graham Jameson)*

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